Properties and consequences of the Schrödinger-Newton Equation

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"Gravitational decoherence"

Motivation
- Carlip 2006
- Schrödinger
- Penrose

What's known
- COW & Co.
- EEP
- uff a theorem

SNE
- as non-rel. limit
- dimensionless
- symmetries
- collapse
- stationary states
- generalisation
- multi particle
- separation
- approximation
- consequences

Summary

Supplementary
- UFF and UGR
- KG inertial
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IOP PUBLISHING CLASSICAL AND QUANTUM GRAVITY

Class. Quantum Grav. **25** (2008) 154010 (6pp) doi:10.1088/0264-9381/25/15/154010

Is quantum gravity necessary?

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Abstract
In view of the enormous difficulties we seem to face in quantizing general relativity, we should perhaps consider the possibility that gravity is a fundamentally classical interaction. Theoretical arguments against such mixed classical–quantum models are strong, but not conclusive, and the question is ultimately one for experiment. I review some work in progress on the possibility of experimental tests, exploiting the nonlinearity of the classical–quantum coupling, which could help settle this question.

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Schrödinger 1927

Der Energieimpulssatz der Materiewellen;
von E. Schrödinger

. . . . . .

Fragt man sich nun, ob diese in sich geschlossene Feldtheorie — von der vorläufigen Nichtberücksichtigung des Elektronendralles abgesehen — der Wirklichkeit entspricht in der Art, wie man das früher von dergleichen Theorien erhofft hatte, so ist die Frage zu *verneinen.* Die durchgerechneten Beispiele, vor allem das H-Atom, zeigen nämlich, daß man in die Wellengleichung (1) *nicht* diejenigen Potentiale einzusetzen hat, welche sich aus den Potentialgleichungen (15′) mit dem Viererstrom (9) ergeben. Vielmehr hat man bekanntlich beim H-Atom in (1) für die φ_α die *vorgegebenen* Potentiale des Kerns und eventueller „äußerer“ elektromagnetischer Felder einzutragen und die Gleichung nach ψ aufzulösen. Aus (9) berechnet sich dann die von diesem ψ „erzeugte“ Stromverteilung, aus ihr nach (15′) die von ihr erzeugten Potentiale. Diese ergeben dann, zu den vorgegebenen Potentialen hinzugefügt, diejenigen Potentiale, mit denen das Atom als ganzes nach außen wirkt. Man

. . . . . .

Gerade die *Geschlossenheit* der Feldgleichungen erscheint somit in eigenartiger Weise durchbrochen. Man kann das heute wohl noch nicht ganz verstehen, hat es aber mit folgenden zwei Dingen in Zusammenhang zu bringen.

. . . . . .

Ob die Lösung der Schwierigkeit wirklich nur in der von einigen Seiten[2]) vorgeschlagenen bloß *statistischen* Auffassung der Feldtheorie zu suchen ist, müssen wir wohl vorläufig dahingestellt sein lassen. Mir persönlich erscheint diese Auffassung heute nicht mehr[1]) endgültig befriedigend, selbst wenn sie sich praktisch brauchbar erweist. Sie scheint mir einen allzu prinzipiellen Verzicht auf das Verständnis des Einzelvorgangs zu bedeuten.

- Schrödinger “closes” the set of Schrödinger-Maxwell equations by letting ψ source the electromagnetic potentials to which ψ couples, thereby introducing non-linearities, similar to radiation-reaction in the classical theory.
- He asserts that “computations” for the H-atom lead to discrepancies which refute such a self-coupling.
- He wonders why in Quantum Mechanics the closedness of the system of field equations is violated in such a peculiar fashion (“in eigenartiger Weise durchbrochen”) and comments of possible impact of probability interpretation on classical concepts of local exchange of energy and momentum.

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- dimensionless
- symmetries
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R. Penrose (Found. Phys. Jan. 2014)

"So why give quantum theory pride of place in this proposed union?"

