Axiomatic Thinking in Physics
– Use and Fallacy –

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Axiomatic Thinking
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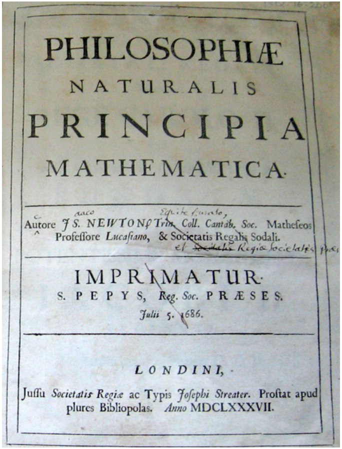
Introduction

Axiomatic thinking – in one form or another – is omnipresent in physics. This includes its "classical" and "non-classical" parts:

- **Mechanics:** Newton, Lagrange, Thomson-Tait, Hamel, Arnold, Lange, Frege,
- **Thermodynamics:** Carathéodory, Giles, Lieb-Yngvason, ...
- **Electrodynamics:** Maxwell, Mie, Post, Hehl-Obukov, ...
- **Special Relativity:** Ignatowski, Rothe, Robb, Reichenbach, Berzi-Gorini, Alexandrov, Zeeman, Benz ...
- **General Relativity:** Hilbert, Weyl, Ehlers-Pirani-Schild, Schelb, Pfister ...
- **Quantum Theory:** Dirac, Neumann, Birkhoff, Mackey, Piron, Ludwig, ...
- **Quantum Field Theory:** Wightman-Gårding, Osterwalder-Schrader, Araki-Haag-Kastler, Hollands-Wald, Fredenhagen ...

It is impossible to do justice to all of them in this talk. Hence I will pick a few according to my own expertise and prejudice.

Origins of axiomatic thinking in modern physics


PHILOSOPHIÆ
NATURALIS
PRINCIPIA
MATHEMATICA

Autore J S. NEWTON, Trin. Coll. Cantab. Soc. Matheseos Professore Lucasiano, & Societatis Regalis Sodali.

IMPRIMATUR
S. PEPYS, Reg. Soc. PRÆSES.
Julii 5. 1686.

LONDINI,

Jussu Societatis Regiæ ac Typis Josephi Streater. Prostat apud plures Bibliopolas. Anno MDCLXXXVII.

- Ever since Newton's *"Principia"* (Philosophiae Naturalis Principia Mathematica (1686), theories for selected parts of the phenomenological world have been presented in a more or less axiomatic form.
- The value of an axiomatic presentation of physical theories in not unanimously judged as high amongst physicists. Some think its a mere matter of taste and sometimes criticise it as dispensable or "excess baggage".
- However, it is commonly accepted (if only implicitly) that **falsification is the essence of progress in physics:**

$$\boxed{A \rightarrow B \Rightarrow \bar{B} \rightarrow \bar{A}} \qquad (1)$$

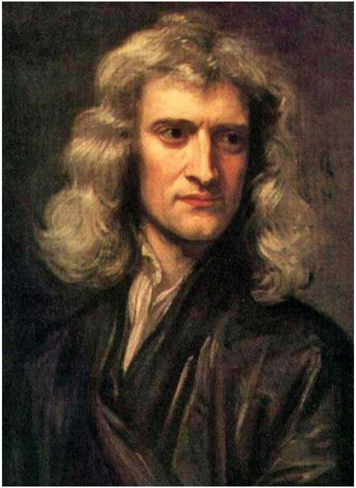
Part I: Voices

- In the first part I wish to take a few examples from the history of physics, where eminent authors have expressed opinions, ex- or implicitly, on the axiomatic method.
- The examples are picked according to my own expertise and prejudice. In particular, no ranking whatsoever is implied.
- I regret to not to talk about axiomatic QM and QFT; but that's essentially outside my field of expertise.

Einstein: Geometry and Experience (1921)

- "Insofar as the statements of mathematics refer to reality they are not certain, and insofar as they are certain, they do not refer to reality."
- "Full clarity on the state of affairs in question (of the relation between mathematical thinking and experience of reality) is brought to the general community by that direction in mathematics which is known under the name of 'axiomatics'."
- "The progress brought about by axiomatics consist in a clear separation of the logically-formal from the contentual aspects. Only the logically-formal forms, according to the axioms, are the object (german: Gegenstand) of mathematics, not however those imaginative contents that are connected with them."

