Equivalence Principle and self-gravitating quantum systems

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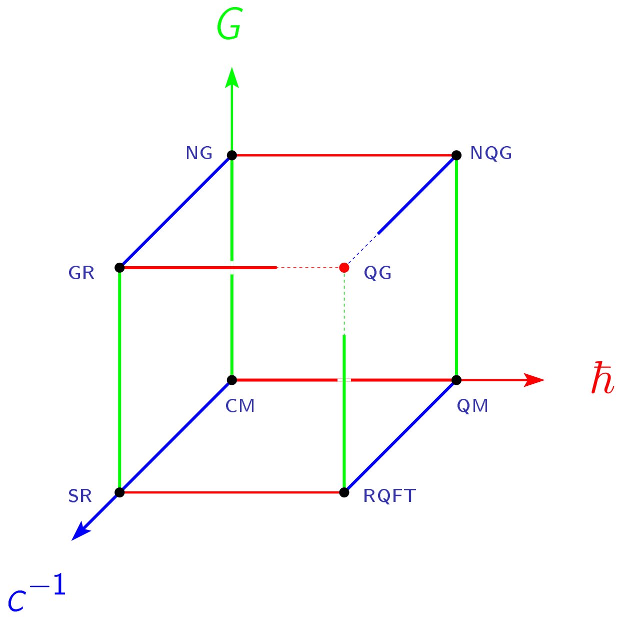
The magic cube



History: From Galilei's Discorsi (1638), first day - 1

Salviati:

"Finally I took two balls, one made from lead one from cork, that one a 100 times heavier than this one, and suspended them on two equal and fine threads of about 2-3 yards length; if I elongated them from the vertical position and released them simultaneously ... it became clearly apparent that the heavier body coincided so much with the lighter one that neither in a 100 nor in a 1000 periods the smallest discrepancy could be noted; they moved in perfect step. To be sure, there was an effect of the medium that presents a resistance to the motion, diminishing the oscillations of the cork much more than that of the lead, but that does not make them more or less frequent, even if the arcs covered by the cork are just 5-6 degrees and that of the lead cover 50-60 degrees; the arcs will be covered in one and the same time span."

- However: The period T of mathematical pendulum with maximal amplitude α is given by an elliptic integral, whose expansion is

$$T(\alpha) = 2\pi\sqrt{\frac{l}{g}}\left\{1 + \frac{1}{4}\sin^2(\alpha/2) + \frac{9}{64}\sin^4(\alpha/2) + \cdots\right\} \qquad (1)$$

$$\Longrightarrow N\frac{T(25°) - T(2.5°)}{T} \approx \frac{N}{4}\sin^2(12,5°) = \frac{N}{85} \qquad (2)$$

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History: From Galilei's Discorsi (1638), first day - 2

Salviati:

> *"In fact, without much experiments, we can prove convincingly how impossible it it that a heavier mass would move swifter than a smaller one...*
> *If we have two bodies whose natural speeds [accelerations] were different, than it is clear that if we combine the slower and the swifter one, the latter would be decelerated by the former, and the former, the swifter one, would be accelerated by the latter.*
> *Hence you see: from the assumption that a heavier body would have a greater speed [acceleration] than a smaller one, I could urge you to conclude further that a greater body moves slower than a smaller one."*

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Three types of masses

In Newtonian physics we have to distinguish between three types of masses:

1. The **inertial mass** determines the force, that acts against an imposed acceleration:
$$\vec{F}_{\mathrm{inertial}} = -m_i \vec{a}\,. \tag{3}$$

2. The **passive gravitational mass** determines the force, by which a body is acted upon in an external gravitational field $\vec{g}$:
$$\vec{F}_{\mathrm{gravitational}} = m_{pg}\,\vec{g}\,. \tag{4}$$

3. The **active gravitational mass** determines the gravitational field produced by a body; e.g. outside a spherical mass distribution centred at $\vec{x}'$
$$\vec{g}(\vec{x}) = -\,G\,m_{ag}\,\frac{\vec{x}-\vec{x}'}{\|\vec{x}-\vec{x}'\|^3}\,. \tag{5}$$

The impact of *actio* = *reactio*

$$\vec{F}_{12} = m_{pg}^{(1)}\,\vec{g}_2(\vec{x}_1) = \left(m_{pg}^{(1)} m_{ag}^{(2)}\right)\, G\, \frac{\vec{x}_2 - \vec{x}_1}{\|\vec{x}_2 - \vec{x}_1\|^3} \tag{6}$$

$$\vec{F}_{21} = m_{pg}^{(2)}\,\vec{g}_1(\vec{x}_2) = \left(m_{pg}^{(2)} m_{ag}^{(1)}\right)\, G\, \frac{\vec{x}_1 - \vec{x}_2}{\|\vec{x}_1 - \vec{x}_2\|^3} \tag{7}$$

$$\vec{F}_{12} = -\vec{F}_{21} \Leftrightarrow \frac{m_{pg}^{(1)}}{m_{ag}^{(1)}} = \frac{m_{pg}^{(2)}}{m_{ag}^{(2)}} = \text{universal constant} = 1 \tag{8}$$

- What remains unexplained is the equality of inertial and gravitational mass; that is, why should

$$\frac{m_i}{m_g} = \text{universal constant?} \tag{9}$$

The first person to clearly state that this presented a big challenge to fundamental physics was Heinrich Hertz in 1884.

Weak equivalence principle for pointlike test masses

- The motion of a **pointlike test masses** is determined through

$$\vec{F}_i + \vec{F}_g = 0 \iff -m_i\,\ddot{\vec{x}}(t) + m_g\,\vec{g}\big(\vec{x}(t)\big) = 0 \qquad (10)$$

- If $m_i = m_g$ this is equivalent to

$$\ddot{\vec{x}}(t) = \vec{g}\big(\vec{x}(t)\big) = -\vec{\nabla}\phi\big(\vec{x}(t)\big) \qquad (11)$$

- **Weak equivalence principle:** The motion of a *pointlike test mass* in an external gravitational field depends only on the initial position and velocity.

Q1 How small is "pointlike"?

A1 Much smaller than typical length over which $\vec{g}$ varies appreciably.