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SNE
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- symmetries
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- stationary states
- generalisation
- multi particle
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- approximation
- consequences

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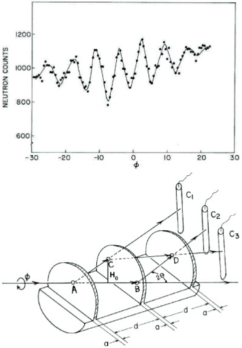
- UFF and UGR
- KG inertial
- KG accelerating
- collapses

QM & Gravity: Tested so far



Colella Overhauser Werner, PRL 1975

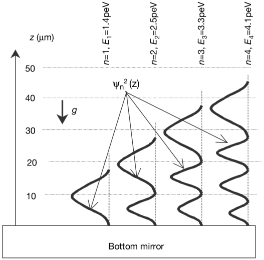


Figure 1 Wavefunctions of the quantum states of neutrons in the potential well formed by the Earth's gravitational field and the horizontal mirror. The probability of finding neutrons at height z, corresponding to the nth quantum state, is proportional to the square of the neutron wavefunction $\psi_n^2(z)$. The vertical axis z provides the length scale for this phenomenon. E_n is the energy of the nth quantum state.

Nesvizhevsky et al., Nature 2002

$$i\hbar\dot{\Psi} = -\frac{\hbar^2}{2m_i}\Delta\Psi + V_{\rm grav}\Psi$$

$$V_{\rm grav} = m_g g z$$

How do you derive this from first principles?

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- stationary states
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- separation
- approximation
- consequences
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Supplementary
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- KG inertial
- KG accelerating
- collapses

Einstein's Equivalence Principle (EEP)

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What's known
- COW & Co.
- EEP
- uff a theorem

SNE
- as non-rel. limit
- dimensionless
- symmetries
- collapse
- stationary states
- generalisation
- multi particle
- separation
- approximation
- consequences

Summary

Supplementary
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- collapses

- **Universality of Free Fall (UFF):** "Test bodies" determine path structure on spacetime (not necessarily of Riemannian type). UFF-violations are parametrised by the Eötvös factor

$$\eta(A,B) := 2\,\frac{|a(A)-a(B)|}{|a(A)+a(B)|} \tag{1}$$

- **Local Lorentz Invariance (LLI):** Local non-gravitational experiments exhibit no preferred directions in spacetime, neither timelike nor spacelike. Possible violations of LLI concern, e.g., variations in $\Delta c/c$.

- **Universality of Gravitational Redshift (UGR):** "Standard clocks" are universally affected by the gravitational field. UGR-violations are parametrised by the α-factor

$$\frac{\Delta\nu}{\nu} = (1+\alpha)\frac{\Delta U}{c^2} \tag{2}$$

Consequences and difficulties of the equivalence principle

- Gravity can be geometrised and hence ceases to be a force (in the Newtonian sense). This only works if *all* dynamical aspects of gravity can be encoded in space-time geometry and if *all* matter components see the *same* geometry to which they *universally* couple.

- This universal coupling scheme translates to special-relativistic (Poincaré invariant) field theories, but not in an obvious fashion to “non-relativistic” (Galilei invariant) Quantum Mechanics.

- Three approaches are followed in the literature:
  1. Redo “Schrödinger Quantisation” for relativistic particles in curved spacetime in a post-newtonian expansion (thus also taking account of vector- and tensor parts of Einsteinian g-field).
  2. Derive post-newtonian expansions of relativistic field equations (Klein-Gordan, Dirac, etc.).
  3. Start from QFT in curved spacetime.

- Unless all this is understood much better, there is no obvious meaning to “Quantum tests of *the* equivalence principle. The many confusions in recent years on various claims concerning such “quantum-tests” reflect the variation of such meanings and the absence of hard criteria to compare them.

