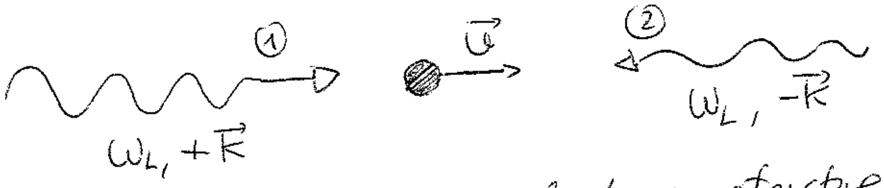
COOLING AND TRAPPING

In this set of lectures we will analyze some of the interesting physics related to cold atomic gases.

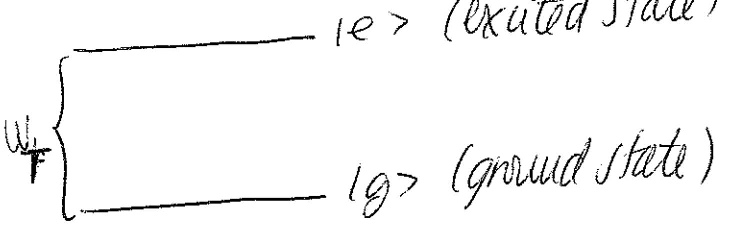
Since all this physics would not be possible without the appropriate cooling and trapping techniques it is perhaps a good idea to briefly introduce the ideas of cooling (laser and evaporative cooling) and trapping.

LASER COOLING

* Laser cooling techniques employ different physical effects (involved in the interaction of light with matter) to cool atomic samples.
* The simplest laser cooling technique is Doppler cooling. We will just briefly renew the idea here.
* Let's consider an atom moving with an initial velocity $\vec{v}$. Let's consider also two counterpropagating laser beams of frequency ω_L and momentum $\pm\vec{k}$:



Let's assume that the electronic structure of the atom can be reduced to two states (two-level atom):



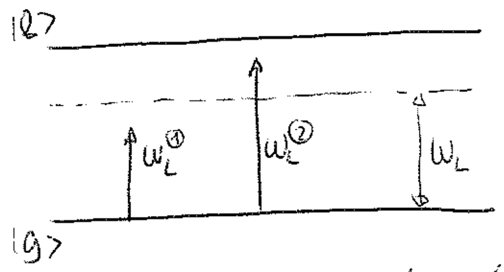
· If the atom is not moving ($v=0$), it sees both lasers with the same frequency $\omega_L = kc$. But if $v \neq 0$, then the atom experiences the Doppler effect, and as a consequence sees an effective frequency

$$\omega_L' = \omega_L - \vec{k}\cdot\vec{v}$$

Hence $\omega_L^{(1)} = \omega_L - kv$

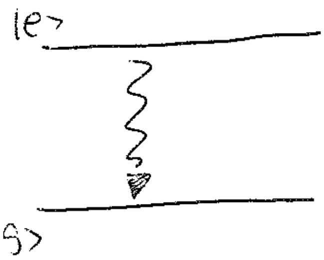
$\omega_L^{(2)} = \omega_L + kv$

· Let's assume that the laser frequency $\omega_L < \omega_T$ (this is what is called red detuning)

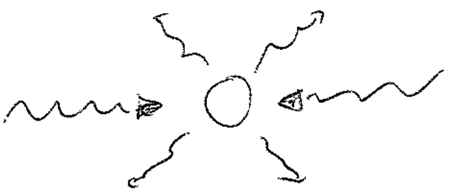


Clearly the photon (2) which opposes to the motion of the atom is closer to resonance with the atomic transition. Since the closer ~~to~~ resonance the larger is the probability that the photon is absorbed, then the counterpropagating photon is absorbed with larger probability.

· After absorbing the photon the atom is in the excited state from which it will eventually decay via spontaneous emission



The spontaneous emission is produced in a random direction. When the photon is emitted it leads to a corresponding random recoil of the atom

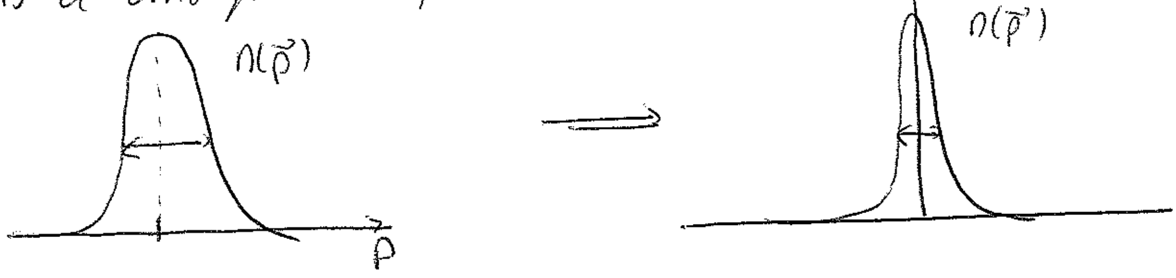


· So the atom experiences a selective absorption against its motion and a subsequent random recoil. After that the process starts again.

- The combination of selective absorption and randomly-oriented spontaneous emission leads to a net friction force for the atom in the direction of the lasers

$$\dot{v} = -\alpha v$$

- As a consequence of that the momentum distribution gets narrower



- Remember from statistical mechanics that a thermal gas presents a Maxwell-Boltzmann momentum distribution

$$n(\vec{p}) \propto e^{-(\vec{p}-\vec{p}_0)^2/2mk_BT}$$

Hence the width of the momentum distribution is directly related with $\sqrt{T}$.

- Therefore if we are able to reduce the width of $n(\vec{p})$ we are effectively <u>cooling</u> the sample.

- So we have seen one example in which a simple physical process involved in the interaction of lasers with atoms, the Doppler cooling, lead to cooling.
- There are other methods which allow to reach even lower temperatures as e.g. Sisyphus cooling, VSCPT, sideband cooling, etc, but we will not discuss them in these lectures.

• TRAPPING OF NEUTRAL ATOMS

• Experiments with atomic samples are of course only possible if one can confine atoms.

• Trapping is relatively easy for charged particles, as ions. However this is not the typical case in cold atoms, where one deals with neutral particles. One needs a relatively sophisticated combination of magnetic fields (basically via the Zeeman effect), optical fields (due to the Stark effect) or a combination of both (magneto-optical traps).

• As an example of trapping we will have a brief look to the basics of the <u>magneto-optical traps</u>.

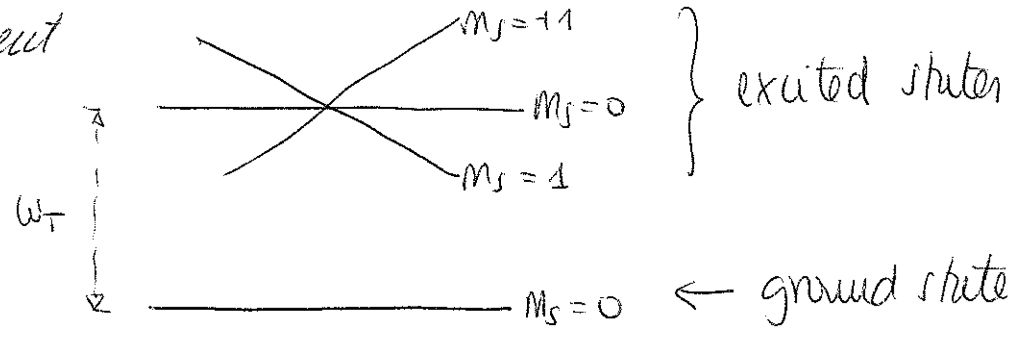
• We consider again 2 counterpropagating lasers, but this time we will assume them as having opposite circular polarizations

• We will assume that the atom has a ground-state level with spin $S=0$, and an excited state with $S=1$, and hence three possibilities $m_s = \pm 1, 0$ ← Zeeman sublevels.

• In the presence of a magnetic field B the levels experience a Zeeman shift $\Delta E \propto m_s B$.

• We shall now assume a spatially inhomogeneous magnetic field $B = \alpha X$

* As a consequence the states present a shift which is spatially dependent

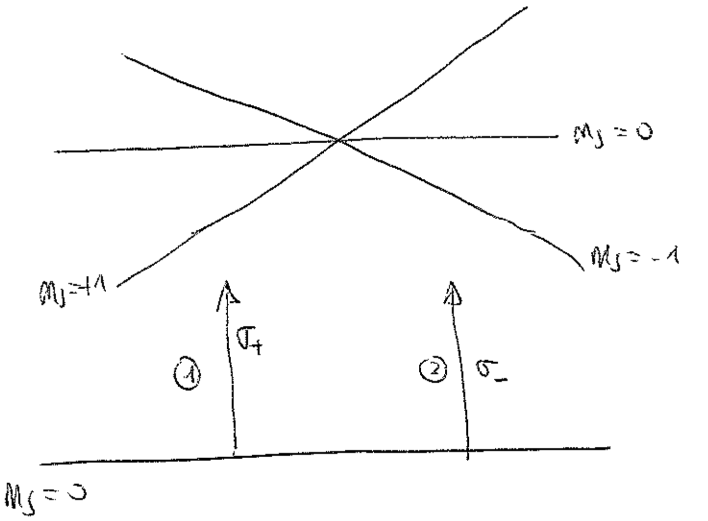


* Let's assume that the lasers are red-detuned $\omega_L < \omega_T$.

Remember that according to selection rules

* σ_+ produces a transition $m_J=0 \longrightarrow m_J=+1$
* σ_- produces a transition $m_J=0 \longrightarrow m_J=-1$

*



It is clear that the laser 1 linking $m_J=0$ and $m_J=+1$ is closer to resonance at the left whereas 2 is closer to resonance at the right.

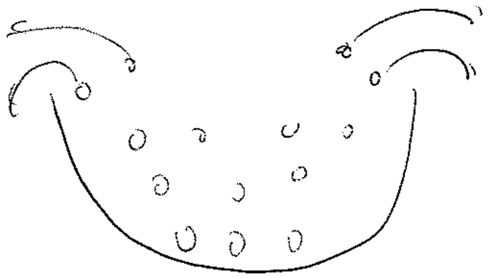
* Hence when the atoms are at the right absorb preferentially photons 2, which are moving to the left. Hence at the right the atom is pushed to the left. On the contrary at the left the atom absorbs preferentially 1 and it's hence pushed to the right. As a consequence the atom experiences a restoring force

$$F = -kx$$

i.e. the atom gets trapped in a harmonic potential!

• EVAPORATIVE COOLING

- laser cooling can be pushed to very low temperatures (even below the photon-recoil limit) but in dense samples the reabsorption of emitted photon drastically reduces the cooling efficiency.
- As a consequence another type of cooling must be employed, the so called evaporative cooling.
- Contrary to laser cooling (which is mainly a one-atom effect) evaporative cooling is based on the collisions between atoms. Basically it consists in ~~letting~~ letting escape the most energetic atoms from the trap. Then the energy per particle obviously decreases. After a rethermalization of the rest of the atoms the sample reaches hence a lower temperature.



- Of course some important technical details are important. In particular, in order to ensure the rethermalization, the collisional rate must be large enough (this requires sufficiently large samples and collisional cross sections). But we will not enter into all these details here.
- What is important for us here is that by using the previously mentioned cooling and trapping techniques it's possible to realize extremely low temperatures ($T < 100\,nK$).

* ULTRA COLD ATOMS

* What happens when the temperature is reduced and reduced?