Newton's Principia: Background doubts



"That gravity should be innate inherent and essential to matter so that one body may act upon another at a distance through a vacuum without the mediation of anything else by and through which their action of force may be conveyed from one to another, is to me so great an absurdity that I believe no man who has in philosophical matters any competent faculty of thinking can ever fall into it. Gravity must be caused by an agent acting constantly according to certain laws, but whether this agent be material or immaterial is a question I have left to the consideration of my readers [of the Principia]".

Newton to Bentley, 25. Feb. 1692

Heinrich Hertz (1857-1894): Principles of Mechanics

DIE

PRINZIPIEN DER MECHANIK

IN NEUEM ZUSAMMENHANGE DARGESTELLT

VON

HEINRICH HERTZ.

MIT EINEM VORWORTE

VON

H. VON HELMHOLTZ.

LEIPZIG, 1894
JOHANN AMBROSIUS BARTH
(ARTHUR MEINER)

Hertz' Mechanics: Introduction

- *"We form for ourselves images or symbols of external objects; and the form which we give them is such that the necessary consequents of the images in thought are always the necessary consequents in nature of the things pictured."*

- *"The images which we here speak of are our conceptions of things. With the things themselves they are in conformity in* **one** *important respect, namely, in satisfying the above-mentioned requirement. For our purpose it is not necessary that they should be in conformity with the things in any other respect whatever".*

- *"The images which we may form of things are not determined without ambiguity by the requirement, that the consequents of the images must be the images of the consequents."*

- *"Of two images of the same object that is the more appropriate which pictures more of the essential relations of the object, – the one which we may call the more distinct. Of two images of equal distinctness the more appropriate is the one which contains, in addition to the essential characteristics, the smaller number of superfluous or empty relations – the simpler of the two".*

Hertz' Mechanics: Table of Contents

Hertz' Mechanics: Time, Space, Mass

Vorbemerkung. Den Überlegungen des ersten Buches 1
bleibt die Erfahrung völlig fremd. Alle vorgetragenen Aussagen sind Urteile a priori im Sinne KANT's. Sie beruhen auf den Gesetzen der inneren Anschauung und den Formen der eigenen Logik des Aussagenden und haben mit der äufseren Erfahrung desselben keinen anderen Zusammenhang, als ihn diese Anschauungen und Formen etwa haben.

Abschnitt 1. Zeit, Raum, Masse.

Erläuterung. Die Zeit des ersten Buches ist die Zeit 2
unserer inneren Anschauung. Sie ist daher eine Gröfse, von deren Änderung die Änderungen der übrigen betrachteten Gröfsen abhängig gedacht werden können, während sie selbst stets unabhängig veränderlich ist.

Der Raum des ersten Buches ist der Raum unserer Vorstellung. Er ist also der Raum der EUKLID'schen Geometrie mit allen Eigenschaften, welche diese Geometrie ihm zuspricht. Es ist gleichgültig für uns, ob man diese Eigenschaften ansieht als gegeben durch die Gesetze der inneren Anschauung, oder als denknotwendige Folgen willkürlicher Definitionen.

Die Masse des ersten Buches wird eingeführt durch eine Definition.

Vorbemerkung. In diesem zweiten Buch werden wir unter 296
Zeiten, Räumen, Massen Zeichen für Gegenstände der äufseren Erfahrung verstehen, deren Eigenschaften übrigens den Eigenschaften nicht widersprechen, welche wir vorher den gleichbenannten Gröfsen als Formen unserer inneren Anschauung oder durch Definition beigelegt hatten. Unsere Aussagen über die Beziehungen zwischen Zeiten, Räumen und Massen sollen daher nicht mehr allein den Ansprüchen unseres Geistes genügen, sondern sie sollen zugleich auch möglichen, insbesondere zukünftigen Erfahrungen entsprechen. Diese Aussagen stützen sich daher auch nicht mehr allein auf die Gesetze unserer Anschauung und unseres Denkens, sondern aufserdem auf vorangegangene Erfahrung. Den Anteil der letzteren aber, soweit er nicht schon in den Grundbegriffen enthalten ist, werden wir zusammenfassen in eine einzige allgemeine Aussage, welche wir als Grundgesetz voranstellen. Eine spätere, nochmalige Berufung auf die Erfahrung findet dann nicht mehr statt. Die Frage nach der Richtigkeit unserer Aussagen fällt also zusammen mit der Frage nach der Richtigkeit oder Allgemeingültigkeit jener einzigen Aussage.