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- uff a theorem
SNE
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- dimensionless
- symmetries
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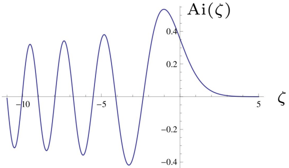
Homogeneous static gravitational field: Bound states

- Time independent Schrödinger equation in linear potential $V(z) = m_g g z$ is equivalent to:

$$\left(\frac{d^2}{d\zeta^2} - \zeta\right)\psi = 0\,, \quad \zeta := \kappa z - \varepsilon \tag{3}$$

where

$$\kappa := \left[\frac{2m_i\, m_g\, g}{\hbar^2}\right]^{\frac{1}{3}}\,, \quad \varepsilon := E\cdot\left[\frac{2m_i}{m_g^2\, g^2\, \hbar^2}\right]^{\frac{1}{3}} \tag{4}$$



- Complement by hard (horizontal) wall $V(z) = \infty$ for $z \le 0$ get energy eigenstates from boundary condition $\psi(z=0) = 0$, hence $\varepsilon = -z_n$:

$$E(n) = -z_n\left[\frac{m_g^2}{m_i}\cdot\frac{g^2\hbar^2}{2}\right]^{\frac{1}{3}} \tag{5}$$

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- EEP
- uff a theorem

SNE
- as non-rel. limit
- dimensionless
- symmetries
- collapse
- stationary states
- generalisation
- multi particle
- separation
- approximation
- consequences

Summary

Supplementary
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- KG accelerating
- collapses

UFF in QM

The following proposition states precisely the extent to which UFF is valid within QM.

- We consider a particle of mass m in spatially homogeneous force field $\vec{F}(t)$. The classical trajectories solve

$$\ddot{\vec{\xi}}(t) = \vec{F}(t)/m\,. \tag{6}$$

Let $\xi(t)$ denote a solution with $\vec{\xi}(0) = \vec{0}$ and some initial velocity. Its flow-map $\Phi : \mathbb{R}^4 \to \mathbb{R}^4$ defines a *freely-falling frame*:

$$\Phi(t, \vec{x}) = \big(t, \vec{x} + \xi(t)\big)\,. \tag{7}$$

- **Proposition:** ψ solves the forced Schrödinger equation

$$i\hbar\partial_t\psi = \left(-\frac{\hbar^2}{2m_i}\Delta - \vec{F}(t)\cdot\vec{x}\right)\psi \tag{8}$$

iff

$$\psi = \big(\exp(i\alpha)\,\psi'\big)\circ\Phi^{-1}\,, \tag{9}$$

where ψ' solves the free Schrödinger equation and

$$\alpha(t,\vec{x}) = \frac{m_i}{\hbar}\left\{\dot{\vec{\xi}}(t)\cdot\big(\vec{x}+\vec{\xi}(t)\big) - \frac{1}{2}\int^t dt'\,\|\dot{\vec{\xi}}(t')\|^2\right\}\,. \tag{10}$$

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- stationary states
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Schrödinger-Newton equation

- Consider, e.g., Einstein – Klein-Gordon system

$$R_{ab} - \tfrac{1}{2} g_{ab} R = \tfrac{8\pi G}{c^4}\, T^{KG}_{ab}(\phi)\,, \quad (\Box_g + m^2)\phi = 0\,. \tag{11}$$

- Make WKB-like ansatz

$$\phi(\vec{x},t) = \exp\left(\frac{ic^2}{\hbar} S(\vec{x},t)\right) \sum_{n=0}^{\infty} \left(\frac{\sqrt{\hbar}}{c}\right)^n a_n(\vec{x},t), \tag{12}$$

  and perform $1/c$ expansion (D.G. & A. Großardt 2012).
- Obtain

$$i\hbar\partial_t\psi = \left(-\frac{\hbar^2}{2m}\Delta + mV\right)\,\psi \tag{13}$$

  where

$$\Delta V = 4\pi G\big(\rho + m|\psi|^2\big)\,. \tag{14}$$

- Ignoring self-coupling, this just generalises previous results and conforms with expectations.