Q2 What is a test mass?

A2 No higher multipoles in mass distribution (has nothing to do with size), no charge, no spin, no significant gravitational self-energy (not too small).

- **Being a "test mass" is a contextual property.**

Einstein's Equivalence Principle (EEP)

- **Universality of Free Fall (UFF)** Requires existence of sufficiently general "test bodies" to determine a path structure on spacetime (not necessarily of pseudo Riemannian type). Possible violations of UFF are parametrised by the Eötvös factor

$$\eta(A,B) := 2\,\frac{|a(A)-a(B)|}{|a(A)+a(B)|} \approx \sum_\alpha \eta_\alpha \left(\frac{E_\alpha(A)}{m_i(A)c^2} - \frac{E_\alpha(B)}{m_i(B)c^2}\right) \tag{12}$$

- **Local Lorentz Invariance (LLI)** Local non-gravitational experiments exhibit no preferred directions in spacetime, neither timelike nor spacelike. Possible violations of LLI concern, e.g., variations in $\Delta c/c$.
- **Universality of Gravitational Redshift (UGR)** Requires existence of sufficiently general "standard clocks" whose rates are universally affected by the gravitational field. Possible violations of UFF are parametrised by the α-factor

$$\frac{\Delta\nu}{\nu} = (1+\alpha)\frac{\Delta U}{c^2} \tag{13}$$

$\Rightarrow$ **Geometrisation of gravity and unification with inertial structure. Far reaching consequences. One of them is: Gravity is not a force!**

Levels of verification of EEP

- **UFF:** Typical results from torsion-balance experiments by the "Eöt-Wash" group between 1994-2008 are

$$\eta(Al, Pt) = (-0.3 \pm 0.9) \times 10^{-12}, \quad \eta(Be, Ti) = (0.3 \pm 1.8) \times 10^{-13} \quad (14)$$

  Planned improved levels are $5 \cdot 10^{-16}$ (MICROSCOPE) and 10^{-18} (STEP).

- **LLI:** Currently best Michelson-Morley type experiments give (Herrmann *et al.* 2005)

$$\frac{\Delta c}{c} < 3 \cdot 10^{-16} \quad (15)$$

- **UGR:** Absolute redshift with H-maser clocks in space (1976, $h = 10\,000\,\mathrm{Km}$) and relative redshifts using precision atomic spectroscopy (2007) give

$$\alpha_{\mathrm{abs}} < 2 \times 10^{-4} \qquad \alpha_{\mathrm{rel}} < 4 \times 10^{-6} \quad (16)$$

- In Feb. 2010 Müeller *et. al.* claimed improvements by 10^4. This is not widely accepted (see below). Long-term expectation in future space missions is to get to 10^{-10} level.
- In Sept. 2010 Chou *et al.* report measurability of gravitational redshift on Earth for $h = 33\,\mathrm{cm}$ using Al^+-based optical clocks ($\Delta t/t < 10^{-17}$).

UFF, UGR, and energy conservation

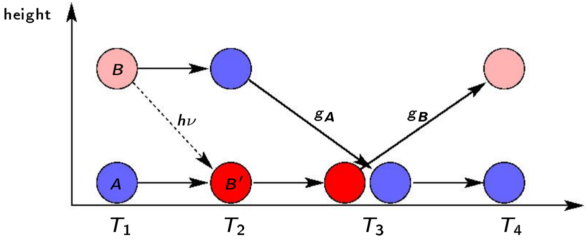
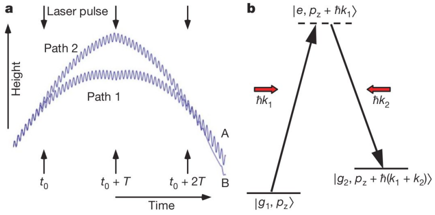


Figure: Gedankenexperiment by Nordtvedt to show that energy conservation connects violations of UFF and UGR. Considered are two copies of a system that is capable of 3 energy states A, B, and B' (blue, pink, and red), with $E_A < E_B < E_{B'}$. Initially system 2 is in state B and placed a height h above system 1 which is in state A. At time T_1 system 2 makes a transition $B \to A$ and sends out a photon of energy $h\nu = E_B - E_A$. At time T_2 system 1 absorbs this photon, which is now blue-shifted, and makes a transition $A \to B'$. At T_3 system 2 has been dropped from height h with acceleration g_A, has hit system 1 inelastically, leaving one system in state A and at rest, and the other system in state B with an upward motion with kinetic energy $E_{\text{kin}} = M_A g_A h + (E_{B'} - E_B)$. The latter motion is decelerated by g_B, which may differ from g_A. At T_4 the system in state B has climbed to the same height h by energy conservation. Hence have $E_{\text{kin}} = M_B g_B h$ and therefore $M_A g_A h + M_{B'} c^2 = M_B c^2 + M_B g_B h$, from which we get

$$\frac{\delta\nu}{\nu} = \frac{(M_{B'} - M_A) - (M_B - M_A)}{M_B - M_A} = \frac{g_B h}{c^2}\left[1 + \frac{M_A}{M_B - M_A}\,\frac{g_B - g_A}{g_B}\right] \tag{17a}$$

$$\Rightarrow \alpha = \frac{M_A}{M_B - M_A}\,\frac{g_B - g_A}{g_B} =: \frac{\delta g/g}{\delta M/M} \tag{17b}$$

A brief comment on Müller *et al.* (Nature 2010)



Have

$$\Delta\phi = \kappa T^2 g^{(Cs)} = \kappa T^2 \frac{m_g^{(Cs)}}{m_i^{(Cs)}} g^{(Earth)} = \kappa T^2 \frac{m_g^{(Cs)}}{m_i^{(Cs)}} \frac{m_i^{(Ref)}}{m_g^{(Ref)}} g^{(\mathrm{Ref})} \tag{18}$$
$$= \eta(Cs, Ref)\, \kappa T^2 g^{(\mathrm{Ref})}$$

- Proportional to (1+Eötvös-factor) in *UFF*-violating theories.
- $\mathbb{Q}$ How does it depend on α in *UGR*-violating theories? Müller *et al.* argue for $\propto (1+\alpha)$ by *representation dependent* interpretation of $\Delta\phi$ as a mere redshift.
- Refutation of this interpretation does not answer $\mathbb{Q}$.