Abschnitt 1. Zeit, Raum, Masse.

Zeit, Raum und Masse schlechthin sind unserer Erfahrung 297
in keinem Sinne zugänglich, sondern nur bestimmte Zeiten, bestimmte räumliche Gröfsen, bestimmte Massen. Jede be-

Carathéodory (1873-1950)

Untersuchungen über die Grundlagen der Thermodynamik.

Von

C. Carathéodory in Hannover.

Inhalt.

Einleitung.

Zu den bemerkenswertesten *Ergebnissen* der *Forschung* des letzten Jahrhunderts über Thermodynamik muß wohl die Erkenntnis gezählt werden, daß sich diese Disziplin frei von jeder Hypothese begründen läßt, die man nicht experimentell verifizieren kann. Der Standpunkt, auf welchen sich die meisten Autoren seit fünfzig Jahren nach den großen Entdeckungen von R. Mayer, den Messungen von Joule und den grundlegenden Arbeiten von Clausius und von W. Thomson stellen, ist etwa folgender:

Es gibt eine physikalische Größe, die mit den mechanischen Größen (Masse, Kraft, Druck usw.) nicht identisch ist, deren Änderungen man durch kalorimetrische Messungen bestimmen kann und die man Wärme

Math. Ann. 67 (1909) 355

Lieb-Yngvason

PHYSICS REPORTS

ELSEVIER

Physics Reports 310 (1999) 1–96

The physics and mathematics of the second law of thermodynamics

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Contents

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A Fresh Look at Entropy and the Second Law of Thermodynamics

The existence of entropy, and its increase, can be understood without reference to either statistical mechanics or heat engines.

Elliott H. Lieb and Jakob Yngvason

In days long gone, the second law of thermodynamics (which predated the first law) was regarded as perhaps the most perfect and unassailable law in physics. It was even supposed to have philosophical import: It has been hailed for providing a proof of the existence of God (who started the universe off in a state of low entropy, from which it is constantly degenerating); conversely, it has been rejected as being incompatible with dialectical materialism and the perfectibility of the human condition.

Alas, physicists themselves eventually demoted the second law to a lesser position in the pantheon—because (or so it was declared) it is "merely" statistics applied to the mechanics of large numbers of atoms. Willard Gibbs wrote: "The laws of thermodynamics may easily be obtained from the principles of statistical mechanics, of which they are the incomplete expression"[1]—and Ludwig Boltzmann expressed similar sentiments.

Is that really so? Is it really true that the second law is merely an "expression" of microscopic models, or could it exist in a world that was featureless at the 10^{-8} cm level? We know that statistical mechanics is a powerful tool for understanding physical phenomena and calculating many quantities, especially in systems at or near equilibrium. We use it to calculate entropy, specific and latent heats, phase transition properties, transport coefficients, and so on, often with good accuracy. Important examples abound, such as Max Planck's realization that by staring into a furnace he could find Avogadro's number, and Linus Pauling's highly accurate back-of-the-envelope calculation of the residual entropy of ice. But is statistical mechanics essential for the second law?

In any event, it is still beyond anyone's computational ability (except in idealized situations) to account for a very precise, essentially infinitely accurate law of physics from statistical mechanical principles. No exception to the second law of thermodynamics has ever been found—not even a tiny one. Like conservation of energy (the "first" law), the existence of a law so precise and so independent of details of models must have a logical foundation that is independent of the fact that matter is composed of interacting particles. Our aim here is to explore that foundation. The full details can be found in reference 2.

As Albert Einstein put it, "A theory is the more impressive the greater the simplicity of its premises, the more different kinds of things it relates, and the more extended its area of applicability. Therefore the deep impression which classical thermodynamics made upon me. It is the only physical theory of universal content concerning which I am convinced that, within the framework of the applicability of its basic concepts, it will never be overthrown."[3]

In an attempt to reaffirm the second law as a pillar of physics in its own right, we have returned to a little-noticed movement that began in the 1950s with the work of Peter Landsberg,[4] Hans Buchdahl,[5] Gottfried Falk, Herbert Jung,[6] and others[2] and culminated in the book of Robin Giles,[7] which must be counted one of the truly great, but unsung works in theoretical physics. It is in these works that the concept of "comparison" (explained below) emerges as one of the key underpinnings of the second law. The approach of these authors is quite different from lines of thought in the tradition of Sadi Carnot, which base thermodynamics on the efficiency of heat engines. (See reference 8, for example, for modern expositions of the latter approach.)