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- uff a theorem

SNE
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- symmetries
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- stationary states
- generalisation
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- separation
- approximation
- consequences

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- KG accelerating
- collapses

Schrödinger-Newton equation

- Without external sources get **"Schrödinger-Newton equation"** (Diósi 1984, Penrose 1998):

$$\boxed{i\hbar\,\partial_t\psi(t,\vec{x}) = \left(-\frac{\hbar^2}{2m}\Delta - Gm^2\int\frac{|\psi(t,\vec{y})|^2}{\|\vec{x}-\vec{y}\|}\,d^3y\right)\psi(t,\vec{x})} \qquad (15)$$

- It can be derived from the action

$$\begin{aligned}\mathcal{S}[\psi,\psi^*] = \int dt\biggl\{\frac{i\hbar}{2}\int d^3x\Bigl(\psi^*(t,\vec{x})\dot{\psi}(t,\vec{x}) - \psi(t,\vec{x})\dot{\psi}^*(t,\vec{x})\Bigr)\\ -\frac{\hbar^2}{2m}\int d^3x\bigl(\vec{\nabla}\psi(t,\vec{x})\bigr)\cdot\bigl(\vec{\nabla}\psi^*(t,\vec{x})\bigr)\\ +\frac{Gm^2}{2}\iint d^3x\,d^3y\,\frac{|\psi(t,\vec{x})|^2\,|\psi(t,\vec{y})|^2}{\|\vec{x}-\vec{y}\|}\biggr\}.\end{aligned} \qquad (16)$$

- Alternative local form through introduction of gravitational potential $\Phi(t,\vec{x})$.

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- EEP
- uff a theorem

SNE
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- dimensionless
- symmetries
- collapse
- stationary states
- generalisation
- multi particle
- separation
- approximation
- consequences

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- collapses

SNE: Dimensionless form

- Introducing a length-scale ℓ we can use dimensionless coordinates

$$\vec{x}' := \vec{x}/\ell\,, \qquad t' := t \cdot \frac{\hbar}{2m\ell}\,, \qquad \psi' = \ell^{3/2}\psi \tag{17}$$

and rewrite the SNE as

$$i\,\partial_{t'}\psi'(t',\vec{x}') = \left(-\Delta' - K \int \frac{|\psi'(t',\vec{y}')|^2}{\|\vec{x}' - \vec{y}'\|}\, d^3y'\right)\psi'(t',\vec{x}')\,, \tag{18}$$

with dimensionless coupling constant

$$K := 2 \cdot \frac{Gm^3\ell}{\hbar^2} = 2 \cdot \left(\frac{\ell}{\ell_P}\right)\left(\frac{m}{m_P}\right)^3 \approx 6 \cdot \left(\frac{\ell}{100\,\mathrm{nm}}\right)\left(\frac{m}{10^{10}\,\mathrm{u}}\right)^3 \tag{19}$$

- Here we used Planck-length and Planck-mass

$$\ell_P := \sqrt{\frac{\hbar G}{c^3}} = 1.6 \times 10^{-26}\,\mathrm{nm}\,, \quad m_P := \sqrt{\frac{\hbar c}{G}} = 1.3 \times 10^{19}\,\mathrm{u}\,. \tag{20}$$

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- Penrose

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- COW & Co.
- EEP
- uff a theorem

SNE
- as non-rel. limit
- dimensionless
- symmetries
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- stationary states
- generalisation
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- approximation
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Supplementary
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Symmetries and scaling properties of SNE

The SNE has all the symmetries as ordinary Schrödinger equation:

- Proper orthochronous Galilei transformations

$$(t, \vec{x}) \to \tau_g(t, \vec{x}) := (t + b\,,\, \mathbf{R} \cdot \vec{x} + \vec{v}t + \vec{a})\,, \tag{21}$$

  acting via proper ray-representations

$$\Psi \to T_g \Psi := \exp(i\beta_g)\big(\Psi \circ \tau_{g^{-1}}\big) \tag{22}$$

  with multiplier-phases

$$\beta_g(t, \vec{x}) = \frac{m}{\hbar}\Big[\vec{v} \cdot (\vec{x} - \vec{a}) - \tfrac{1}{2}\vec{v}^2(t - b)\Big]\,, \tag{23}$$

  extended by parity and anti-linear time-reversal transformations.
- Global $U(1)$ phase transformations.
- Scaling *covariance*: Let

$$S_\lambda[\psi](t, \vec{x}) := \lambda^{9/2}\psi(\lambda^5 t\,,\, \lambda^3 \vec{x})\,, \tag{24}$$

  then $S_\lambda[\psi]$ satisfies the SNE for mass parameter λm iff ψ satisfies SNE for mass parameter m.

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- Carlip 2006
- Schrödinger
- Penrose
What's known
- COW & Co.
- EEP
- uff a theorem
SNE
- as non-rel. limit
- dimensionless
- symmetries
- collapse
- stationary states
- generalisation
- multi particle
- separation
- approximation
- consequences
Summary
Supplementary
- UFF and UGR
- KG inertial
- KG accelerating
- collapses

Collapse: Naive estimate

- Free Gaussian

$$\Psi_{\text{free}}(r,t) = \left(\pi a^2\right)^{-3/4}\left(1+\frac{i\,\hbar\,t}{m\,a^2}\right)^{-3/2}\exp\left(-\frac{r^2}{2a^2\left(1+\frac{i\,\hbar\,t}{m\,a^2}\right)}\right). \tag{25}$$

- Radial probability density, $\rho(r,t) = 4\pi\, r^2\, |\Psi_{\text{free}}(r,t)|^2$, has a global maximum at

$$r_p = a\sqrt{1+\frac{\hbar^2 t^2}{m^2 a^4}} \quad\Rightarrow\quad \ddot{r}_p = \frac{\hbar^2}{m^2\, r_p^3}\,. \tag{26}$$

- At time $t = 0$ (say) this outward acceleration due to dispersion, $\ddot{r}_p = \frac{\hbar^2}{m^2\, r_p^3}$, equals gravitational inward acceleration $\frac{G\,m}{r_p^2}$ at time $t = 0$ if (compare (19))

$$m^3 a = m_P^3 \ell_P. \tag{27}$$

- For $a = 500\,\mathrm{nm}$ this yields a naive estimate for the threshold mass for collapse of about $4\times 10^9\mathrm{u}$.

SNE

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- COW & Co.
- EEP
- uff a theorem

SNE
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- dimensionless
- symmetries
- collapse
- stationary states
- generalisation
- multi particle
- separation
- approximation
- consequences

Summary

Supplementary
- UFF and UGR
- KG inertial
- KG accelerating
- collapses

Stationary states: Analytical existence and numerical values

- Note that outward acceleration due to dispersion is $\propto r^{-3}$ and inward acceleration due to gravity $\propto r^{-2}$. Hence there will be no collapse to a δ-singularity.
- An analytic proof for the existence of a stable ground state has been given by E. Lieb in 1977 in the context of the Choquard equation for one-component plasmas, which is, however, formally identical.
- Tod et al. investigated bound states numerically and found the (unique) stable ground state at Energy E_0 and width a_0, given by

$$\begin{aligned} E_0 &= -0.163\,\frac{G^2 m^5}{\hbar^2} = -0.163 \cdot mc^2 \cdot \left(\frac{m}{m_P}\right)^4 \\ &\approx -mc^2 \cdot 10^{-36}\, m^4[10^{10}\,u]\,, \end{aligned} \tag{28a}$$

$$\begin{aligned} a_0 &= \frac{2\hbar^2}{Gm^3} = 6 \cdot 10^6\,\mathrm{ly} \cdot m^{-3}[u] \\ &\approx 10^{-6}\,\mathrm{cm} \cdot m^{-3}[10^{10}\,u]\,. \end{aligned} \tag{28b}$$