Bohr & Einstein

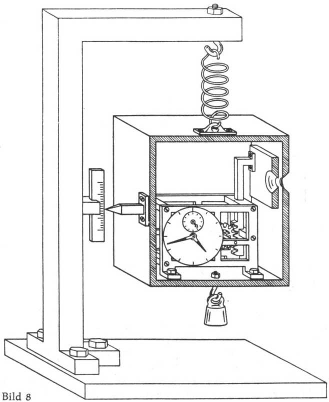


Bild 8

- Einstein argues to be able to violate $\Delta E \Delta T > \hbar$.
- Bohr argues that inequality holds due to UGR:

$$\text{QM:} \qquad \Delta q > \frac{\hbar}{\Delta p} > \frac{\hbar}{T g \Delta m}$$

$$\text{ART:} \qquad \Delta T = \frac{gT}{c^2}\,\Delta q$$

$$\Longrightarrow \qquad \Delta T > \frac{\hbar}{\Delta m\, c^2} = \frac{\hbar}{\Delta E}$$

- Bohr's argument can be (and has been) criticised on various accounts, but its underlying logic (QT needs GR for consistency) seems truly remarkable.

QFT needs GR: Gravity as regulator?

- Consider thin mass shell of Radius R, inertial rest-mass M_0, gravitational mass M_g, and electric charge Q. Its total energy is

$$E = M_0 c^2 + \frac{Q^2}{2R} - G\,\frac{M_g^2}{2R} \tag{19}$$

- Now use the following two principles:

$$\begin{aligned} E &= M_i c^2 \\ M_g &= M_i \end{aligned} \tag{20}$$

- Get quadratic equation for mass $M := M_i = M_g$:

$$\Rightarrow \quad M := \frac{E}{c^2} = M_0 + \frac{Q^2}{2c^2 R} - G\,\frac{M^2}{2c^2 R} \tag{21}$$

Gravity as regulator (contd.)

- The solution is

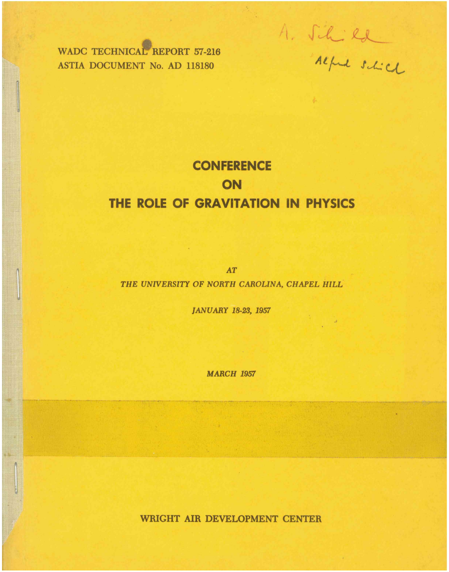
$$M(R) = \frac{Rc^2}{G}\left\{-1+\sqrt{1+\frac{2G}{Rc^2}\left(M_0+\frac{Q^2}{2c^2R}\right)}\ \right\} \tag{22}$$

- Its $R \longrightarrow 0$ limit exits

$$\lim_{R\to O} M(R) = \sqrt{\frac{2Q^2}{G}} = \sqrt{2\alpha}\cdot\frac{|Q|}{e}\cdot M_{\mathrm{Planck}} \tag{23}$$

but its small-G approximation is not uniform in R at $R=0$:

$$\begin{aligned} M \ =& \left(m_0+\frac{Q^2}{2c^2R}\right) \\ &+\sum_{n=1}^{\infty}\frac{(2n-1)!!}{(n+1)!}\cdot\left(-\frac{G}{Rc^2}\right)^n\cdot\left(m_0+\frac{Q^2}{2c^2R}\right)^{n+1} \end{aligned} \tag{24}$$



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A. Schild
Alfred Schild

CONFERENCE
ON
THE ROLE OF GRAVITATION IN PHYSICS

AT
THE UNIVERSITY OF NORTH CAROLINA, CHAPEL HILL

JANUARY 18-23, 1957

MARCH 1957

WRIGHT AIR DEVELOPMENT CENTER

DE WITT called attention to the fact that there is an ambiguity in the choice of field variables in terms of which one may make an expansion about the Minkowski metric $\eta_{\mu\nu}$. For example, one might use either $\psi_{\mu\nu}$ or $\varphi_{\mu\nu}$ where $g_{\mu\nu} = \eta_{\mu\nu} + \psi_{\mu\nu}$, $\mathfrak{g}^{\mu\nu} = \eta_{\mu\nu} + \varphi_{\mu\nu}$. To be consistent in a self-energy calculation one should expand out to the second order, and the difference in choice leads to a difference in the graviton-vacuum expectation value of the interaction term which is proportional to the trace of the non-gravitational (or matter) stress tensor. There are some fields, notably the electromagnetic field, for which this trace vanishes. In this case, you get the same result in the second order, no matter what you expand in terms of. It is a curious thing in the electromagnetic case (although, as Utiyama has pointed out, you do have the derivative coupling and therefore the divergence is of the second or third order), that if you include second order terms in the interaction, the old-fashioned self-energy will be exactly compensated. In fact, one gets no photon self-interaction up to the second order. MISNER asked if the computation has also been done for the neutrino field. DE WITT replied that although it should be done, it has not been done for two reasons: (1) The mathematics of the spinor problem up to the second order is considerably more complicated and has not yet been fully worked out. (2) Interest has in the meantime shifted to the problem of tackling the complete nonlinear gravitational field.

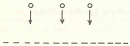
WHEELER pointed out that the linear starting point is incompatible with any topology other than a Euclidean one; that if one has curved space or "worm holes," you just can't start expanding this way. BELINFANTE asked if anyone had found it possible to have hole theory in a curved space; i.e., can one make a covariant distinction between positive and negative energy states?

ROSENFELD said that Dirac, just after he had formulated hole theory and was faced with the objection that there is an infinite energy associated with these states, had attempted unsuccessfully to overcome this difficulty by introducing a closed universe.