The basic question

The paradigmatic event that the second law deals with can be described as follows. Take a macroscopic system in an equilibrium state X and place it in a room along with a gorilla equipped with arbitrarily complicated machinery (a metaphor for the rest of the universe), and a weight—and close the door. As in the old advertisement for indestructible luggage, the gorilla can do anything to the system—including tearing it apart. At the end of the day, however, when the door is opened, the system is found to be in some other equilibrium state, Y, the gorilla and machinery are found in their original state, and the only other thing that has possibly changed is that the weight has been raised or lowered. Let us emphasize that although our focus is on equilibrium states, the processes that take one such state into another can be arbitrarily violent. The gorilla knows no limits. (See figure 1.)

The question that the second law answers is this: What distinguishes those states Y that can be reached from X in this manner from those that cannot? The

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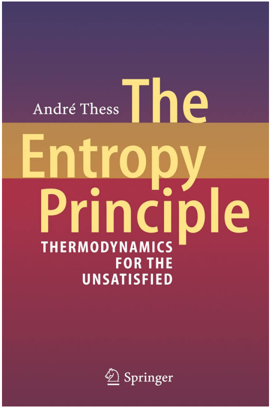
Lieb-Yngvason in action



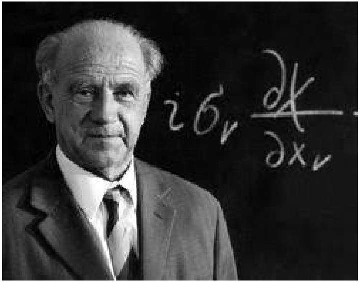
Is it possible to define entropy in classical thermodynamics in a way that is mathematically accurate and at the same time easy to understand? I have often asked this question to myself first when I was a student and later when I became a professor of mechanical engineering and had to teach thermodynamics. Unfortunately, I never got a satisfactory answer. In textbooks for physicists I often found the claim that entropy can only be "really" understood when one has recourse to statistical physics. But it appeared strange to me that a physical law as perfect as the second law of thermodynamics, which is closely related to entropy, should depend on tiny details of the molecular structure of the matter that surrounds us. By contrast, in textbooks for engineers entropy was most often defined on the basis of temperature and heat. However, I never felt comfortable with the idea that such a fundamental quantity as entropy should be determined on the basis of two concepts which cannot be accurately defined without entropy. Given this state of affairs, I came close to resignation and was on the verge of believing that an accurate and logically consistent definition of entropy in the framework of a macroscopic theory was altogether impossible.

In the spring of the year 2000, I came across an article entitled "A Fresh Look at Entropy and the Second Law of Thermodynamics" written by the physicists Elliott Lieb and Jakob Yngvason which appeared in the journal *Physics Today*. Their idea that the concept of adiabatic accessibility rather than temperature or heat is the logical basis of thermodynamics appealed to me immediately. For the first time in my academic life I began to feel that I really understood the entropy of classical thermodynamics. However, it took considerable effort to study and understand the article "The Physics and Mathematics of the Second Law of Thermodynamics" (*Physics Reports*, vol. 310, 1999, pp. 1–96) by the same authors in which the full "Lieb-Yngvason theory" is presented. Once I had finished the work, however, I was convinced that the Lieb-Yngvason theory represents the ultimate formulation of classical thermodynamics. Although the theory is mathematically complex, it is based on an idea so simple that each student of science or engineering should be able to understand it.