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- Penrose
What's known
- COW & Co.
- EEP
- uff a theorem
SNE
- as non-rel. limit
- dimensionless
- symmetries
- collapse
- stationary states
- generalisation
- multi particle
- separation
- approximation
- consequences
Summary
Supplementary
- UFF and UGR
- KG inertial
- KG accelerating
- collapses

Stationary states: Rough estimates

- A rough energy-estimate for the ground state is obtained, as usual, by setting
$$E \approx \frac{\hbar^2}{2ma^2} - \frac{Gm^2}{2a}\,. \tag{29}$$

- Minimising in a then gives rough estimates for ground state
$$a_0 = \frac{2\hbar^2}{Gm^3} = 2\ell_P \cdot \left(\frac{m_p}{m}\right)^3\,, \qquad E_0 = -\frac{1}{8}\frac{G^2m^5}{\hbar^2}\,. \tag{30}$$

- Sanity check for applicability of Newtonian gravity (weak field approximation) is that diameter of mass distribution is much larger than its Schwarzschild radius
$$a_0 = \frac{2\hbar^2}{Gm^3} \gg \frac{2Gm}{c^2} \quad \Leftrightarrow \quad \left(\frac{m}{m_p}\right)^4 \ll 1 \tag{31}$$

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- uff a theorem

SNE
- as non-rel. limit
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- consequences

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Supplementary
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General SNE

- SNE is of form

$$i\hbar\partial_t\psi = \left(-\frac{\hbar^2}{2m}\Delta + \left(\phi\star|\psi|^2(t,\vec{x})\right)\right)\psi(t,\vec{x}) \tag{32}$$

where

$$\phi\star|\psi|^2(t,\vec{x}) = -Gm^2\int\frac{|\psi(t,\vec{x})|^2}{\|\vec{x}-\vec{y}\|}\,d^3y \tag{33}$$

i.e.

$$\phi(\vec{x}) = -\frac{Gm^2}{r}\,. \tag{34}$$

- Equation (32) is still valid with modified ϕ for separated centre-of-mass wave-function. For example, for homogeneous spherically-symmetric matter distribution get

$$\phi(r) = \begin{cases} -\frac{Gm^2}{R}\left(\frac{3}{2}-\frac{r^2}{2R^2}\right) & \text{for } r < R \\ -\frac{Gm^2}{r} & \text{for } r \geq R \end{cases} \tag{35}$$

- **This equation can be derived for the centre-of-mass wavefunction of an N-particle system obeying the original n-particle SNE of Diósi (1984).**

The N-particle SNE

Principle: *Each particle is under the influence of the Newtonian gravitational potential that is sourced by an active gravitational mass-density to which each particle contributes proportional to its probability density in position space as given by the marginal distribution of the total wave function.*

- Hence

$$\rho(t;\vec{x}) = \sum_{j=0}^{N} m_j P_j(t;\vec{x}) = \sum_{j=0}^{N} m_j \int |\Psi_N(t;\vec{y}_1,\cdots,\vec{y}_N)|^2 \, \delta^{(3)}(\vec{y}_j - \vec{x}) \, d^{3N}y \tag{36}$$

giving rise to the gravitational potential

$$\begin{aligned} U_G(t;\vec{y}_1,\cdots,\vec{y}_N) &= -G \sum_{i=0}^{N} \int \frac{m_i \rho(t;\vec{x})}{\|\vec{y}_i - \vec{x}\|} \, d^3x \\ &= -G \sum_{i=0}^{N} \sum_{j=0}^{N} \int \frac{m_i m_j P_j(t;\vec{x})}{\|\vec{y}_i - \vec{x}\|} \, d^3x \end{aligned} \tag{37}$$

- Note that the mutual gravitational interaction is not local and includes self interaction, in contrast to what we usually assume in electrodynamics. It is this difference that implies modifications of the dynamics for the centre-of-mass wavefunction. These modifiations are like for the 1-particle SNE if the width of the wave function is large compared to the support of the matter distribution (D.G. & A. Großardt 2014).