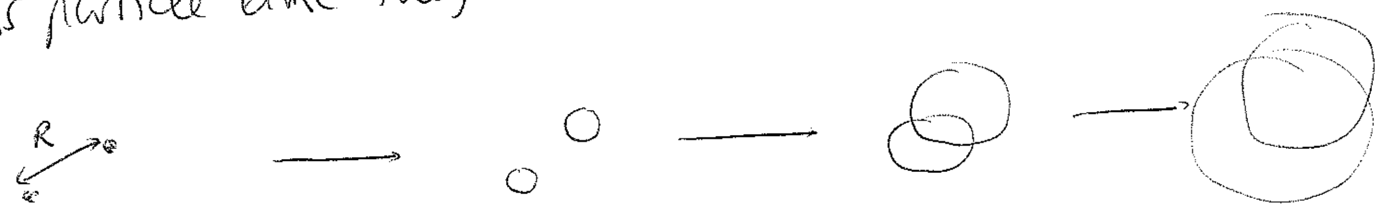
* We have just said that the width of the momentum distribution $\Delta p \sim \sqrt{2mK_BT}$

* Remember that from the Heisenberg uncertainty principle

$$\Delta x \, \Delta p \leq \hbar/2$$

* As a consequence if $\Delta p \sim \sqrt{T}$, $\Delta x \sim \frac{1}{\sqrt{T}}$

The colder the atoms the more spread they are, i.e. the less particle like they are, and the more wave-like they become.



* At some temperature the width of the "atomic bubbles" becomes comparable to the interparticle distance ($R \sim 1/\text{density}^{1/3}$).

The width of the "bubbles" is provided by the thermal de-Broglie wavelength $\lambda_T = \left(\frac{2\pi\hbar^2}{mK_BT}\right)^{1/2}$

So something remarkable occurs when $\lambda_T \gtrsim \frac{1}{\text{density}^{1/3}}$

The wavepackets then overlap, and the particles become indistinguishable. Quantum statistics (i.e. whether we deal with bosons or fermions) will then play a crucial role. Let's see now how.

We will consider mainly the case of bosons in the following.

• THE IDEAL BOSE GAS

* In this section we will consider an ideal Bose gas. From quantum statistical mechanics we know that the average occupation of a level of energy ϵ_i is provided by (we use the grand-canonical ensemble)

$$\bar{n}_i = \frac{1}{e^{\beta(\epsilon_i-\mu)}-1} \qquad \text{where } \beta = 1/k_B T \;,\; T \equiv \text{Temperature},\; k_B \equiv \text{Boltzmann's constant}$$

(For Fermions we have $+1$ instead of -1) $\quad \mu \equiv$ chemical potential

The total number of particles is then

$$N = \sum_i \bar{n}_i$$

and the energy $E = \sum_i \epsilon_i \bar{n}_i$

* Since $\bar{n}_i \geq 0$, then $\mu < \epsilon_0$ where $\epsilon_0 \equiv$ lowest eigenenergy.

Clearly when $\mu \to \epsilon_0$ the occupation number

$$N_0 \equiv \bar{n}_0 = \frac{1}{e^{\beta(\epsilon_0-\mu)}-1}$$

becomes increasingly large. This is the mechanism behind the Bose-Einstein condensation (BEC).

· Let $N = N_0 + N_T$

where $N_T(T,\mu) = \sum_{i\neq 0} \bar{n}_i(T,\mu) \Rightarrow$ Thermal component (non-condensed part)

· For a fixed value of T, $N_T(T,\mu)$ reaches its maximum at $N_C(T) = N_T(T, \mu=\epsilon_0)$. N_C grows with the temperature. There's a critical temperature T_C for which $N_C(T_C) = N$. When this happens something remarkable occurs. For $T<T_C$,

$N_c < N$. As a consequence, the number of non-condensed atoms is smaller than the total number of atoms. Therefore the rest of the atoms must go into the ground state! This is the BEC.

+ Ideal Bose gas in the box

* Let's consider an ideal Bose gas in a box of volume $V = L^3$
The eigenenergies are $\epsilon = p^2/2m$ with $\vec{p} = \frac{2\pi\hbar}{L}\vec{n}$, $\vec{n} = \{n_x, n_y, n_z\}$,
where $n_{x,y,z} = 0, \pm 1, \pm 2 \ldots$
Clearly the lowest energy is given by $p=0$. Hence

$$N_T = \sum_{p\neq 0} \frac{1}{e^{\beta\left(\frac{p^2}{2m}-\mu\right)}-1} \Longrightarrow \frac{V}{(2\pi\hbar)^3}\int d^3p \, \frac{1}{e^{\beta\left(\frac{p^2}{2m}-\mu\right)}-1} = \frac{V}{\lambda_T^3} g_{3/2}(e^{\beta\mu})$$

where $g_p(z) = \sum_{\ell=1}^{\infty} \frac{z^\ell}{\ell^p}$, and $\Rightarrow \lambda_T = \left(\frac{2\pi\hbar^2}{mk_BT}\right)^{1/2} \equiv$ Thermal de-Broglie wavelength

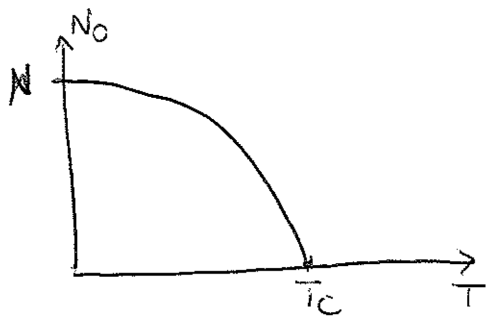
* In this case $\epsilon_0 = 0$, hence the critical temperature is given by

$$N_c(T_c) = \frac{V}{\lambda_T(T_c)^3} g_{3/2}(1) = N \qquad \mu = 0 \qquad g_{3/2}(1) \simeq 2.612$$

Hence
$$k_B T_c = \frac{2\pi\hbar^2}{m}\left(\frac{n}{g_{3/2}(1)}\right)^{2/3}$$ where $n = \frac{N}{V} \equiv$ particle density

* For $T < T_c$: $$N_T(T) = \frac{V}{\lambda_T(T)^3} g_{3/2}(1) = N \frac{\lambda_T(T_c)^3}{\lambda_T(T)^3} = N\left(\frac{T}{T_c}\right)^{3/2}$$

Hence the number of condensed particles goes as



$$\boxed{N_0(T) = N\left[1 - \left(T/T_c\right)^{3/2}\right]}$$

- Ideal Bose gas in the harmonic trap

- In the following we consider a Bose gas trapped by a harmonic potential:

$$V(\vec{r}) = \frac{m\omega_x^2}{2}x^2 + \frac{m\omega_y^2}{2}y^2 + \frac{m\omega_z^2}{2}z^2$$

The eigenenergies are then:

$$\epsilon_{n_x n_y n_z} = \hbar\omega_x(n_x + 1/2) + \hbar\omega_y(n_y + 1/2) + \hbar\omega_z(n_z + 1/2)$$

In this case the lowest energy is then

$$\epsilon_0 = \hbar(\omega_x + \omega_y + \omega_z)/2$$

Then:

$$N_T = \sum_{n_x n_y n_z \neq 0} \frac{1}{e^{\beta(\epsilon_{\vec{n}} - \mu)} - 1}$$

$$N_c(T) = N_T(T, \mu = \epsilon_0) = \sum_{n_x n_y n_z} \frac{1}{e^{\beta[\hbar\omega_x n_x + \hbar\omega_y n_y + \hbar\omega_z n_z]} - 1}$$

$$\overset{k_BT \gg \hbar\omega_{x,y,z}}{\simeq} \int dn_x\, dn_y\, dn_z \frac{1}{e^{\beta\hbar(\omega_x n_x + \omega_y n_y + \omega_z n_z)} - 1} = \zeta(3)\left(\frac{k_BT}{\hbar\bar{\omega}}\right)^3$$

($\zeta(3) \equiv$ Riemann function)

Equating $N_c(T_c) = N$ we find T_c:

$$k_BT_c = \hbar\bar{\omega}\left(\frac{N}{\zeta(3)}\right)^{1/3} = 0.94\, N^{1/3}\hbar\omega_{ho}$$

Hence for $T<T_c$ $N_T(T) = \left(\frac{T}{T_c}\right)^3$, and hence

$$\boxed{\frac{N_0}{N} = 1 - \left(\frac{T}{T_c}\right)^3}$$

* THE OFF-DIAGONAL LONG-RANGE ORDER

· The one-body density matrix is ~~characterized~~ defined as:

$$\rho(\vec{r},\vec{r}\,') = \langle \hat{\varphi}^{+}(\vec{r})\,\hat{\varphi}(\vec{r}\,')\rangle$$

where $\hat{\varphi}^{+}(\vec{r})$, $\hat{\varphi}(\vec{r})$ are the field operators creating (annihilating) a particle at $\vec{r}$.

* For a non-interacting system, the eigenstates of the density matrix are exactly the eigenstates of the single-particle Hamiltonian, and hence

$\bar{n}_i$ = average occupation.

$$\rho(\vec{r},\vec{r}\,') = \sum_i \bar{n}_i\, \varphi_i^{*}(\vec{r})\,\varphi_i(\vec{r}\,')$$

For a free gas (in absence of trapping) there is translational invariance, and hence $\rho(\vec{r},\vec{r}\,') = \rho(\vec{r}-\vec{r}\,')$.

Using the plane-wave solutions $\varphi_{\vec{p}}(\vec{r}) = \frac{1}{\sqrt{V}}\, e^{i\vec{p}\cdot\vec{r}/\hbar}$

then for

$n_0 = N_0/V$

· $T<T_C \Rightarrow \rho(s) = n_0 + \frac{1}{(2\pi\hbar)^3}\int d^3p\, \frac{e^{i\vec{p}\,\vec{s}/\hbar}}{e^{\beta p^2/2m}-1}$

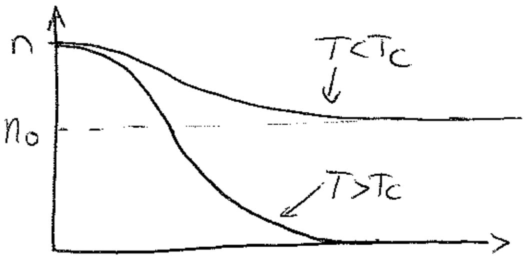
· $T>T_C \Rightarrow \rho(s) = \frac{1}{(2\pi\hbar)^3}\int d^3p\, \frac{e^{i\vec{p}\cdot\vec{s}/\hbar}}{e^{\beta\left(\frac{p^2}{2m}-\mu\right)}-1}$

It's interesting to discuss the behavior of $\rho(s)$ for large s. Without entering into all the details of the calculation one can see that

· $T<T_C: \rho(s) \simeq n_0 + \frac{e^{\beta\epsilon_0}}{(2\pi)^3\lambda_T^2}\,\frac{1}{s}$

· $T>T_C: \rho(s) \simeq n\, e^{-\pi s^2/\lambda_T^2}$

* Hence there's a dramatic difference. For $T > T_c$ $\rho(s)$ decays to zero exponentially with s (actually like a gaussian), whereas for $T < T_c$ $\rho(s)$ decays towards a finite value. This finite value is called the <u>off-diagonal long range order</u> which coincides with the condensate fraction.



* This is a very important concept, since these considerations also hold in the presence of interactions.

In the presence of interactions (we will introduce them in a moment)

$$\rho(\bar{r},\bar{r}') = \sum_i n_i\, \chi_i^*(\bar{r})\, \chi_i(\bar{r}')$$

where the eigenfunctions $\chi_i(\bar{r})$ are not now the ones of the single-particle Hamiltonian. These are the so called <u>natural states</u>

BEC occurs when one of the natural states, say $i=0$, acquires a macroscopic population $n_0 = N_0$ of the order of N.

• BRIEF REVIEW OF SCATTERING THEORY

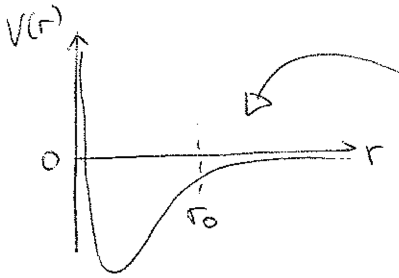
* In this section we recall briefly some basic features of the theory of elastic scattering of slow particles.