MISNER said that one could get a quite good qualitative idea of what happens in curved space (the metric being externally impressed) by using the Feynman prescription. Since the action is still quadratic in the interesting field variables, there is no difficulty. In a spherical space there exist states of excitation which do not scatter each other; but as soon as the space has "bumps" in it, photons in one state get scattered into other states. Hole theory is possible if the metric is static; however, a time dependent metric causes electrons to go to positive energy states.

BERGMANN pointed out that, on this account, one does not have to exclude hole theory, because if the electrons get excited, there is occasional pair production.

SALECKER introduced a thought experiment, involving a stream of particles falling on a diffraction grating. On account of the de Broglie relation for the waves associated with the stream, $\lambda = \frac{h}{mv}$, one expects that particles of different mass will scatter differently if they fall from a given height. According to general relativity, one expects the same behavior for different masses with the same initial state of motion. Therefore, we arrive at a contradiction with the principle of equivalence.



FEYNMAN asked if the grating is here allowed to exert forces on the particles which are non-gravitational. DE WITT said that one needs rather a grating (made, for example, out of planets) which acts only through its gravitational field on the stream. FEYNMAN then said that he did not believe that the principle of equivalence denies the possibility of distinguishing between two different masses. Of course, the principle of equivalence would prevent one from distinguishing between masses by means of this particular experiment if only classical laws were operative. However, the introduction of Planck's constant into the scheme of things introduces new possibilities, which are not necessarily in contradiction with the principle of equivalence. As far as this particular experiment is concerned, all that the principle of equivalence would say would be that if one performs the experiment in an elevator, he will obtain the same result as in a corresponding gravitational field. FEYNMAN also emphasized that the quantities G and c by themselves do not lead to a unit of mass, whereas such a unit exists if $\hbar$ is included.

WHEELER pointed out that the principle of equivalence only denies the possibility of distinguishing between the gravitational and inertial masses of a single body, but definitely does not prevent one from distinguishing the masses of two different bodies, even when only gravitational forces are involved. For example, we know the relative sizes of the masses of the sun and the various planets solely from observation of their gravitational interactions. BERGMANN added that the principle of equivalence makes a statement about local conditions only. Therefore you can do one of two things: either (1) use a small diffraction grating that is not gravitational, or (2) use a diffraction grating made of planets. In this case, the conditions are certainly not local.

FEYNMAN characterized the point which Salecker had raised as an interesting point and a true point, but not necessarily a paradoxical one. If the falling particles are not allowed to react back on the grating, then according to the classical theory they will all follow the same paths. Whereas, in the quantum theory they will give rise to different diffraction patterns depending on their masses.

SALECKER then raised again the question why the gravitational field needs to be quantized at all. In his opinion, charged quantized particles already serve as sources for a Coulomb field which is not quantized. [Editor's Note: Salecker did not make completely clear what he meant by this. If he meant that some forces could be represented by actions-at-a-distance, then, although he was misunderstood, he was right. For the corresponding field can then be eliminated from the theory and hence remain unquantized. He may have meant to imply that one should try to build up a completely action-at-a-distance theory of gravitation, modified by the relativistic necessities of using both advanced and retarded interactions and imbedded in an "absorber theory of radiation" to preserve causality. In this case, gravitation per se could remain unquantized. However, these questions were not discussed until later in the session.]

BELINFANTE insisted that the Coulomb field is quantized through the ψ-field. He then repeated DeWitt's argument that it is not logical to allow an "expectation value" to serve as the source of the gravitational field. There are two quantities which are involved in the description of any quantized physical system. One of them gives information about the general dynamical behavior of the system, and is represented by a certain operator (or operators). The other gives information about our

The gravitational "H-atom"

- Centrifugal force equals gravitational attraction

$$m_i \omega^2 r = G \frac{m_g M_g}{r^2} . \tag{25}$$

- Angular momentum ($\propto m_i$) is quantised

$$m_i \omega r^2 = n\hbar \tag{26}$$

- Bohr radii and frequencies

$$r(n) = \left(\frac{1}{m_i m_g} \right) \cdot \frac{n^2 \hbar^2}{G M_g} , \qquad \omega(n) = \left(m_i m_g^2 \right) \cdot \frac{G^2 M_g^2}{n^3 \hbar^3} , \tag{27}$$

and energies

$$E(n) = \tfrac{1}{2} m_i \omega^2(n) r^2(n) - \frac{G \, m_g M_g}{r(n)} = -\left(m_i m_g^2 \right) \cdot \frac{G^2 M_g^2}{2 \, n^2 \hbar^2} . \tag{28}$$

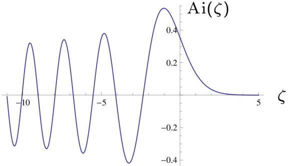
Homogeneous static gravitational field

- Time independent Schrödinger equation in linear potential $V(z) = m_g gz$ is equivalent to:

$$\left(\frac{d^2}{d\zeta^2} - \zeta\right)\psi = 0\,, \quad \zeta := \kappa z - \varepsilon \tag{29}$$

where

$$\kappa := \left[\frac{2m_i\, m_g\, g}{\hbar^2}\right]^{\frac{1}{3}}\,, \quad \varepsilon := E\cdot\left[\frac{2m_i}{m_g^2\, g^2\, \hbar^2}\right]^{\frac{1}{3}}\,. \tag{30}$$



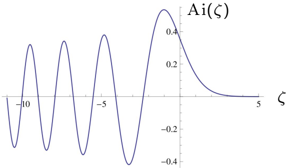
- Complement by hard (horizontal) wall $V(z) = \infty$ for $z \leq 0$ get energy eigenstates from boundary condition $\psi(z=0) = 0$, hence $\varepsilon = -z_n$ (Abele *et al.* 2002 → today's 11 o'clock lecture, Kajari *et al.* 2010, ...):

$$E(n) = -z_n\left[\frac{m_g^2}{m_i}\cdot\frac{g^2\,\hbar^2}{2}\right]^{\frac{1}{3}}\,. \tag{31}$$