I then decided to involve my students in order to test whether the Lieb-Yngvason theory is as convincing as I believed. I have been teaching a one-year thermodynamics course for the undergraduate mechanical engineering students of Ilmenau University of Technology since 1998. I use Moran and Shapiro's textbook "Fundamentals of Engineering Thermodynamics" (Wiley and Sons), and I introduce entropy as is currently most often done in engineering courses, namely via the Carnot process cycle and the Clausius inequality. One week after having introduced entropy in the regular lecture,

- *"Every mathematician knows it is impossible to understand an elementary course in thermodynamics".* (V. Arnold: Gibbs Symposium 1990)

Heisenberg: Unified Field Theory



Einführung
in die einheitliche Feldtheorie
der Elementarteilchen

von Werner Heisenberg

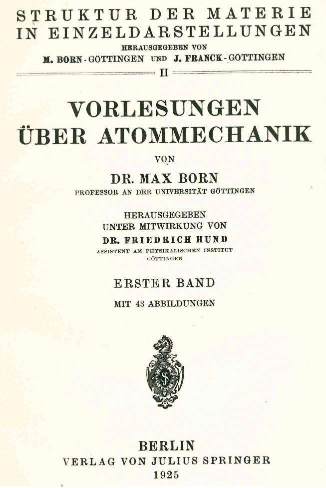
S. Hirzel Verlag Stuttgart

1967

Heisenberg: Introduction to “Einheitliche Feldtheorie”

- *“The idea, according to which elementary particles appear as dynamical systems, comparable to the stationary states of a complicated atom or molecule and as determined universally by quantum mechanics, has for a long time found little response by physicists.”*
- *“At the current state of the theory it would be premature to start with a set of well defined axioms and deduce the theory by means of exact mathematical methods. What we need is a mathematical description, which fits the experimental situation, which does not seem to contain contradictions and which, therefore, may perhaps be later completed into an exact mathematical scheme. History of physics teaches us that, usually, a new theory can only then be given precise mathematical expression if all essential physical problems have been solved.”*

Max Born (1882-1970): Mechanics of the Atom


STRUKTUR DER MATERIE
IN EINZELDARSTELLUNGEN
HERAUSGEGEBEN VON
M. BORN - GÖTTINGEN UND J. FRANCK - GÖTTINGEN
II

VORLESUNGEN
ÜBER ATOMMECHANIK

VON

DR. MAX BORN
PROFESSOR AN DER UNIVERSITÄT GÖTTINGEN

HERAUSGEGEBEN
UNTER MITWIRKUNG VON
DR. FRIEDRICH HUND
ASSISTENT AM PHYSIKALISCHEN INSTITUT
GÖTTINGEN

ERSTER BAND

MIT 43 ABBILDUNGEN

BERLIN
VERLAG VON JULIUS SPRINGER
1925

Born: Introduction to "Atommechanik"

- *"The title 'Atommechanik' of this lecture, which I delivered in the winter-semester 1923/24 in Göttingen, is formed after the label 'Celestial Mechanics'. In the same way as the latter labels that part of theoretical astronomy which is concerned with the calculation of trajectories of heavenly bodies according to the laws of mechanics, the word 'Atommechanik' is meant to express that here we deal with the facts of atomic physics from the particular point of view of applying mechanical principles. This means that we are attempting a deductive presentation of atomic theory. The reservations, that the theory is not sufficiently developed (matured), I wish to disperse with the remark that we are dealing with a test case, a logical experiment, the meaning of which just lies in the determination of the limits to which the principles of atomic- and quantum physics succeed, and to pave the ways which shall lead us beyond that limits. I called this book 'Volume I' in order to express this programme already in the title; the second volume shall then contain a higher approximation to the 'final' mechanics of atoms."*

Part II: Examples

- In the second half of this talk I wish to present some mathematical details connected with axiomatisation in modern physics. I picked the examples of special and general relativity.
- I will start with special relativity, which is mathematicall less complex but far from trivial. Ever since Einstein's 1905 motivation/derivation of the Lorentz transformations, starting from the two explicit (and many implicit) assumptions: "relativity principle" and "constancy of the speed of light in vacuum", physicists have asked how one can reduce the set of hypotheses. Ignatowski (1910), Rothe (1911), and Bezi-Gorini (1969) showed how to arrive at the (one parameter family) of Lorentz groups without the c-postulate. Here I will mention results in the opposite direction.
- In general relativity I will only mention the most famous developments, that are considered classic today.