* The problem of calculating the scattering amplitude of two colliding atoms reduces to the solution of the time-independent Schrödinger equation

$$\left(\frac{-\hbar^2}{2m}\nabla^2 + V(r) - E\right)\psi(r) = 0$$

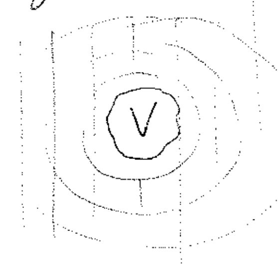
(I assume the collision of 2 particles which are identical)

This Hamiltonian is in the relative coordinate and actually $m \equiv$ reduced mass



Typically $V(r)$ looks like this

. One needs to find a solution of this equation in the asymptotic region $r \gg r_0$, where $r_0 \equiv$ range of the potential V.



In the asymptotic region the solution can be written as

$$\psi(\vec{r}) = e^{ikz} + f(\theta)\,\frac{e^{ikr}}{r}$$

e^{ikz}: incoming plane wave

$\frac{e^{ikr}}{r}$: scattered spherical wave

$f(\theta)$: Scattering amplitude for an angle θ between $\vec{r}$ and z.

Here $k = \sqrt{\frac{2mE}{\hbar^2}}$

. The scattering amplitude $f(\theta)$ is in general a function of E, and determines the scattering cross section

$$d\sigma = |f(\theta)|^2 d\Omega$$

(i.e. the probability to be scattered in a given solid angle between Ω and $\Omega + d\Omega$)

Actually this expression is only valid for non-identical particles. For identical particles the orbital part of the wave function must be symmetric or antisymmetric, depending on whether the total spin of the 2 particles is even or odd, respectively.

Hence:

$$d\sigma = |f(\theta) \pm f(\pi-\theta)|^2 d\Omega \qquad \theta \in [0, \pi/2]$$

For polarized bosons ⟶ total spin ⇒ even ⟶ +

polarized fermions ⟶ total spin ⇒ odd ⟶ −

* We expand the wavefunction in the form:

$$\psi(\vec{r}) = \sum_{l=0}^{\infty} P_l(\cos\theta)\,\frac{\chi_{kl}(r)}{kr} \qquad P_l \equiv \text{Legendre polynomials}$$

then:

$$\frac{d^2\chi_{kl}}{dr^2} - \underbrace{\frac{l(l+1)}{r^2}\chi_{kl}}_{\text{centrifugal barrier}} + \frac{2m}{\hbar^2}(E - V(r))\,\chi_{kl} = 0 \qquad \begin{pmatrix}\text{radial}\\ \text{equation}\end{pmatrix}$$

For large distances we may neglect $V(r)$ and the centrifugal barrier and hence $\frac{d^2\chi_{kl}}{dr^2} + \frac{2mE}{\hbar^2}\chi_{kl} = 0 \rightarrow \chi_{kl} = A_l \sin(kr - \frac{\pi l}{2} + \delta_l)$

* Without entering into all details, we write the final result for the scattering amplitude as a function of k and the dephases δ_l:

$$f(\theta) = \frac{1}{2ik}\sum_{l=0}^{\infty}(2l+1)\,P_l(\cos\theta)\left[e^{2i\delta_l} - 1\right]$$

• Note that since $\cos(\pi-\theta) = -\cos\theta$, and $P_l(-\cos\theta) = (-1)^l \cos\theta$

Then:

* For identical bosons only even l contributes $\begin{pmatrix} l=0 \rightarrow s \text{ wave} \\ l=2 \rightarrow d \text{ wave} \\ \vdots \end{pmatrix}$

For identical fermions only odd l contributes $\begin{pmatrix} l=1 \rightarrow p \text{ wave} \\ l=3 \\ \vdots \end{pmatrix}$

* The phase shifts δ_ℓ are crucial, and to calculate them one has to solve the Schrödinger equation. The situation is simple for $kr_0 \ll 1$ (low energies)

For $r \ll 1/k$ we can set $E=0$ in the radial equation.

For the s-wave ($l=0$):

$$\frac{d^2\chi_{k0}}{dr^2} - \frac{2m}{\hbar^2}V(r)\,\chi_{k0} = 0$$

In the region $r_0 \ll r \ll 1/k$ $V(r) \simeq 0$, and hence

$$\frac{d^2\chi_{k0}}{dr^2} = 0 \longrightarrow \chi_{k0} = C_0(1-\kappa r)$$

We have to match this solution with that for $r<r_0$, and this gives us κ, which depends on $V(r)$

Actually $\kappa = -k\cot\delta_0$ where δ_0 is the phase shift for s-wave.

* For the s-wave

$$\left.\begin{array}{l} f(\theta) = \dfrac{1}{2ik}\left(e^{2i\delta_0}-1\right) \\ \cot\delta_0 = -\dfrac{\kappa}{k} \xrightarrow[k\to 0]{} \delta_0 \simeq -\dfrac{k}{\kappa} \end{array}\right\} f(\theta) \simeq -\frac{1}{\kappa} = -a = \frac{\delta_0}{k}$$

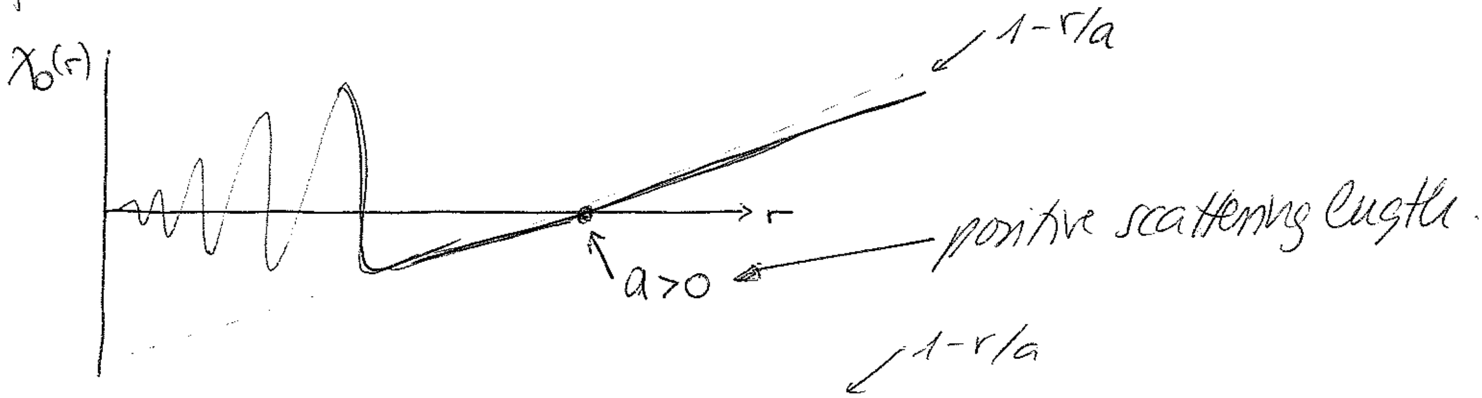
· The phase shifts at higher l behave as $\delta_\ell \propto k^{2\ell}$, and hence vanish quickly for $k\to 0$. This isn't surprising because for $l>0$ there's a centrifugal barrier that prevents the scattering.

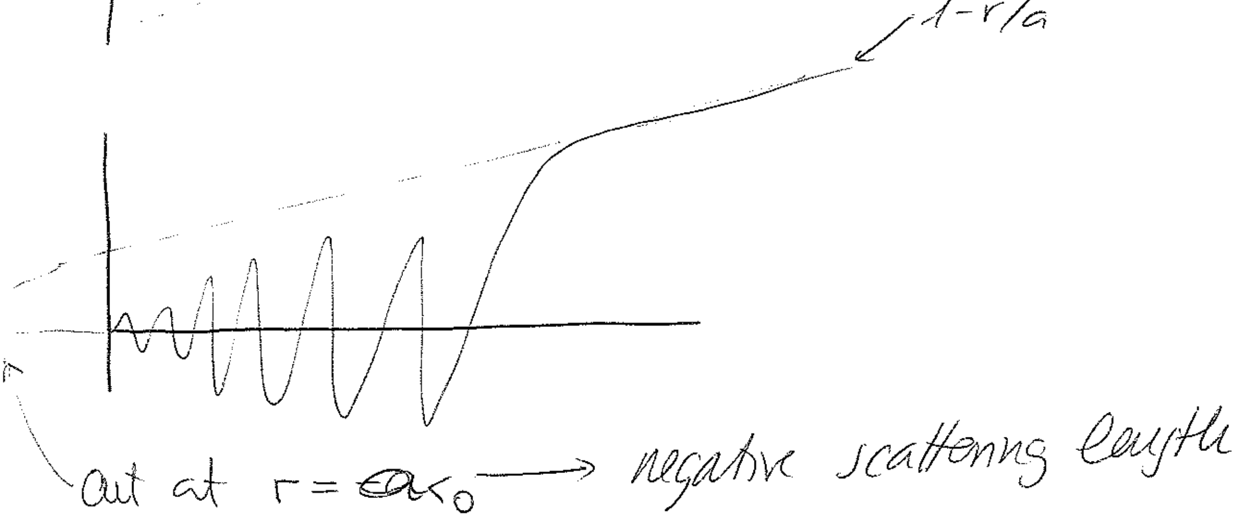
* Hence for $k \to 0$

$$f(\theta) = -a \begin{array}{l} \nearrow \sigma = 8\pi a^2 \text{ for bosons} \\ \searrow 0 \text{ for fermions.} \end{array}$$

The value a is the so-called <u>s-wave scattering length</u> which, as we will see, plays a crucial role in the physics of cold gases.

* let's have a look to two differente cases and the form of χ_0 for $k \to 0$:





* I would like to point 2 important issues concerning interactions that we ~~will~~ employ later.

a) For the determination of a it's just important the asymptotic behavior. Hence one can substitute $V(r)$ by any other potential as long as it gives the same scattering length a.

In particular we will employ the so called pseudopotential

$$V_{eff}(r) \cong \frac{4\pi\hbar^2 a}{m}\delta(r)$$

which provides the same scattering length.

b) The value of the scattering length may be modified by means of scattering resonances, e.g. the so-called Feshbach-resonances. Without entering in details at this point these resonances allow to change a from >0 to <0.

* THE WEAKLY-INTERACTING BOSE GAS

- Let's consider a dilute Bose gas in which the average interparticle distance $d = 1/n^{1/3} \gg r_0 \equiv$ range of the interaction. Hence we can limit ourselves to binary collisions (since 3 body processes are safely negligible).
- The distance d is large enough to use the asymptotic expressions for the scattering. Since we will deal with very low energies $kr_0 \ll 1$ this means that only the s-wave scattering will be of relevance.
- In the following we will always consider the diluteness condition

$$n|a|^3 \ll 1$$

- The Hamiltonian describing a weakly-interacting Boson gas in free space is given by

$$\hat{H} = \int d^3r \, \frac{\hbar^2}{2m} \vec{\nabla}\hat{\psi}^\dagger(\vec{r}) \cdot \vec{\nabla}\hat{\psi}(\vec{r}) + \frac{1}{2}\int d^3r \int d^3r' \, \hat{\psi}^\dagger(\vec{r})\hat{\psi}^\dagger(\vec{r}\,') V(\vec{r}\,'-\vec{r}) \hat{\psi}(\vec{r}\,')\hat{\psi}(\vec{r})$$

For a uniform gas occupying a volume V the field operators are

$$\hat{\psi}(\vec{r}) = \sum_{\vec{p}} \hat{a}_{\vec{p}} \frac{e^{i\vec{p}\cdot\vec{r}/\hbar}}{\sqrt{V}}$$

$\hat{a}_{\vec{p}} \equiv$ annihilates a particle with momentum $\vec{p}$.