Homogeneous static gravitational field

- Classical turning point $z_{\rm turn}$

$$m_g\, g\, z_{\rm turn} = E \Leftrightarrow z_{\rm turn} = \frac{E}{m_g\, g} = \frac{\varepsilon}{\kappa} \Leftrightarrow \zeta = 0\,. \tag{32}$$



- Large $(-\zeta)$ - expansion of Airy function gives decomposition of ingoing and outgoing waves with phase delay of

$$\Delta\theta(z) = \frac{4}{3}\Big[\kappa\big(E/m_g\, g - z\big)\Big]^{3/2} - \pi/2 \tag{33}$$

corresponding to a "Peres time of flight" (Davies 2004)

$$T(z) := \hbar\frac{\partial\Delta\theta}{\partial E} = 2\,\frac{\hbar\kappa^{\frac{3}{2}}}{m_g\, g}\sqrt{z_{\rm turn} - z} = 2\sqrt{\frac{m_i}{m_g}}\cdot\sqrt{2\cdot\frac{z_{\rm turn} - z}{g}} \tag{34}$$

- For other than linear potential we will *not* get *classical* return time.

S & KG: Inertial motion

- Galilei symmetry is a suitable $1/c \to 0$ limit (contraction) of Poincaré symmetry. Likewise, the Schrödinger equation for ψ is a suitable $1/c \to 0$ limit of the Klein-Gordon equation for ϕ if we set

$$\phi(t,\vec{x}) = \exp\{-imc^2\, t/\hbar\}\, \psi(t,\vec{x})\,. \tag{35}$$

- The Klein-Gordon field transforms as scalar

$$\phi'(t',\vec{x}') = \phi(t,\vec{x})\,. \tag{36}$$

Hence (35) implies

$$\psi'(t',\vec{x}') = \exp\{-imc^2\,(t-t')/\hbar\}\, \psi(t,\vec{x})\,. \tag{37}$$

- Using

$$t = \frac{t' + \vec{x}'\cdot\vec{v}/c^2}{\sqrt{1-v^2/c^2}} = t' + c^{-2}\bigl(\vec{x}'\cdot\vec{v} + t'v^2/2\bigr) + \mathcal{O}(1/c^4)\,, \tag{38}$$

The $1/c \to 0$ limit of Poincare symmetry by proper representations turns into Galilei symmetry by non-trivial ray representations

$$\psi'(t',\vec{x}') = \exp\{-im(\vec{x}'\cdot\vec{v} + t'v^2/2)/\hbar\}\, \psi(t,\vec{x})\,. \tag{39}$$

S & KG: Rigid accelerations

- In Minkowski space, rigid motions in x-direction and of arbitrary acceleration of a body parametrised by ξ are given by family of timelike lines $\tau \mapsto \big(ct(\tau,\xi)\,,\,x(\tau,\xi)\big)$, where

$$ct(\tau,\xi) = c\int^{\tau} d\tau' \cosh\chi(\tau') + \xi\,\sinh\chi(\tau)$$
$$x(\tau,\xi) = c\int^{\tau} d\tau' \sinh\chi(\tau') \,+ \xi\,\cosh\chi(\tau)\,,$$

  Here τ is eigentime of body element $\xi = 0$ and $\chi(\tau) = \tanh^{-1}(v/c)$ is rapidity of all body elements at τ.

- The Minkowski metric in co-moving coordinates (τ,ξ) reads $(g := c\dot{\chi})$

$$ds^2 = c^2\,dt^2 - d\vec{x}^2 = \left(1 + \frac{g(\tau)\,\xi}{c^2}\right)\,c^2\,d\tau^2 - d\vec{\xi}^2\,. \qquad (40)$$

- Write down Klein-Gordon equation in co-moving coordinates

$$\left\{\Box_g + m^2\right\}\phi = \left\{(-\det g)^{-1/2}\;\partial_a\left[(-\det g)^{1/2}\;g^{ab}\partial_b\right] + m^2\right\}\phi = 0\,. \qquad (41)$$

S & KG: Rigid accelerations

- In analogy to (35) write

$$\phi(t,\vec{x}) = \exp\{-imc^2\,\tau/\hbar\}\,\psi(t,\vec{x}) \tag{42}$$

and take $1/c^2 \to 0$ limit

$$i\hbar\partial_\tau\psi = \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial\vec{\xi}^2} + mg(\tau)\xi\right)\psi\,. \tag{43}$$

This corresponds to particle in homogeneous but time-dependent gravitational field pointing in negative ξ-direction.

- Note that again ϕ transformed as scalar (compare (36))

$$\phi^{\mathrm{inert}}(t,\vec{x}) = \phi^{\mathrm{acc}}(\tau,\vec{\xi}) \tag{44}$$

but that again this is not true for ψ, where (compare (35))

$$\begin{aligned}\phi^{\mathrm{inert}}(t,\vec{x}) &= \exp\{-imc^2\,t/\hbar\}\,\psi^{\mathrm{inert}}(t,\vec{x})\\ \phi^{\mathrm{acc}}(\tau,\vec{\xi}) &= \exp\{-imc^2\,\tau/\hbar\}\,\psi^{\mathrm{acc}}(\tau,\vec{\xi})\,,\end{aligned} \tag{45}$$

- Hence (compare (37))

$$\psi^{\mathrm{acc}}(\tau,\vec{\xi}) = \exp\{-imc^2\,(t-\tau)/\hbar\}\;\psi^{\mathrm{inert}}(t,\vec{x})\,. \tag{46}$$

Schrödinger-Newton equation

- Consider Einstein – Klein-Gordon system

$$R_{ab} - \tfrac{1}{2} g_{ab} R = \tfrac{8\pi G}{c^4}\; T^{KG}_{ab}(\phi)\,, \quad (\Box_g + m^2)\phi = 0 \qquad (47)$$

- Make WKB-like ansatz

$$\phi(\vec{x},t) = \exp\left(\frac{ic^2}{\hbar} S(\vec{x},t)\right) \sum_{n=0}^{\infty} \left(\frac{\sqrt{\hbar}}{c}\right)^n a_n(\vec{x},t), \qquad (48)$$

and perform $1/c$ expansion (D.G. & A. Großardt 2012).