SR: Causality implies the Lorentz group

- Let $\mathbb{R}^{n+1}$ be endowed with quadratic form

$$Q(x) = (x_0)^2 - \sum_{k=1}^{n} (x_k)^2 \tag{2}$$

- We define relations $\ll$ (partial odering) and $\lessdot$ (not transitive) by

$$x \ll y \quad \Leftrightarrow \quad y_0 > x_0 \ \wedge \ Q(y-x) > 0 \tag{3a}$$
$$x \lessdot y \quad \Leftrightarrow \quad y_0 > x_0 \ \wedge \ Q(y-x) = 0 \tag{3b}$$

- **Theorem [A.D. Alexandrov (1950), E.C. Zeeman (1963)]:** Let $n \geqslant 2$ and $f : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}$ be a bijection such that either $x \ll y \Leftrightarrow f(x) \ll f(y)$ or $x \lessdot y \Leftrightarrow f(x) \lessdot f(y)$, then f is the composition of a time-orientation preserving Lorentz transformation, a translation, and a positive dilation ($x \mapsto \lambda x$, $\lambda > 0$).
- Note: Bijectivity needs to be assumed, but continuity follows. The result does not extend to $n = 1$ (much more causal automorphisms exist).

Beckman-Quarles analogs

- **Theorem [F.S. Beckman & D.A. Quarles (1953)]:**
Let f be a self map of Euclidean space $\big(\mathbb{R}^n, \langle\cdot,\cdot\rangle\big)$, where $n \geqslant 2$. Suppose there exists a positive real number, r, such that $\|x-y\| = r \Rightarrow \|f(x)-f(y)\| = r$. Then f is a Euclidean motion.

- **Theorem [W. Benz (1980), J.A. Lester (1981)]:**
Let f be a self map of $\mathbb{R}^{n+1}$, where $n \geqslant 1$, with Minkowskian quadratic form (2). Suppose there exists a non-zero real number, r, such that $Q(y-x) = r \Rightarrow Q\big(f(y)-f(x)\big) = r$. Then f is a composition of a Lorentz transformation and a translation.

- Note: Both, bijectivity and continuity, are not assumed but follow. So, mathematically, this result might look stronger than the Alexandrov-Zeeman result. However, preservation of single (timelike or spacelike) length have no obvious physical significance. To physicists the Alexandrov-Zeeman axioms will presumably appear "deeper" due to the fundamental physical significance of causality.

Topologies: Zeeman

- **Theorem [E.C. Zeeman (1966)]:**
  Replace the Euclidean topology of Minkowski space by the finest topology, called "the fine tolology", that induces the Euclidean topology on all timelike straight lines and all spacelike hyperplanes. Any homeomorphisms of that topological space is the composition of a Lorentz transformation, a translation, and a dilation. Continuous timelike paths are picewise linear, consisting of a finite number of straight intervals along time axes, exactly like the path of a freely moving particle under a finite number of collisions.

- *"From a topologist's point of view the fine topology looks technically complicated because, although it is Hausdorff, being finer than the Euclidean topology it is not normal; and although it is connected and locally connected it is not locally compact, nor does any point have a countable base of neighbourhoods. However these disadvantages are outweighed by the physical advantages described above."*

Topologies: Hawking et al.

- Besides being "not nice", Zeeman's fine topology can be criticised for still invoking physically unwarranted assumptions: Spacelike hyperplanes are not accessible. Restriction to *straight* timelike paths invokes inertial structure and neglects non-inertially moving particles under action of force.

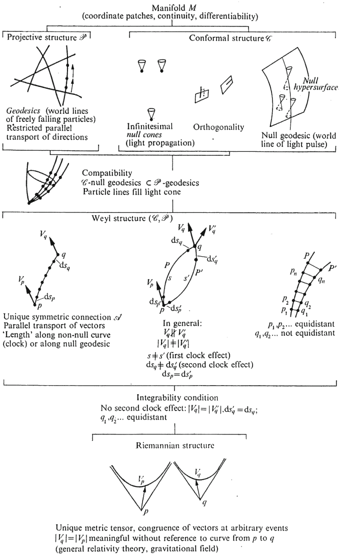
- **Theorem [Hawking & King & McCarthy (1975)]:**
  Replace the Euclidean topology of Minkowski space by the finest topology, called "the path tolology", that induces the Euclidean topology on arbitrary timelike curves (to be defined appropriately). Then any homeomorphisms of that topological space is the composition of a Lorentz transformation, a translation, and a dilation. This topology is Hausdorff, connected, locally connected and (sic!) first countable, though still not normal or locally compact.

- From a physical "operational" point of view, the path topology is much more natural than the fine topology, since a set is open if and only if a general observer – moving on any timelike curve – "times" it to be open.