Then:

$$\hat{H} = \sum_p \frac{p^2}{2m}\hat{a}_p^\dagger \hat{a}_p + \frac{1}{2V}\sum_{p_1,p_2,q} V_{\vec{q}} \, \hat{a}^\dagger_{\vec{p}_1+\vec{q}} \, \hat{a}^\dagger_{\vec{p}_2-\vec{q}} \, \hat{a}_{\vec{p}_2} \, \hat{a}_{\vec{p}_1}$$

where $V_{\vec{q}} = \int V(\vec{r}) e^{-i\vec{q}\cdot\vec{r}/\hbar} d^3r$

Since only small momenta contribute, then we just take $V_{\vec{q}} \simeq V_0$

$$V_0 = \int V_{eff}(\vec{r})\, d\vec{r}$$

Hence:
$$\hat{H} = \sum_{p} \frac{p^2}{2m} \hat{a}_p^+ \hat{a}_p + \frac{1}{2V} \cancel{V_0} \sum_{p_1 p_2 q} \hat{a}^+_{\bar{p}_1+\bar{q}} \hat{a}_{\bar{p}_2-\bar{q}} \hat{a}_{\bar{p}_2} \hat{a}_{\bar{p}_1}$$

* For $T \ll T_c$, the lowest eigenstate ($\bar{p}=0$) is macroscopically populated. Hence we will perform the so-called Bogolubov approximation $\hat{a}_0, \hat{a}_0^+ \simeq \sqrt{N_0}$

In a first approximation we can neglect all $\hat{a}_{\bar{p}\neq 0}$, and we can assume $N_0 \simeq N$, and we can also assume $V_0 \simeq \frac{4\pi\hbar^2 a}{m} \equiv g$ in first approximation.

Then $E_0 = \frac{1}{2V} N^2 g$ ← ground-state energy

Hence $E_0 = \frac{1}{2} N g n$

* The chemical potential $\mu \equiv \frac{\partial E_0}{\partial N} = gn$

Note that for the non-interacting gas μ was zero.

* Up to now we have just considered $\hat{a}_0$ and $\hat{a}_0^+$. Terms with only one particle operator with $\bar{p}\neq 0$ don't enter due to momentum conservation. One may retain all quadratic terms in the particle operators with $p\neq 0$ to obtain (I skip some details of the derivation here):
$$\hat{H} = \text{Constant} + \sum_p \left\{ \frac{p^2}{2m} \hat{a}_p^+ \hat{a}_p + \frac{1}{2} gn \left(2\hat{a}_p^+ \hat{a}_p + \hat{a}_p^+ \hat{a}_{-p}^+ + \hat{a}_p \hat{a}_{-p}\right)\right\}$$

. Note that now in addition to the more familiar terms $\hat{a}_p^+ \hat{a}_p$ we have some "strange" terms $\hat{a}_p \hat{a}_{-p}$ and $\hat{a}_p^+ \hat{a}_{-p}^+$.

* This Hamiltonian can be diagonalized by means of a so-called Bogoliubov transformation

$$\hat{a}_p = u_p \hat{b}_p + v_{-p}^* \hat{b}_{-p}^+$$
$$\hat{a}_p^+ = u_p^* \hat{b}_p^+ + v_{-p} \hat{b}_{-p}$$

This transformation introduces a new set of operators $\hat{b}_p$ and $\hat{b}_p^+$ to which we impose bosonic commutation rules

$$[\hat{b}_p, \hat{b}_{p'}^+] = \delta_{pp'} \longrightarrow$$ since $[a_p, a_{p'}^+] = \delta_{pp'}$ this imposes the condition $|u_p|^2 - |v_{-p}|^2 = 1$

* Just few words about the new operators $\hat{b}_p$ and $\hat{b}_p^+$. They are annihilation and creation operators, but clearly they don't annihilate or create particles. They are excitations, and we will see in a moment what is the energy of these excitations but we cannot associate these excitations to individual excited particles, but rather to "quasiparticles".

* Ok, let's come back to $|u_p|^2 - |v_{-p}|^2 = 1$. Remember that for the hyperbolic functions $\cosh^2\alpha - \sinh^2\alpha = 1$. Hence we can assign:

$$u_p = \cosh\alpha_p$$
$$v_{-p} = \sinh\alpha_p$$

* If now we substitute the Bogoliubov transformation into $\hat{H}$ we find something like:

$$\hat{H} = \text{constant} + \sum_p \left\{ A(p)\, b_p^+ b_p + B(p)\left[b_p^+ b_{-p}^+ + b_p b_{-p} \right] \right\}$$

where $B(p) = \frac{gn}{2}\left[|u_p|^2 + |v_{-p}|^2\right] + \left(\frac{p^2}{2m} + gn\right) u_p v_{-p}$

$$= \frac{gn}{2}\cosh 2\alpha_p + \frac{1}{2}\left(\frac{p^2}{2m} + gn\right)\sinh 2\alpha_p$$

* We impose $B_p = 0$, and this gives us

$$\coth 2\alpha_p = -\left[\frac{p^2/2m + gn}{gn}\right]$$

$$u_p, v_{-p} = \pm\left[\left(\frac{p^2/2m + gn}{2\epsilon(p)}\right) \pm \frac{1}{2}\right]^{1/2}$$

and finally

$$\hat{H} = \text{CONSTANT} + \sum_p \epsilon(p)\, \hat{b}_p^+ \hat{b}_p$$

where $\boxed{\epsilon(p) = \left[\frac{gn}{m} p^2 + \left(\frac{p^2}{2m}\right)^2\right]^{1/2}}$ → Bogoliubov spectrum

. Note that the original system of independent particles can be described by independent quasiparticles of energy $\epsilon(p)$. This is a very important result with far-reaching consequences as we will see very soon.

* The CONSTANT in $\hat{H}$ is the ground-state energy (i.e. in absence of quasiparticles carrying excitations) and it's of the form

$$E_0 = \frac{N}{2} gn \left[1 + \underbrace{\frac{128}{15\sqrt{\pi}} (na^3)^{1/2}}_{\text{correction due to interactions}}\right]$$

; remember that $n|a|^3 \ll 1$

and hence

$$\mu = gn\left[1 + \frac{32}{3\sqrt{\pi}} (na^3)^{1/2}\right]$$

* In the following we will try to understand the important physics behind the Bogoliubov spectrum.

* THE BOGOLIUBOV SPECTRUM

* We have just seen that the excited states of an interacting Bose gas can be described in terms of a gas of non-interacting quasiparticles with a dispersion law

$$\epsilon(p) = \left[\left(\frac{p^2}{2m}\right)^2 + p^2 c_s^2 \right]^{1/2} \qquad \text{where } c_s = \left(\frac{gn}{m}\right)^{1/2}$$

* For small momenta $p \ll mc_s$, we can approximate

$$\epsilon(p) \simeq c_s p$$

This is the same linear dispersion law that you have for example for photon (remember that for a photon $\omega = kc \rightarrow \hbar\omega = \hbar k c \Rightarrow E = pc$). It's also the dispersion law that one has for phonons (remember that a phonon is the collective excitation of a crystalline structure).

For phonons $\epsilon = c_s p$ where c_s is the sound velocity.

Phonons are nothing else than sound waves.

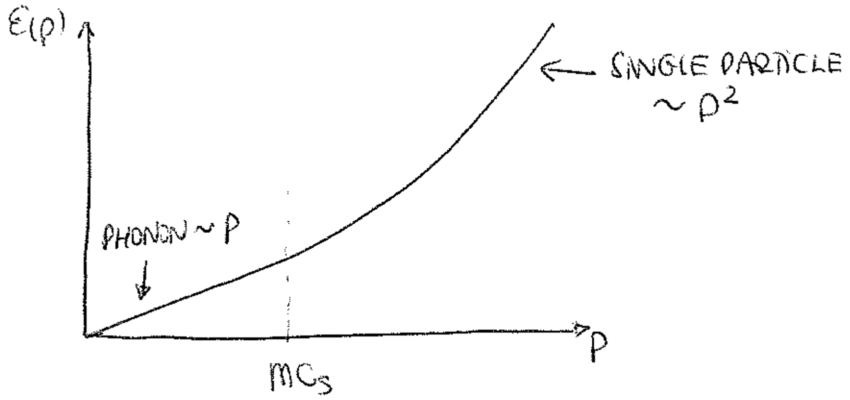
· Then small momenta (long wave length) excitations are sound waves.

· For large momenta $p \gg mc_s$

$$\epsilon(p) \approx \frac{p^2}{2m} + mc_s^2$$

and we recover the typical dispersion law of a massive particle, i.e. the typical kinetic energy we all now. This is the so-called single-particle part of the spectrum.

· The transition between the phonon and the single-particle regime occurs for $p \sim mc_s$.



- By setting $p^2/2m = gn$ with $p = \hbar/\xi$ we can define a characteristic length

$$\xi = \sqrt{\frac{\hbar^2}{2mgn}} = \frac{1}{\sqrt{2}}\frac{\hbar}{mc_s}$$

⟶ This is the so-called <u>healing-length</u>. This is an important length scale as we will see in the future.

- Just a final remark concerning the excitations we just discussed. Remember that we are dealing here with quasiparticles and not with particles. This is particularly clear from the following brief discussion. Since the quasiparticles are bosons of energy $\varepsilon(p)$ we can easily write the occupation number

$$N_p \equiv \langle \hat{b}_p^\dagger \hat{b}_p \rangle = \frac{1}{e^{\beta\varepsilon(p)} - 1}$$

(we set $\mu = 0$ because we deal here with an ideal gas of quasiparticles)

- By means of the Bogoliubov transform we can calculate $n_p = \langle \hat{a}_p^\dagger \hat{a}_p \rangle$ i.e. the <u>particle</u> occupation number:

$$n_p = \langle \hat{a}_p^\dagger \hat{a}_p \rangle = |v_{-p}|^2 + |u_p|^2 \langle \hat{b}_p^\dagger \hat{b}_p \rangle + |v_{-p}|^2 \langle \hat{b}_{-p}^\dagger \hat{b}_{-p} \rangle$$

(holding for $p \neq 0$)

The number of condensed atoms is hence

$$N_0 = N - \sum_{p\neq 0} n_p = \left[N - \sum_{p\neq 0} |v_{-p}|^2 \right] - \sum_{p\neq 0} \left(|u_p|^2 \langle \hat{b}_p^\dagger \hat{b}_p \rangle + |v_{-p}|^2 \langle \hat{b}_{-p}^\dagger \hat{b}_{-p} \rangle \right)$$

At absolute zero, $\langle b_p^\dagger b_p \rangle = 0$ for $p \neq 0$. However the interaction lead to a depletion of the condensate:

$$N_0 \xrightarrow[T\to 0]{} N - \sum_{p\neq 0} |v_{-p}|^2$$

Even at $T=0$ not all particles are in the condensate!

This is the so-called quantum depletion

* A proper calculation yields for a uniform gas at $T=0$:

$$n_0 \equiv \frac{N_0}{V} = n\left[1 - \frac{8}{3\sqrt{\pi}}(na^3)^{1/2}\right]$$

Note that for $na^3 \ll 1 \rightarrow n_0 \underset{T=0}{\simeq} n$ as we had before.

But when the interaction increases (i.e. at a Feshbach resonance) the depletion of the condensate is larger and larger. For sufficiently large interactions we cannot talk anymore of a condensate and we enter into the strongly-interacting regime where we have to employ a rather different (and more involved) theory.

* The Bogoliubov spectrum will play a crucial role in our discussion of superfluidity, as we will see later on.

• NON-UNIFORM BOSE GASES AT ZERO TEMPERATURE

• THE ORDER PARAMETER. BOGOLIUBOV APPROXIMATIONS

• Let's recall our discussion about the off-diagonal long-range order. Remember that we defined the density matrix as
$\rho(\vec{r},\vec{r}\,') = \langle \hat{\psi}^{+}(\vec{r})\, \hat{\psi}^{+}(\vec{r}\,')\rangle$

• We can always diagonalize this matrix (also for interacting systems) to get $\rho(\vec{r},\vec{r}\,') = \sum_i n_i \varphi_i^*(\vec{r})\, \varphi_i(\vec{r}\,')$

where n_i are the eigenvalues and $\varphi_i(\vec{r})$ the eigenfunctions.