- Obtain

$$i\hbar\partial_t\psi = \left(-\frac{\hbar^2}{2m}\Delta + mV\right)\psi \qquad (49)$$

where

$$\Delta V = 4\pi G\left(\rho + m|\psi|^2\right). \qquad (50)$$

- Ignoring self-coupling, this just generalises previous results and conforms with expectations.

Schrödinger-Newton equation

- Without external sources get **"Schrödinger-Newton equation"** (Diosi 1984, Penrose 1998):

$$i\hbar\partial_t\psi(t,\vec{x}) = \left(-\frac{\hbar^2}{2m}\Delta - Gm^2\int\frac{|\psi(t,\vec{y})|^2}{\|\vec{x}-\vec{y}\|}\,d^3y\right)\psi(t,\vec{x}) \qquad (51)$$

- It can be derived from the action

$$\begin{aligned}\mathcal{S}[\psi,\psi^*] = \int dt\Bigg\{&\frac{i\hbar}{2}\int d^3x\Big(\psi^*(t,\vec{x})\dot{\psi}(t,\vec{x}) - \psi(t,\vec{x})\dot{\psi}^*(t,\vec{x})\Big)\\ &-\frac{\hbar^2}{2m}\int d^3x(\vec{\nabla}\psi(t,\vec{x}))\cdot(\vec{\nabla}\psi^*(t,\vec{x}))\\ &+\frac{Gm^2}{2}\iint d^3x\,d^3y\,\frac{|\psi(t,\vec{x})|^2\,|\psi(t,\vec{y})|^2}{\|\vec{x}-\vec{y}\|}\Bigg\}. \qquad (52)\end{aligned}$$

SNE: Dimensionless form

- Introducing a length-scale ℓ we can use dimensionless coordinates

$$\vec{x}' := \vec{x}/\ell\,, \qquad t' := t \cdot \frac{\hbar}{2m\ell}\,, \qquad \psi' = \ell^{3/2}\psi \tag{53}$$

and rewrite the SNE as

$$i\,\partial_{t'}\psi'(t',\vec{x}') = \left(-\Delta' - K\int \frac{|\psi'(t',\vec{y}')|^2}{\|\vec{x}'-\vec{y}'\|}\,d^3y'\right)\psi'(t',\vec{x}')\,, \tag{54}$$

with dimensionless coupling constant

$$K := 2\cdot\frac{Gm^3\ell}{\hbar^2} = 2\cdot\left(\frac{\ell}{\ell_P}\right)\left(\frac{m}{m_P}\right)^3 \approx 6\cdot\left(\frac{\ell}{100\,\mathrm{nm}}\right)\left(\frac{m}{10^{10}\,\mathrm{u}}\right)^3 \tag{55}$$

- Here we used Planck-length and Planck-mass

$$\ell_P := \sqrt{\frac{\hbar G}{c^3}} = 1.6\times 10^{-26}\,\mathrm{nm}\,, \quad m_P := \sqrt{\frac{\hbar c}{G}} = 1.3\times 10^{19}\,\mathrm{u}\,. \tag{56}$$

Symmetries and scaling properties of SNE

- The SNE has the same symmetries as ordinary Schrödinger equation: Full inhomogeneous Galilei group, including parity and time reversal, and global $U(1)$ phase transformations.
- Also it has the following scaling covariance: Let

$$S_\lambda[\psi](t,\vec{x}) := \lambda^{9/2}\psi(\lambda^5 t\,,\,\lambda^3\vec{x})\,, \tag{57}$$

then $S_\lambda[\psi]$ satisfies the SNE for mass parameter λm iff ψ satisfies SNE for mass parameter m

Collapse: Naive estimate

- Free Gaussian

$$\Psi_{\text{free}}(r,t) = (\pi a^2)^{-3/4} \left(1 + \frac{i\,\hbar\,t}{m\,a^2}\right)^{-3/2} \exp\left(-\frac{r^2}{2a^2\left(1+\frac{i\,\hbar\,t}{m\,a^2}\right)}\right). \tag{58}$$

- Radial probability density, $\rho(r,t) = 4\pi\, r^2\, |\Psi_{\text{free}}(r,t)|^2$, has a global maximum at

$$r_p = a\sqrt{1 + \frac{\hbar^2\, t^2}{m^2\, a^4}} \quad \Rightarrow \quad \ddot{r}_p = \frac{\hbar^2}{m^2\, r_p^3}\,. \tag{59}$$

- At time $t=0$ (say) this outward acceleration due to dispersion, $\ddot{r}_p = \frac{\hbar^2}{m^2\, a^3}$, equals gravitational inward acceleration $\frac{G\,m}{r^2}$ at time $t=0$ if (compare (55))

$$m^3\, a = m_P^3 \ell_p\,. \tag{60}$$

- For $a = 500\,\mathrm{nm}$ this yields a naive estimate for the threshold mass for collapse of about $4 \times 10^9\mathrm{u}$.

Stationary states: Analytical existence and numerical values

- Note that outward acceleration due to dispersion is $\propto r^{-3}$ and inward acceleration due to gravity $\propto r^{-2}$. Hence there will be no collapse to a δ-singularity.
- An analytic proof for the existence of a stable ground state has been given by E. Lieb in 1977 in the context of the Choquard equation for one-component plasmas, which is, however, formally identical.
- Tod et al. investigated bound states numerically and found the (unique) stable ground state at

$$E_0 = -0.163 \frac{G^2 m^5}{\hbar^2} = -0.163 \cdot mc^2 \cdot \left(\frac{m}{m_P}\right)^4 \tag{61}$$

Stationary states: Rough estimates

- A rough energy-estimate for the ground state is obtained, as usual, by setting

$$E \approx \frac{\hbar^2}{2ma^2} - \frac{Gm^2}{2a}\,. \tag{62}$$

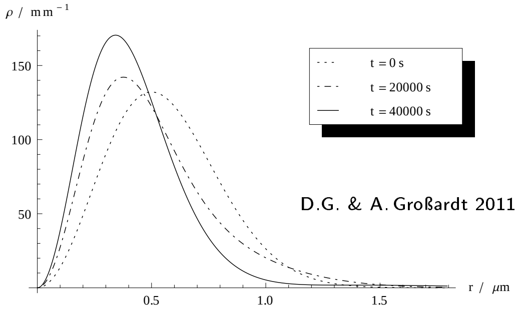
- Minimising in a then gives rough estimates for ground state

$$a_0 = \frac{2\hbar^2}{Gm^3} = 2\ell_P \cdot \left(\frac{m_p}{m}\right)^3\,, \qquad E_0 = -\frac{1}{8}\,\frac{G^2 m^5}{\hbar^2} \tag{63}$$