GR: Clocks, Rods, Clocks, Particles, and Light-Rays

- Attempts to axiomatise General Relativity go back to its hour of birth, namely Hilberts *Die Grundlagen der Physik* (Gött. Nachr. Nov. 20th 1915 and Dec. 23rd 1916; slightly modified version Math. Ann. 1924.)
- Hilbert intimately tight up the axioms of GR (gravitation) with those of what he believed was an appropriate candidate for all matter interactions: Gustav Mie's 1913 non-quantum theory of non-linear electrodynamics (taken up again in 1934 by Born-Infeld; with less ambitious motivation).
- Whereas Mie's theory is not any longer believed to have that significance, the axiomatisation of GR, taken with a minimum of primitive matter representatives, is still taken as relevant and persued actively by some.
- Primitive matter representative may be idealised "clocks" and "rods", or "test particles" and "light-rays".
- If (M, g) is a spacetime, a "clock" is a (piecewise C^2) map $\gamma : I \rightarrow M$ with timelike $\dot{\gamma}$, whereas a "particle" is an unparametrised class of timelike geodesic curves (autoparallels). A "light ray" is an unparametrised class of lightlike (null) geodesics curves.
- Hilbert gave a prescription how to determine g from the reading of 10 independently moving (light-) clocks. It was the idea of Hermann Weyl to excluseively use particles and light-rays as primitive elements. Particles would set the projective, light-rays ther conformal structure.

Axiomatising GR: Ehlers-Pirani-Schild (1972)



- Primitive elements are a set M of "events" and two sets of subsets $\mathcal{L}$ and $\mathcal{P}$ of "light-rays" and "particles".

D A set $D_1, \cdots, D_4$ of four axioms characterise the differential-topological structure of M.

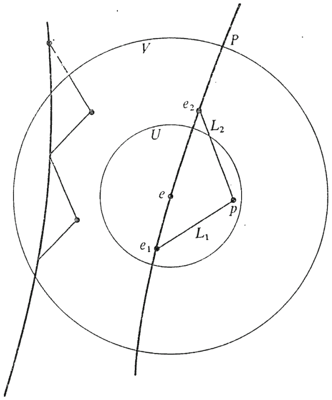
L On top of [D], a set L_1, L_2 of two axioms fix the *causal* structure with an underlying C^3 manifold M and a C^2 *conformal structure* of Lorentzian metrics.

P On top of [D], a set P_1, P_2 of two axioms characterise a *projective structure* (the class of free-fall worldlines).

C A last axiom, C, ensures causal-compatibility between conformal structure (light-cones) and particle trajectories (always inside the light cone). Froms all this, a *Weyl geometry* $(M, [g]_c, \nabla)$ results.

R In order to reduce this to a Semi-Riemannian geometry, additional physical imput is needed; like: no 2nd-clock-effect, or compatibility of projective structure with WKB-limit of massive-wave propagation (Audretsch 1983, Audretsch-Gähler-Straumann 1984).

Example: How Finsler metrics get kicked out



L_1) "Any event e has a neighbourhood V such that each event p in V can be connected within V to a particle P by at most two light rays. Moreover, given such a neighbourhood and a particle P through e, there is another neighbourhood $U \subset V$, such that any event p in U can, in fact, be connected with P within V by precisely two light rays L_1 and L_2 and these intersect P in two distinct events e_1, e_2 if $p \neq P$. If t is a coordinate on $P \cap V$ with $t(e) = 0$, then $g : p \mapsto -t(e_1)t(e_2)$ is a function of class C^2 on U".

Conclusion

- Hilbert's axiomatisation programme is persued - in one form or another - in many branches of classical and modern physics.
- Opinions diverge as regards its heuristic value, that is, concerning its use and power in the creative process of developing "insight" into the laws of Nature.
- One of the most interesting but also most difficult question intimately associated to this programme is how to interpret Hilbert's term "deepening" (german: "Tieferlegung"). There is no natural objective measure for "depth" and often, in physics, the number of axioms is reduced at the price of a priori inbuilt physical limitations (e.g., Hilbert's connection of GR with Mie's theory).
- In physics this is related to the problem of "fundamentality", which is often passionately discussed with too many ideologically motivated preconceptions. I suggest to follow Max Born and regard axiomatic approaches pragmatically as "logical experiments", which contribute to our understanding just as much as experiments in the lab. Both should go hand in hand!

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Conclusion

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