• We can use these eigenfunction to write the field operator $\hat{\psi}(\vec{r})$ in the form $\hat{\psi}(\vec{r}) = \sum_i \varphi_i \hat{a}_i$

where $\hat{a}_i$ ($\hat{a}_i^+$) are the annihilation (creation) operators of a particle in the state φ_i and which obey the usual bosonic commutation relations $[\hat{a}_i, \hat{a}_j^+] = \delta_{ij}$, $[\hat{a}_i, \hat{a}_j] = [\hat{a}_i^+, \hat{a}_j^+] = 0$

• Hence

$\rho(\vec{r},\vec{r}\,') = \sum_{ij} \varphi_i^*(\vec{r})\, \varphi_j(\vec{r}\,') \langle \hat{a}_i^+ \hat{a}_j \rangle \Rightarrow \langle \hat{a}_i^+ \hat{a}_j \rangle = \delta_{ij}\, n_i$

Remember that for $T<T_c$, there is a macroscopically large eigenvalue $n_0 = N_0$ (which remember was the off-diagonal long-range order). Hence the corresponding wave function $\varphi_0(\vec{r})$ plays a crucial role in the BEC theory, being the so-called condensate wave function.

* Hence: $\hat{\psi}(\vec{r}) = \varphi_0(\vec{r})\hat{a}_0 + \sum_{i\neq 0} \varphi_i(\vec{r})\hat{a}_i$ (actually we have seen it already in a previous discussion)

* Now we will perform a rather crucial approximation, known as the Bogoliubov approximation, which consists on replacing

$$\hat{a}_0, \hat{a}_0^\dagger \longrightarrow \sqrt{N_0} \quad \text{which is a c-number.}$$

Note that by doing this we are forgetting the non-commutativity of the operators. Why? Well, because $[\hat{a}_0, \hat{a}_0^\dagger] = 1$, but since $\langle \hat{a}_0^\dagger \hat{a}_0 \rangle = N_0 \gg 1$, this means that $\hat{a}_0$ and $\hat{a}_0^\dagger$ are of the order of $\sqrt{N_0} \gg 1$. Hence, we can neglect the operator character.

· As a consequence, we can re-write the field operator in the form:

$$\hat{\psi}(\vec{r}) \cong \psi_0(\vec{r}) + \delta\hat{\psi}(\vec{r})$$

where $\psi_0(\vec{r}) = \sqrt{N_0}\,\varphi_0(\vec{r}) \longleftarrow$ condensed part (c-number)

$\delta\hat{\psi}(\vec{r}) = \sum_{i\neq 0} \varphi_i(\vec{r})\hat{a}_i \longleftarrow$ non-condensed part (operator)

At very low temperatures we can forget the non-condensed part and hence $\hat{\psi}(\vec{r}) \simeq \psi_0(\vec{r})$

(<u>Note</u>: this is equivalent to what one does in quantum optics when substituting the quantum fields by the classical electromagnetic field, for fields with large photon occupation).

· The function $\psi_0(\vec{r})$ is called the <u>condensate wavefunction</u> and clearly plays the role of an order parameter characterizing the BEC phase, since it vanishes for $T > T_c$.

* The function $\psi_0(\vec{r})$ is a complex quantity

$$\psi_0(\vec{r}) = |\psi_0(\vec{r})|\, e^{iS(\vec{r})} \longrightarrow \text{phase}$$

$|\psi_0(\vec{r})|^2 \longrightarrow$ density distribution of the condensate

· The phase $S(\vec{r})$ is quite important as we will see later in these lectures

(Note: ψ_0 is defined up to a phase factor $e^{i\alpha}$ without changing the physics. This (so-called gauge) symmetry is broken when choosing a particular phase of the BEC, i.e. there's a spontaneous symmetry breaking here.)

* Just a final point before leaving our discussion on the order parameter

One can interpret

$\psi_0(\vec{r}) = \langle \hat{\psi}(\vec{r}) \rangle$ because $\hat{\psi}(\vec{r}) \cong \varphi_0(\vec{r})\hat{a}_0$

but $\langle \hat{\psi}(\vec{r}) \rangle$ means here $\langle N-1 | \hat{\psi} | N \rangle$ (i.e. it destroys one particle). But the states $|N\rangle$ and $|N-1\rangle$ are physically equivalent up to corrections of the order $1/N_0 \ll 1$.

· Let $|N\rangle$ be ~~the~~ stationary state for N particles

then $|N\rangle(t) = |N\rangle(0)\, e^{-iE(N)t/\hbar}$

Hence $\psi_0(\vec{r},t) = e^{-i[E(N)-E(N-1)]t/\hbar}\, \psi_0(\vec{r},0)$

The chemical potential is defined as the change of the energy of the system when adding a particle $\longrightarrow \mu = \frac{\partial E}{\partial N} \simeq E(N) - E(N-1)$

Hence $\boxed{\psi_0(\vec{r},t) = e^{-i\mu t/\hbar}\, \psi_0(\vec{r})}$ → The chemical potential μ is, as we will see, a crucial parameter here.

(Note: the ~~energy~~ evolution of the ψ_0 is not given by the energy but by the chemical potential)

- THE GROSS-PITAEVSKII EQUATION (GPE)

- Remember that the Hamiltonian describing a weakly-interacting Bose gas is given by:

$$\hat{H} = \int d^3r \; \hat{\psi}^{\dagger}(\vec{r}) \left[\frac{-\hbar^2}{2m} \nabla^2 + V_{ext}(\vec{r},t) \right] \hat{\psi}(\vec{r})$$
$$+ \frac{1}{2} \int d^3r \int d^3r' \; \hat{\psi}^{\dagger}(\vec{r}) \hat{\psi}^{\dagger}(\vec{r}') \hat{\psi}(\vec{r}') \hat{\psi}(\vec{r})$$

(Note: the only difference with the Hamiltonian discussed in page 17 is that now we consider an external potential $V_{ext}(\vec{r})$. This is e.g. the trapping potential)

- In the Heisenberg representation the operator $\hat{\psi}(\vec{r},t)$ fulfills the equation:

using the commutation rules $[\hat{\psi}(\vec{r}), \hat{\psi}^{\dagger}(\vec{r}')] = \delta(\vec{r}-\vec{r}')$

$$i\hbar \frac{\partial}{\partial t} \hat{\psi}(\vec{r},t) = [\hat{\psi}(\vec{r},t), \hat{H}] =$$
$$= \left[\frac{-\hbar^2}{2m} \nabla^2 + V_{ext}(\vec{r},t) + \int d^3r' \; \hat{\psi}^{\dagger}(\vec{r}',t) V(\vec{r}'-\vec{r}) \hat{\psi}(\vec{r}',t) \right] \hat{\psi}(\vec{r},t)$$

- Doing the Bogoliubov approximation: $\hat{\psi}(\vec{r},t) \longrightarrow \psi_0(\vec{r},t)$ we arrive to the Gross-Pitaevskii equation [assuming as in a previous discussion) that $V(\vec{r}'-\vec{r}) \simeq g\,\delta(\vec{r}'-\vec{r})$]:

$$\boxed{i\hbar \frac{\partial}{\partial t} \psi_0(\vec{r},t) = \left[\frac{-\hbar^2}{2m} \nabla^2 + V_{ext}(\vec{r},t) + g|\psi_0(\vec{r},t)|^2 \right] \psi_0(\vec{r},t)}$$

This equation is the main theoretical tool for investigating non-uniform dilute Bose gases at low temperatures. This equation is a particular example of a nonlinear Schrödinger equation (similar to that appearing in nonlinear optics in Kerr media) and hence the BEC physics is inherently nonlinear. We will have a look to nonlinear phenomena later.

* Note, that since the stationary solution fulfills

$$\psi_0(\vec{r},t) = e^{-i\mu t/\hbar}\,\psi_0(\vec{r})$$

then the time-independent Schrödinger equation is of the form:

$$\mu\,\psi_0(\vec{r}) = \left[\frac{-\hbar^2}{2m}\vec{\nabla}^2 + V_{ext}(\vec{r}) + g|\psi_0(\vec{r})|^2\right]\psi_0(\vec{r})$$

the value of the chemical potential μ is fixed by the normalization condition

$$\int |\psi_0(\vec{r})|^2\, d^3r = N$$

* For a uniform system $V_{ext}(\vec{r}) = 0$, $|\psi_0(\vec{r})|^2 = n$ = uniform density

and hence $\mu = gn$ as we already knew.

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* BEC IN A BOX POTENTIAL

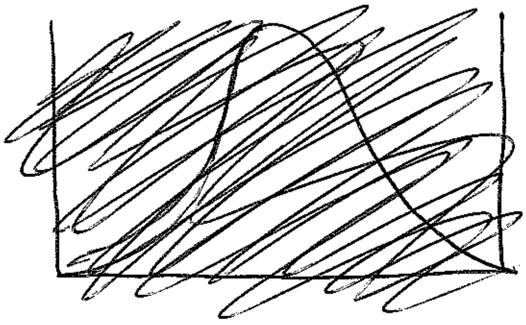
* In the following we consider a very simple but instructive problem, namely that of N interacting bosons (with $g>0 \rightarrow$ repulsive gas) confined in a box potential of the form

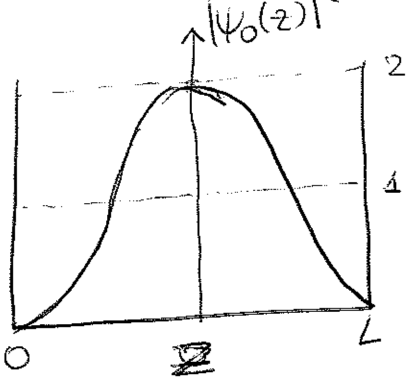
$$V_{ext}(z) = \begin{cases} 0 & 0<z<L \\ \infty & \text{otherwise} \end{cases} \longrightarrow \psi_0(0) = \psi_0(L) = 0$$

Hence, in absence of any interactions ($g=0$), the GPE becomes a usual Schrödinger equation, and hence

$$\psi_0(\vec{r}) = \sqrt{2\bar{n}}\,\sin\frac{\pi}{L}z \qquad \text{with } \bar{n} = \frac{N}{V}$$

$V \equiv$ volume of the box (we assume periodic boundary conditions in x and y)





* Let's see what happens if the interactions are present.
Now the system is controlled by the GPE

$$\mu \psi_0(z) = \left[-\frac{\hbar^2}{2m} \frac{d^2}{dz^2} + g |\psi_0(z)|^2 \right] \psi_0(z)$$

* Since $g>0$, the system minimizes the energy by making the density uniform in the interior of the box. Clearly the maximal density is minimized (and with it the interaction energy $g|\psi_0|^2$) if we spread $|\psi_0|$ as uniform as possible in the box. Only at the walls of the box something will happen, because obviously at the walls we have to fulfill $|\psi_0|^2 = 0$.
* So except close to the walls we can consider a uniform density n.
* Remember the definition of healing length associated with the density n: $\xi = \sqrt{\frac{\hbar}{2mgn}}$

Let $\tilde{\psi}_0(z) = \psi_0(z)/\sqrt{n}$, hence (introducing $\tilde{z} = z/\xi$)

$$\mu \tilde{\psi}_0(z) = \left[\frac{-\hbar^2}{2m\xi^2} \frac{d^2}{d\tilde{z}^2} + gn |\tilde{\psi}_0(z)|^2 \right] \tilde{\psi}_0(z)$$

Assuming that the box size L is much larger than the border region at which the density is different than n, we can the forget the border and approximate

$$\mu \approx gn = \frac{\hbar^2}{2m\xi^2}$$

Hence

$$\tilde{\psi}_0(\tilde{z}) = -\frac{d^2}{d\tilde{z}^2} \tilde{\psi}_0(\tilde{z}) + \tilde{\psi}_0(\tilde{z})^3$$

* let's see what happens close to the border $z=0$.