- Sanity check for applicability of Newtonian gravity (weak field approximation) is that diameter of mass distribution is much larger than its Schwarzschild radius

$$a_0 = \frac{2\hbar^2}{Gm^3} \gg \frac{2Gm}{c^2} \quad \Leftrightarrow \quad \left(\frac{m}{m_p}\right)^4 \ll 1 \tag{64}$$

The time-dependent SN-Equation-1



- Time evolution of rotationally symmetric Gauß packet of initial width $500\,\mathrm{nm}$. Collapse sets in for masses $m > 4 \times 10^9\,\mathrm{u}$, but collapse times are of many hours (recall scaling laws, though).
- This is a 10^6 correction to earlier simulations by Carlip and Salzman (2006).

The time-dependent SN-Equation-2

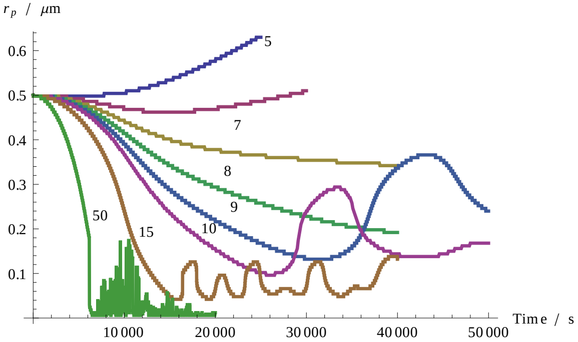


Figure: Time evolution of peak of radial probability density for increasing masses. First bounces back from minimal contraction are seen within shown interval of time above masses of 9×10^{9} u. (D.G. and A. Großardt 2011)

The time-dependent SN-Equation-3

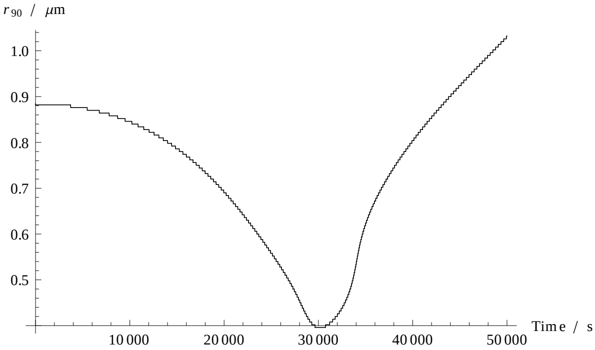


Figure: Radius r_{90} within which 90% of the probability is located as function of time for $m = 10^{10}\,\mathrm{u}$. Note that the minimum is *not* zero but around $r_{90} \approx 0.4\mu\mathrm{m}$.(D.G. and A. Großardt 2011)

The time-dependent SN-Equation-4

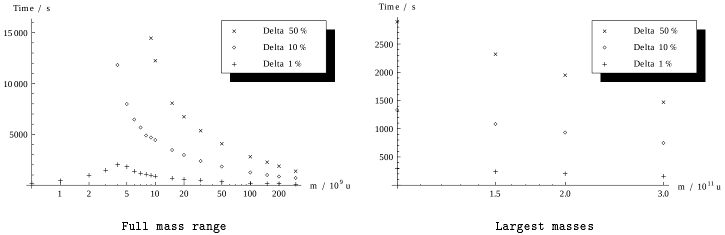


Figure: Time it takes until the gravitationally interacting solution differs from the solution of the free Schrödinger equation in its *full width at half maximum* (FWHM) by a percentage of 1 %, 10 %, and 50 % respectively.

Modifications of SNE

- SNE is of form

$$i\hbar\partial_t\psi = \left(-\frac{\hbar^2}{2m}\Delta + \left(\phi \star |\psi|^2(t,\vec{x})\right)\right)\psi(t,\vec{x}) \tag{65}$$

where

$$\phi \star |\psi|^2(t,\vec{x}) = -Gm^2 \int \frac{|\psi(t,\vec{x})|^2}{\|\vec{x}-\vec{y}\|}\, d^3y \tag{66}$$

i.e.

$$\phi(\vec{x}) = -\frac{Gm^2}{r}\,. \tag{67}$$

- Modifications of SNE have been suggested where ϕ is the potential of extended charge, like for homogeneous solid sphere of radius R (Jääskeläinen 2012)

$$\phi(r) = \begin{cases} -\frac{Gm^2}{R}\left(\frac{3}{2} - \frac{r^2}{2R^2}\right) & \text{for } r < R \\ -\frac{Gm^2}{r} & \text{for } r \geq R \end{cases} \tag{68}$$

- This equation can be derived for the centre-of-mass wavefunction of an N-particle system obeying the original n-particle SNE of Diosi (1984).

The N-particle SNE

- Principle: *Each particle is under the influence of the Newtonian gravitational potential that is sourced by an active gravitational mass-density to which each particle contributes proportional to its probability density in position space as given by the marginal distribution of the total wave function.*
- Hence

$$\rho(\vec{x}) = \sum_{j=1}^{N} m_j P_j(\vec{x}) = \sum_{j=1}^{N} m_j \int |\Psi_N(\vec{y}_1, \cdots, \vec{y}_N)|^2 \, \delta^{(3)}(\vec{y}_j - \vec{x}) \, d^{3N} y \tag{69}$$

giving rise to the gravitational potential

$$\begin{aligned} U_G\left[\Psi(\vec{y}_1, \cdots, \vec{y}_N)\right] &= -G \sum_{i=1}^{N} \int \frac{m_i \rho(\vec{x})}{\|\vec{y}_i - \vec{x}\|} \, d^3x \\ &= -G \sum_{i=1}^{N} \sum_{j=1}^{N} \int \frac{m_i m_j P_j(\vec{x})}{\|\vec{y}_i - \vec{x}\|} \, d^3x \end{aligned} \tag{70}$$

- Note that the mutual gravitational self interaction includes self interaction. Also, the interaction is not local. This differs from what we are used to in electrodynamics and gives rise to contribution to centre-of-mass motion (D.G. & A. Großardt 2013).