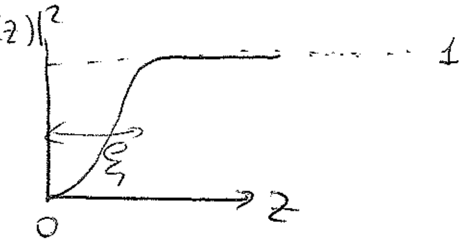
At $z=0 \longrightarrow \Psi_0(z)=0$

Out of the border region $\longrightarrow \Psi_0(z) \xrightarrow[z \gg \text{border}]{} 1$

This equation has an analytic solution, namely:

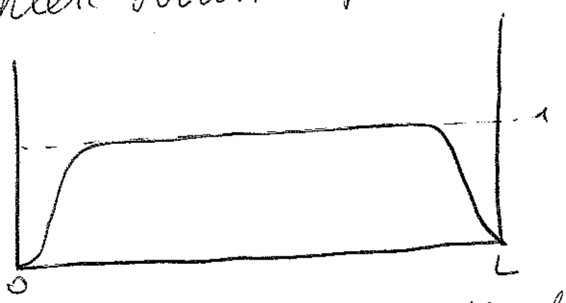
$$\Psi_0(\tilde{z}) = \tanh\left(\tilde{z}/\sqrt{2}\right)$$

Hence $\psi_0(z) = \sqrt{n}\,\tanh\left(\frac{z}{\sqrt{2}\,\xi}\right)$



* Then we can now identify the border region as an interval of the order of the healing length ξ (hence this calculation is ok for $\xi \ll L$)

* the complete solution for the interacting BEC in the box looks



• The <u>healing length</u> it is therefore a crucial length scale in the problem.

The physical meaning of the healing length becomes evident. It's the length scale at which a perturbation of the density (i.e. the border at the box wall) will die out into a uniform density.

Basically the healing length is the result of comparing the typical kinetic energy associated with a variation of the wavefunction in a length scale $\xi \longrightarrow \frac{\hbar^2}{2m\xi^2}$ and the chemical potential $\mu = gn$ (the interaction energy).

* Clearly when the interaction increase (g increases) then ξ decreases, and the border region becomes narrower. On the contrary when g decreases ξ becomes larger.

* We have thus seen that the 2-body interactions can significantly modify the ground state profile of a BEC. We will see later what happens in an harmonic trap.

• HYDRODYNAMIC EQUATIONS. THE THOMAS-FERMI LIMIT

* Recall the GPE

$$i\hbar \frac{\partial}{\partial t}\psi_0(\vec{r},t) = \left[\frac{-\hbar^2\nabla^2}{2m} + V_{ext}(\vec{r}) + g|\psi_0(\vec{r},t)|^2\right]\psi_0(\vec{r},t)$$

remember that $\psi_0(\vec{r},t) = \sqrt{n(\vec{r},t)}\, e^{iS(\vec{r},t)}$

* let's multiply by ψ_0^*

$$i\hbar\,\psi_0^* \frac{\partial}{\partial t}\psi_0 = \psi_0^*\left[-\frac{\hbar^2\nabla^2}{2m}\right]\psi_0 + V_{ext}(\vec{r})|\psi_0|^2 + g|\psi_0|^4$$

let's take the complex conjugate of this equation:

$$-i\hbar\,\psi_0 \frac{\partial}{\partial t}\psi_0^* = \psi_0\left[\frac{-\hbar^2}{2m}\nabla^2\right]\psi_0^* + V_{ext}(\vec{r})|\psi_0|^2 + g|\psi_0|^4$$

let's substract both equations

$$i\hbar\frac{\partial}{\partial t}n = \frac{-\hbar^2}{2m}\left[\psi_0^*\nabla^2\psi_0 - \psi_0\nabla^2\psi_0^*\right] = \frac{-\hbar^2}{2m}\nabla\left[\psi_0^*\nabla\psi_0 - \psi_0\nabla\psi_0^*\right]$$

then $\frac{\partial}{\partial t}n = \frac{i\hbar}{2m}\nabla(\psi_0^*\nabla\psi_0 - \psi_0\nabla\psi_0^*)$

Since $\psi_0 = n^{1/2}e^{iS} \rightarrow \nabla\psi_0 = \left[\frac{1}{2n}\nabla n + i\nabla S\right]n e^{iS} = \left(\frac{\nabla n}{2} + i\nabla S\right)\psi_0$

Then $\psi_0^*\nabla\psi_0 - \psi_0\nabla\psi_0^* = n\left[\frac{\nabla n}{2} + i\nabla S - \frac{\nabla n}{2} + i\nabla S\right] = 2in\nabla S$

Hence: $$\frac{\partial n}{\partial t} = \frac{i\hbar}{2m}\nabla\left[2in\nabla S\right] = -\nabla\left[\frac{\hbar}{m}n\nabla S\right]$$

$$= -\nabla\vec{J}$$

* Then $\boxed{\dfrac{\partial n}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0}$

This has the form of a continuity equation, where $\vec{J}$ plays the role of a density current (recall the notion of current in quantum mechanics)

The physical meaning of this equation is that the number of particles are conserved by the equation (by the GPE).

* The current is number of atoms per surface area per second hence $\boxed{\vec{v}_S = \dfrac{\hbar}{m}\vec{\nabla}S}$ has units of velocity.

It's indeed the velocity of the condensate flow. Note that $\vec{\nabla}\times\vec{v}_S = \dfrac{\hbar}{m}\vec{\nabla}\times(\vec{\nabla}S) = 0$.

$\vec{v}_S$ is an example of a so-called irrotational fluid, something crucial for discussing the properties of superfluids as we will see later.

* Inserting $\psi_0 = \sqrt{n}\, e^{iS}$ into the GPE we obtain (in addition to the continuity equation) a second equation providing the evolution of the phase

$$\boxed{\hbar\frac{\partial}{\partial t}S + \left(\frac{m v_S^2}{2} + V_{ext}(\vec{r}) + gn - \frac{\hbar^2}{2m\sqrt{n}}\nabla^2\sqrt{n}\right) = 0}$$

This equation and the continuity equation provide a closed set of coupled equations equivalent to the GPE.

• Hence:

$$m\frac{\partial}{\partial t}\vec{v}_s = -\vec{\nabla}\left[\frac{m v_s^2}{2} + V_{ext}(\vec{r}) + gn - \frac{\hbar^2}{2m\sqrt{n}}\nabla^2\sqrt{n}\right]$$

If you have a close look to this equation and the continuity equation you will see that $\hbar$ just appears in the last term at the r.h.s. of the last equation. Hence the quantum character is pretty much reduced to that term, which receives the name "quantum pressure"

• The quantum pressure becomes small if the density changes slowly in space. let's see first when this happens.
let's call R the typical length characterizing density variations, typically the condensate size for the ground state configuration
Hence the quantum pressure term scales as:

$$\frac{\nabla^2\sqrt{n}}{\sqrt{n}} \sim \frac{1}{R^2}$$

• Remember that $gn = \frac{\hbar^2}{2m\xi^2}$

Hence if $R \gg \xi$, then $\frac{\hbar^2}{2m\xi^2} \gg \frac{\hbar^2}{2m\sqrt{n}}\nabla^2\sqrt{n}$, and we can neglect the quantum pressure. This is the so-called Thomas-Fermi limit.

In this limit

$$\boxed{m\frac{\partial}{\partial t}\vec{v}_s + \vec{\nabla}\left[\frac{m v_s^2}{2} + V_{ext}(\vec{r}) + gn\right] = 0}$$

* This equation is well known in hydrodynamics, being the Euler equation for a nonviscuous gas with pressure

$P = gn^2/2$

• Note: the sound velocity in a fluid is defined as $mc_s^2 = \frac{\partial P}{\partial n} = gn \longrightarrow c_s = \sqrt{\frac{gn}{m}}$ which is exactly the sound-velocity we got from the Bogoliubov spectrum (so all matches, fortunately!).

* In the ground state configuration $\vec{v}_s = 0$, and hence

$\vec{\nabla}\left[V_{ext}(\vec{r}) + gn \right] = 0 \longrightarrow V_{ext}(\vec{r}) + gn = \text{constant}$

This constant is the chemical potential (as it becomes clearly by removing the kinetic energy in the time-independent GPE).

Hence $\boxed{\mu = V_{ext}(\vec{r}) + gn(\vec{r})}$

as always μ is fixed by setting $\int n(\vec{r})\, d^3r = N$.

* In the following we will employ this general expression for the case of a BEC in an harmonic trap.

* BEC in a harmonic trap

* Let's consider the harmonic confinement:

$$V_{ext}(\vec{r}) = \frac{m\omega_x^2}{2}x^2 + \frac{m\omega_y^2}{2}y^2 + \frac{m\omega_z^2}{2}z^2$$

* We will consider the case $g>0$ (repulsive BEC), and leave the attractive case ($g<0$) for a later discussion.
* Let's consider the Thomas-Fermi limit:

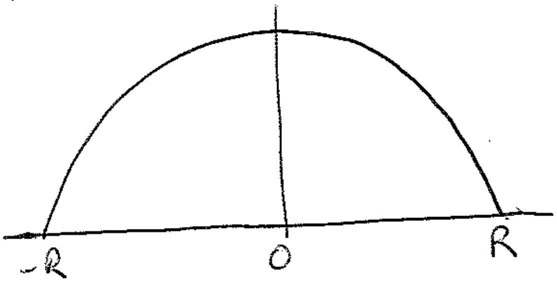
$$\mu = V_{ext}(\vec{r}) + g n(\vec{r})$$

Hence
$$n(\vec{r}) = \frac{1}{g}(\mu - V_{ext}(\vec{r}))$$
$$= \frac{\mu}{g}\left[1 - \frac{m\omega_x^2}{2\mu}x^2 - \frac{m\omega_y^2}{2\mu}y^2 - \frac{m\omega_z^2}{2\mu}z^2\right]$$
$$= n_0\left[1 - (x/R_x)^2 - (y/R_y)^2 - (z/R_z)^2\right]$$

where $n_0 \equiv$ central density

$R_{k=x,y,z} \Rightarrow$ are the so-called Thomas-Fermi Radii.

Therefore the density profile of a BEC in a trap in the Thomas-Fermi limit acquires the form of an inverted parabola



The Thomas-Fermi radius is provided by the classical turning point

$$\mu = \frac{m\omega^2}{2}R^2$$

* It's interesting to have a look to the actual form of μ and R_K as a function of the system parameters.

* We know that

$$N = \int d^3r\, n(\vec{r}) = n_0 \int d^3r \left[1 - \left(\frac{x}{R_x}\right)^2 - \left(\frac{y}{R_y}\right)^2 - \left(\frac{z}{R_z}\right)^2\right]$$

$$= n_0 R_x R_y R_z \int_0^1 r^2 dr\,(1-r^2)\, 4\pi$$

$\bar{\omega} = (\omega_x \omega_y \omega_z)^{1/3}$

$$= \frac{8\pi}{15} n_0 R_x R_y R_z = \frac{8\pi}{15}\frac{\mu}{g}\left(\frac{2\mu}{m\omega_x^2}\right)^{1/2}\left(\frac{2\mu}{m\omega_y^2}\right)^{1/2}\left(\frac{2\mu}{m\omega_z^2}\right)^{1/2} =$$

$$= \frac{8\pi}{15 g}\left(\frac{2}{m\bar{\omega}^2}\right)^{3/2}\mu^{5/2} = \frac{8\pi}{15\cdot\frac{4\pi\hbar^2}{m}a}\left(\frac{2}{m\bar{\omega}^2}\right)^{3/2}\mu^{5/2} =$$

$l_{HO} = \sqrt{\frac{\hbar}{m\bar{\omega}}}$

$$= \left(\frac{2\mu}{\hbar\bar{\omega}}\right)^{5/2}\frac{l_{HO}}{15a} \longrightarrow \boxed{\mu = \frac{\hbar\bar{\omega}}{2}\left(\frac{15aN}{l_{HO}}\right)^{2/5}}$$

As a consequence:

$$\boxed{R = \left(\frac{2\mu}{m\bar{\omega}^2}\right)^{1/2} = l_{HO}\left(\frac{15aN}{l_{HO}}\right)^{1/5}} \longrightarrow R_k = \frac{\bar{\omega}}{\omega_k} R$$

· As a consequence note that the size of the system increases as $N^{1/5}$. For the ideal gas the size of the condensate is fixed by the oscillator length. In the case of a repulsive gas, the size of the system grows when we increase N, because the particles repel each other.

* If we compare R and the healing length:

$$\xi = \left(\frac{\hbar^2}{2m\mu}\right)^{1/2} \longrightarrow \frac{\xi}{R} = \left(\frac{\hbar\bar{\omega}}{2\mu}\right)^2 = \left(\frac{15Na}{l_{HO}}\right)^{-2/5}$$

Remember that the Thomas-Fermi limit is justified if $\xi \ll R$, which is true if $\mu \gg \hbar\bar{\omega}$, which in turn is true for $Na/a_{HO} \gg 1$.

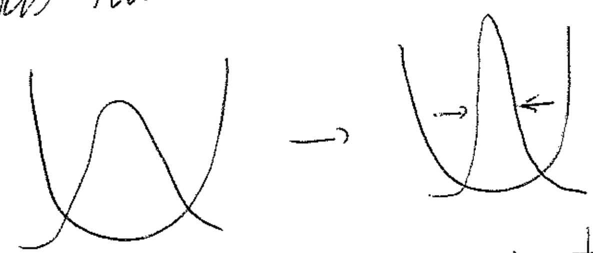
* Typical values of $a/a_{HO} \sim 10^{-3}$, hence $N \gg 10^3$ to get the Thomas-Fermi limit. Typically $N \sim 10^4, 10^5$ so the Thomas-Fermi limit is typically OK (but not allways).

- BEC WITH ATTRACTIVE ~~INT~~ INTERACTIONS

- If $g<0$, the interactions are attractive, and the behaviour of the condensates is very different.
- let's write the GPE:

$$\mu \Psi_0 = \left[\frac{-\hbar^2}{2m}\nabla^2 + V_{ext}(\vec{r}) - |g|\,|\Psi|^2\right]\Psi$$

It's clear that the system minimizes its energy by contracting and increasing its density in the trap center. In a classical gas this would be unstoppable. However, in quantum mechanics we must take into account the zero-point oscillation ~ the correspon-ding to the trap. The zero point kinetic energy tends to compensate the contraction.



$$\underbrace{\left(\frac{-\hbar^2}{2m}\nabla^2 + V_{ext}(\vec{r})\right)\Psi}_{\text{zero point energy: tends to prevent excessive contraction}} \quad \underbrace{-|g|\Psi^3}_{\text{tends to contract the gas towards the trap center}}$$

zero point energy: tends to prevent excessive contraction → This is a single particle effect. It's N independent

tends to contract the gas towards the trap center → This effect grows with N

* As a consequence, for a sufficiently large number of particles the kinetic energy can't compensate the non-linear compression (non linear focusing in the language of nonlinear optics) and as a consequence the condensate collapses!

- Let's see this in more detail.

Let's consider a spherical trap with frequency ω.

Let's use the following Gaussian ansatz for the condensate

$$\phi(\vec{r}) = \left(\frac{N}{\lambda^3 a_{HO}^3 \pi^{3/2}}\right)^{1/2} e^{-r^2/2\lambda^2 a_{HO}^2}$$

λ fixes the width of the condensate and works as our variational parameter.

- Now that we have set a Gaussian "bubble" of width λa_{HO}, let's calculate the energy as a function of λ.

Remember that

$$E = \int d^3r \left[\frac{\hbar^2}{2m}|\nabla\psi|^2 + V_{ext}(\vec{r})|\psi|^2 - |g|\frac{|\psi|^4}{2}\right]$$

After plugging the Ansatz one gets:

$$\frac{E(\lambda)}{N\hbar\omega} = \frac{3}{4}\left(\frac{1}{\lambda^2} + \lambda^2\right) - \frac{N|a|}{\sqrt{2\pi}\, l_{HO}}\lambda^{-3}$$

kinetic energy

trap

interaction energy.

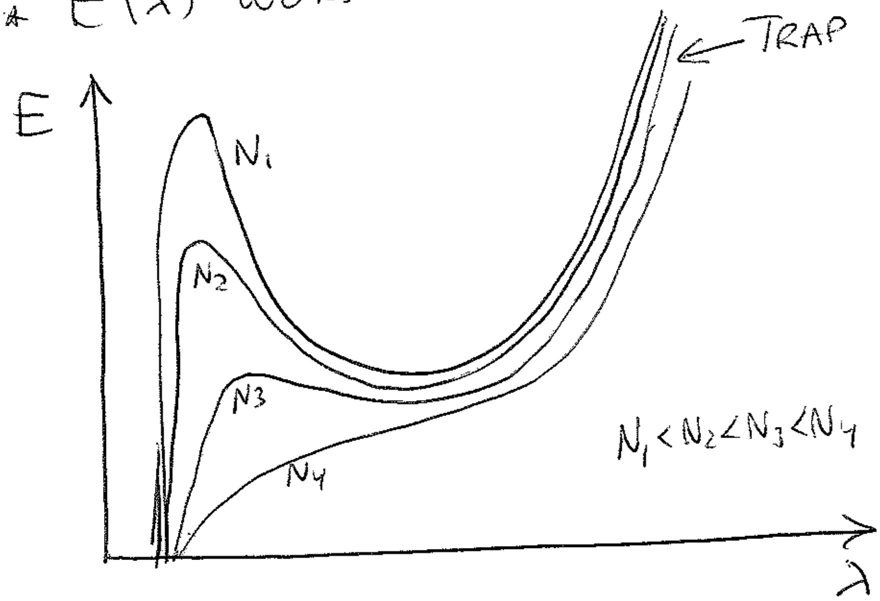
- Here it is quite crucial how the different terms scale with λ.

- The kinetic energy $\sim 1/\lambda^2$ (this was expected because the kinetic energy is a 2nd derivative, and hence scales as $\sim 1/\text{length}^2$)

- The trap energy $\sim \lambda^2$ (also expected because we consider an harmonic potential $\sim x^2$)

- The interaction energy $\sim \frac{1}{\lambda^3}$ (also expected because the interaction energy is proportional to the density & and the density $= \frac{1}{\text{volumen}} \sim \frac{1}{\text{length}^3}$)

* $E(\lambda)$ looks like this:

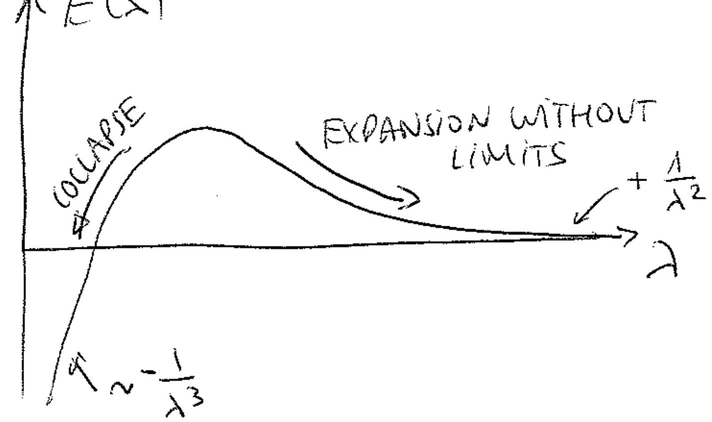


- For small values of N $E(\lambda)$ shows a potential barrier, which prevents the system to collapse.
- For large N, the barrier dissappears and the collapse occurs.

* The numerical value at which collapse occurs is

$$\frac{N_{cr}}{a_{ho}}|a| = 0.575$$

* Note that in absence of a trap one just has kinetic energy and interactions, and the energy curves like this:



* Note that the system either collapses or expands without limits. As a consequence there cannot be a self-bound ("bubble-like") solution for the GPE in a 3D environment.

* The situation is radically different in 1D systems, where self-bound solutions are possible, as we will discuss when talking about solitons

• SUPERFLUIDITY

• LANDAU'S CRITERION

• Before starting with Landau's theory of superfluidity, let's recall the transformation laws of energy and momentum under Galilean transformations.

Let E and $\vec{p}$ the energy and momentum of a fluid in a reference system K.

Let's consider a second reference system K' moving with velocity $\vec{V}$ relative to K. For the system K'

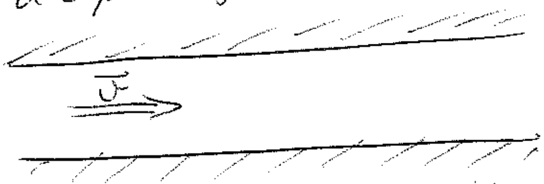
$$E' = E - \vec{P}\cdot\vec{V} + \frac{MV^2}{2} \quad \text{and} \quad \vec{p}\,' = \vec{p} - M\vec{V}$$

like the Doppler effect

where M is the total Mass of the fluid.

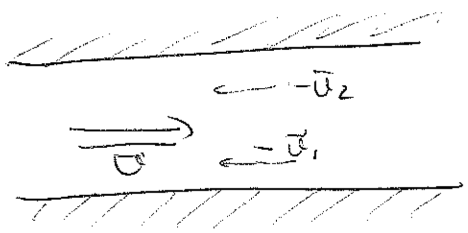
• We will use these results in a second.

• Let us now consider an uniform fluid at zero temperature flowing along a capillary at constant velocity $\vec{v}$

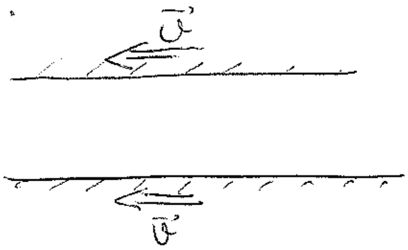


If the fluid is viscous then the motion will produce energy dissipation with the consequent heating and decrease of kinetic energy.

One can understand this process as taking place through the creation of elementary excitation on top of the fluid moving against it



* Let's now describe the process in the reference system moving with the fluid.

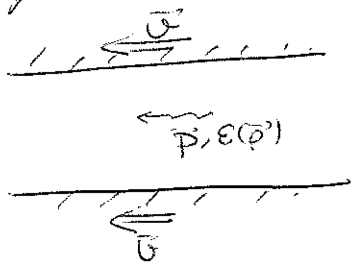


Now the fluid is at rest, and the walls are moving

* Let $E_0 \equiv$ energy of the ground state (with no excitation)

In absence of any excitation the fluid is at rest (in this reference frame) and hence $\vec{P}_0 = 0$

* Let's consider now that a single excitation with momentum $\vec{p}$ appears. The momentum of the system becomes $\vec{P} = \vec{p}$, and the energy is $E = E_0 + \varepsilon(\vec{p})$, where $\varepsilon(\vec{p})$ is the dispersion.



(Remember that for a Bose-Einstein condensate $\epsilon(\vec{p})$ is the Bogoliubov spectrum)

* Let's go back now to the reference frame where the capillary is at rest. This system moves with a velocity $-\vec{v}$ when compared to the system at which the ~~capillary~~ fluid is at rest. Hence, by applying the galilean transformations, we obtain:

$$E' = E_0 + \frac{1}{2} M v^2 + \epsilon(\vec{p}) + \vec{p}\cdot\vec{v}$$

$$\vec{P}' = \vec{P} + M\vec{v}.$$

Hence by creating one excitation on top of the fluid we have changed the momentum by $\vec{P}$ and the energy by an amount $\epsilon(\vec{p}) + \vec{p}\cdot\vec{v}$.

· The crucial point is that the spontaneous creation of excitations (linked with the flow dissipation) is just possible if this change of energy is negative;

$$\epsilon(\vec{p}) + \vec{p}\cdot\vec{v} < 0$$

i.e. if the energy is reduced by creating the excitation.