Collapse for modified SNE

- Modifications of collapse behaviour as compared to ordinary SNE occur once the width of the wave packet exceeds the size of the matter distribution.
- Even for lithium ($\rho = 534\,\mathrm{kg}\cdot\mathrm{m}^{-3}$) the radius of a spherical homogeneous mass of $10^{10}\,\mathrm{u}$ is of about $200\,\mathrm{nm}$. Hence no significant changes occur for the alterations of the SNE suggested by Jääskeläinen (2012).
- The same holds for hollow spheres. In fact, somewhat unexpectedly, the modified SNE can lead to faster collapse (smaller collapse masses for given width).
- This can be understood in terms of the energy functional

$$E = \frac{\hbar^2}{2m}\int d^3x\,|\vec{\nabla}\psi(t,\vec{x})|^2 + \frac{1}{2}\int d^3x|\psi(t,\vec{x})|^2\,(\phi\star|\psi|^2)(t,\vec{x}) \quad (71)$$

Collapse occurs if negative potential energy exceeds positive kinetic energy. This can be related to sign of second time-derivative of second momentum of distribution $|\psi(t,\vec{x})|^2$.

Collapse in ball- and sphere-potential

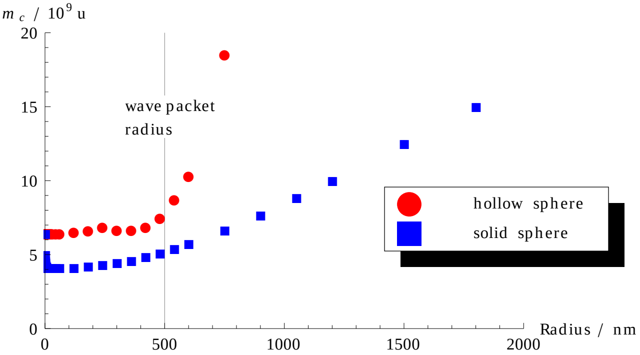


Figure: The critical mass beyond which the wave packet collapses is plotted against the radius of the hollow (red dots) and solid (blue squares) sphere in the potential term. (D.G. and A. Großardt 2013)

Schrödinger 1927

Der Energieimpulssatz der Materiewellen;
von E. Schrödinger

. . . . . .

Fragt man sich nun, ob diese in sich geschlossene Feldtheorie — von der vorläufigen Nichtberücksichtigung des Elektronendralles abgesehen — der Wirklichkeit entspricht in der Art, wie man das früher von dergleichen Theorien erhofft hatte, so ist die Frage zu *verneinen.* Die durchgerechneten Beispiele, vor allem das H-Atom, zeigen nämlich, daß man in die Wellengleichung (1) *nicht* diejenigen Potentiale einzusetzen hat, welche sich aus den Potentialgleichungen (15′) mit dem Viererstrom (9) ergeben. Vielmehr hat man bekanntlich beim H-Atom in (1) für die φ_α die *vorgegebenen* Potentiale des Kerns und eventueller „äußerer" elektromagnetischer Felder einzutragen und die Gleichung nach ψ aufzulösen. Aus (9) berechnet sich dann die von diesem ψ „erzeugte" Stromverteilung, aus ihr nach (15′) die von ihr erzeugten Potentiale. Diese ergeben dann, zu den vorgegebenen Potentialen hinzugefügt, diejenigen Potentiale, mit denen das Atom als ganzes nach außen wirkt. Man

. . . . . .

Gerade die *Geschlossenheit* der Feldgleichungen erscheint somit in eigenartiger Weise durchbrochen. Man kann das heute wohl noch nicht ganz verstehen, hat es aber mit folgenden zwei Dingen in Zusammenhang zu bringen.

. . . . . .

Ob die Lösung der Schwierigkeit wirklich nur in der von einigen Seiten[2]) vorgeschlagenen bloß *statistischen* Auffassung der Feldtheorie zu suchen ist, müssen wir wohl vorläufig dahingestellt sein lassen. Mir persönlich erscheint diese Auffassung heute nicht mehr[1]) endgültig befriedigend, selbst wenn sie sich praktisch brauchbar erweist. Sie scheint mir einen allzu prinzipiellen Verzicht auf das Verständnis des Einzelvorgangs zu bedeuten.

- Schrödinger "closes" the set of Schrödinger-Maxwell equations by letting ψ source the electromagnetic potentials to which ψ couples, thereby introducing non-linearities, similar to radiation-reaction in the classical theory.
- He asserts that "computations" for the H-atom lead to discrepancies which refute such a self-coupling.
- He wonders why in Quantum Mechanics the closedness of the system of field equations is violated in such a peculiar fashion ("in eigenartiger Weise durchbrochen") and comments of possible impact of probability interpretation on classical concepts of local exchange of energy and momentum.

Summary

- On the level of differential equations, SNE can be derived from Einstein-Klein-Gordon or Einstein-Dirac system.
- Describes inhibitions of dispersion for certain mass ranges and widths, like above $6.5 \times 10^9\,\mathrm{u}$ and $500\,\mathrm{nm}$, followed by oscillatory behaviour with radiation and settlement to ground state.
- Scaling law
$$S_\lambda[\psi](t,\vec{x}) := \lambda^{9/2}\psi(\lambda^5 t\,,\,\lambda^3\vec{x})\,, \tag{72}$$
shows: Tenfold mass and 10^{-3} width results in 10^{-5} collapse time.
- Centre-of-mass SNE with modified kernel can be obtained from Diosi's many-particle SNE. Modified kernel leads to same or even more favourable results as long as support diameter of mass distribution does not exceed the width of the centre-of-mass wave function.
- All this *ignores* the possible quantum nature of the gravitational field.

THANK YOU!

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