* Note that the most favourable situation for creating the excitation occurs when $\vec{p}$ and $\vec{v}$ are counterpropagating, i.e. when

$\vec{p}\cdot\vec{v} = -pv$.

Hence we arrive to $\epsilon(\vec{p}) - pv < 0 \longrightarrow v > \dfrac{\epsilon(\vec{p})}{p}$

* If this is the case the flow is unstable and its kinetic energy will be transformed into heat.

But: if the velocity v is smaller than

$$v_c = \min_{\vec{p}} \frac{\epsilon(\vec{p})}{p}$$

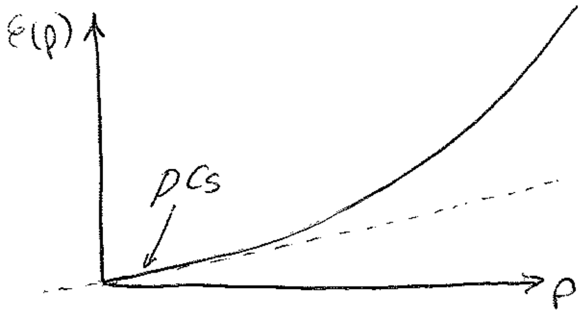
Then the condition for the creation of excitations is never fulfilled and no excitation will spontaneously grow in the fluid. As the consequence of that, the flow presents no dissipation, i.e. the fluid becomes <u>SUPERFLUID</u>

· Landau's criterion for superfluidity can then be written in the form

$$\boxed{v < v_c = \min_{\vec{p}} \frac{\epsilon(\vec{p})}{p}}$$

* let's see now that the weakly-interacting Bose gas fulfills the Landau criterion for superfluidity.

• Remember that for a weakly interacting Bose gas we found that the dispersion law $\epsilon(\vec{p})$ was provided by the Bogoliubov spectrum



$$\epsilon(\vec{p}) = \left[\left(\frac{p^2}{2m} \right)^2 + (p c_s)^2 \right]^2$$

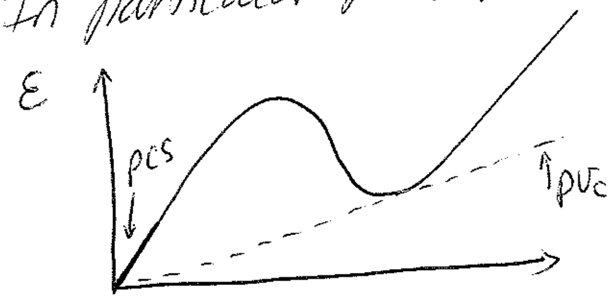
Where $c_s \equiv$ sound velocity

* let's apply now the Landau criterion.

It's clear than $\min\limits_{p} \frac{\epsilon(p)}{p} = c_s$ (see the dashed line in the figure)

Hence the critical velocity is the sound velocity $\boxed{v_c = c_s}$

* For other superfluids the dispersion law is different. In particular for ~~Helium~~ ^4He, the dispersion law looks like this



where the minimum is the so called roton minimum

If we apply Landau's criterion we observe that now $v_c < c_s$ because it's the roton minimum which determines the critical superfluid velocity.

* VORTICES

* Remember that $\vec{v}_S = \frac{\hbar}{m} \vec{\nabla} S$ is the superfluid velocity, where $S \equiv$ phase of the condensate wavefunction.

* ~~This~~ This identification of the superfluid velocity and the gradient of the phase introduces a crucial relationship between BEC and superfluidity.

· However, this relationship doesn't involve $|\psi_0(\vec{r})|^2$ only the phase $S(\vec{r})$.
In particular it would be wrong to confuse the condensate fraction and the superfluid ~~density~~ fraction (fraction of the system which is superfluid). For example at $T=0$, an interacting Bose gas shows a quantum depletion, as we have seen, and hence the condensate fraction is not 100%. On the contrary the whole gas becomes superfluid.

· Similarly in ^4He, the whole gas becomes superfluid, although only 10% is condensed (due to the large interactions which lead to a very large depletion).

* In the following we will have a look to a key feature of superfluids, namely the appearance of quantized vortices.

· Quantized vortices / Quantized circulation

· We have mentioned that the superfluid velocity is irrotational. Hence a superfluid cannot rotate as an ordinary liquid.

· If we start to stir the condensate at low angular velocities the superfluid will remain at rest. However for a sufficiently large angular velocity such at state at rest becomes energetically unfavourable. Let's see why (although it seems intuitively almost obvious)

. Let's move to the reference in which the stirr is at rest (rotating frame). The energy becomes

$$E \longrightarrow \underset{\text{lab. frame}}{E} - \vec{\Omega}\cdot\vec{L}$$

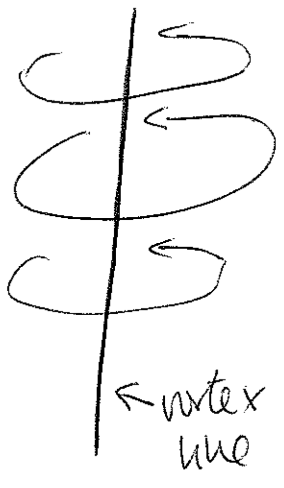
$\vec{\Omega} \longrightarrow$ angular velocity of the stirr

$\vec{L} \longrightarrow$ angular momentum. (in the laboratory frame)

We have to minimize this energy.

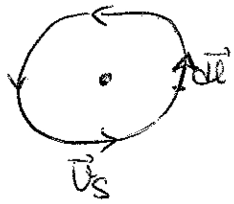
It's clear that for sufficiently large Ω, with $\vec{\Omega}\cdot\vec{L} > 0$, it's better to have angular momentum $\neq 0$ than to have $L=0$ (which is what you would have at rest)

* But, as commented already, since a superfluid cannot rotate in a rigid way, the rotation will eventually be realized through the creation of quantized vortex lines.



· Let's see first why I call these lines quantized.

· Let's see the vortex line from the top.



length element-line of the contour

* The circulation of the superfluid velocity around a contour around the vortex line is defined as:

$$\kappa = \oint_{\text{CONTOUR}} \vec{v}_s \cdot d\vec{l} = \oint \frac{\hbar}{m}\vec{\nabla}S \cdot d\vec{l}$$

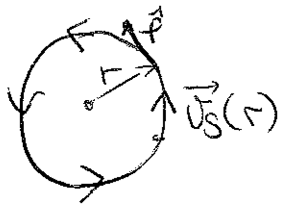
$\oint \vec{\nabla}S \cdot d\vec{l} \longrightarrow$ tells us how much does the condensate phase when moving around the vortex and coming back to the initial point.

But since the condensate wavefunction $\Psi_0(\vec{r},t)$ is univalued then the variation of the phase must be a multiple of 2π: $\Delta S = 2\pi s$

Hence $\kappa = \left(\frac{2\pi\hbar}{m}\right)s$

So this means that the circulation is quantized in multiples of $\left(\frac{2\pi\hbar}{m}\right)$

. Let's consider a straight vortex line. The velocity $\vec{v}_s$ is given by circles lying on the plane perpendicular to the line. So from the top



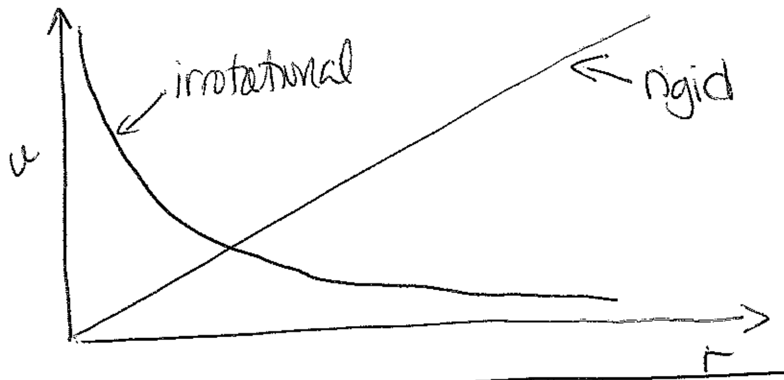
Since $\oint_{\substack{\text{circle}\\ \text{of radius}\\ r}} \vec{v}_s \cdot d\vec{l} = v_s^{(r)}\, 2\pi r = 2\pi \frac{\hbar}{m} s$

then $\boxed{v_s = s\,\frac{\hbar}{mr}} \longrightarrow \vec{v}_s = v_s\,\hat{\varphi}$

Note: $\hat{\varphi} \longrightarrow$ tangential vector (see figure)

* Compare this with the rigid body expression.

For a rigid body $\vec{v} = \vec{\Omega} \times \vec{r} \longrightarrow v = \Omega r$



The difference is very significant indeed!!

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* THE VORTEX CORE

* Remember that $\vec{v}_S = \frac{\hbar}{m} s \frac{1}{r} \hat{\varphi} = \vec{\nabla} S = \frac{1}{r} \frac{dS}{d\varphi} \hat{\varphi}$

Hence the condensate phase is $S = s\varphi$

* Therefore the condensate wavefunction is

$$\psi_0(\vec{r}) = e^{is\varphi} |\psi_0(\vec{r})|$$

* Let's put this into the stationary GPE. We get

$$\mu|\psi_0| = -\frac{\hbar^2}{2m}\left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right]|\psi_0| + \underbrace{\frac{\hbar^2 s^2}{2mr^2}|\psi_0|}_{\text{centrifugal barrier}} + g|\psi_0|^3$$

* Far away from the vortex ($r \to \infty$) it's clear that we recover the uniform density $|\psi_0| \longrightarrow \sqrt{n}$.

Let's proceed like in our discussion of the boundary effects in the box potential.

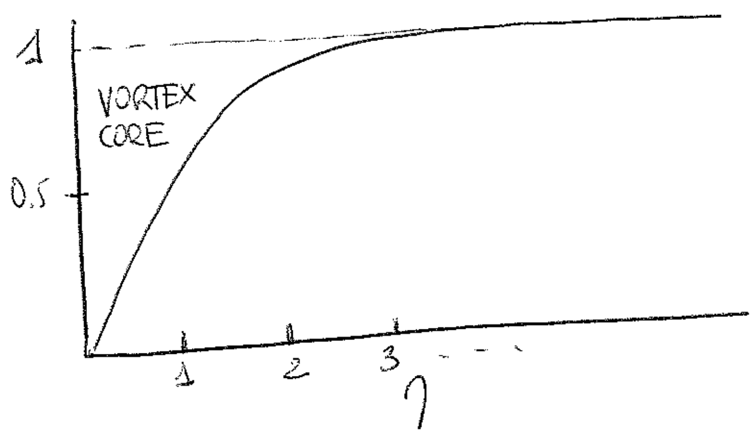
We substitute
$r \longrightarrow r/\xi \equiv \eta$
$|\psi_0| \longrightarrow \sqrt{n}\, f(\eta)$

where ξ is the healing length. $\xi = \frac{\hbar}{\sqrt{2mgn}}$

* When doing this substitution we get to the equation for the vortex core:

$$\frac{1}{\eta}\frac{d}{d\eta}\left[\eta\frac{df}{d\eta}\right] + \left(1 - \frac{S^2}{\eta^2}\right)f - f^3 = 0 \quad \text{with } f(\infty) = 1$$

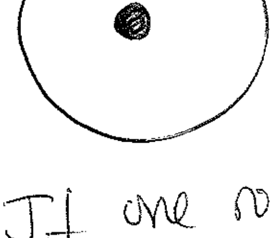
This looks like this:



• You can see that the typical size of the vortex core is again provided by the healing length ξ

(again we have done a perturbation, this time a vortex, and this perturbation "heals" after some distance of the order of some healing lengths towards the constant density)

* So, if you look from the top you will see a hole



* If one rotates faster and faster more vortices which actually place themselves in a lattice configuration (they behave like 2D charged particles) which is called an Abrikosov vortex lattice

It's a triangular vortex lattice.