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- generalisation
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Supplementary
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- collapses

Separation

- Using instead of $\{\vec{x}_i \mid i = 0, 1, \cdots N\}$ centre-of-mass $\vec{c}$ and relative co-ordinates $\{\vec{r}_\alpha \mid \alpha = 1, \cdots N\}$ (thereby distinguishing the 0-th particle),

$$\vec{c} := \frac{1}{M}\sum_{a=0}^{N} m_a\,\vec{x}_a = \frac{m_0}{M}\vec{x}_0 + \sum_{\beta=1}^{N}\frac{m_\beta}{M}\vec{x}_\beta\,, \tag{38a}$$

$$\vec{r}_\alpha := \vec{x}_\alpha - \vec{c} = -\frac{m_0}{M}\,\vec{x}_0 + \sum_{\beta=1}^{N}\left(\delta_{\alpha\beta} - \frac{m_\beta}{M}\right)\vec{x}_\beta \tag{38b}$$

- Get in large N limit with $\Psi(\vec{x}_0, \cdots \vec{x}_N) = \psi(\vec{c})\chi(\vec{r}_1, \cdots \vec{r}_N)$

$$U_G(t; \vec{c}, \vec{r}_1, \cdots, \vec{r}_N) = -G\sum_{\alpha=1}^{N} m_\alpha \int d^3\vec{c}\,' \int d^3\vec{r}\,' \frac{|\psi(t;\vec{c}\,')|^2\rho_c(\vec{r}\,')}{\|\vec{c} - \vec{c}\,' + \vec{r}_\alpha - \vec{r}\,'\|}\,, \tag{39}$$

where

$$\rho_c(t;\vec{r}) := \sum_{\beta=1}^{N} m_\beta \left\{\int \prod_{\substack{\gamma=1\\ \gamma\neq\beta}}^{N} d^3\vec{r}_\gamma\right\} |\chi(t;\vec{r}_1, \cdots, \vec{r}_{\beta-1}, \vec{r}, \vec{r}_{\beta+1}, \cdots, \vec{r}_N)|^2\,. \tag{40}$$

Motivation
- Carlip 2006
- Schrödinger
- Penrose

What's known
- COW & Co.
- EEP
- uff a theorem

SNE
- as non-rel. limit
- dimensionless
- symmetries
- collapse
- stationary states
- generalisation
- multi particle
- separation
- approximation
- consequences

Summary

Supplementary
- UFF and UGR
- KG inertial
- KG accelerating
- collapses

Approximation

- For a separation into centre-of-mass and relative motion we wish to get rid of $\vec{r}_\alpha$-dependence in (39).
- This can, e.g., be achieved by assuming the width of the c.o.m wave function to be much larger than diameter of mass distribution. Then,

$$\begin{aligned} U_G &= -G\sum_{\alpha=1}^{N} m_\alpha \int d^3\vec{c}' \int d^3\vec{r}' \frac{|\psi(t;\vec{c}')|^2 \rho_c(\vec{r}')}{\|\vec{c}-\vec{c}'+\vec{r}_\alpha-\vec{r}'\|} \\ &\approx -GM\int d^3\vec{c}' \int d^3\vec{r}' \frac{|\psi(t;\vec{c}')|^2 \rho_c(\vec{r}')}{\|\vec{c}-\vec{c}'-\vec{r}'\|} = U_G(t;\vec{c}) \end{aligned} \tag{41}$$

- Alternatively one may apply a Born-Oppenheimer approximation that consists of replacing U_G with its expectation-value in the state χ for the relative motion:

$$\begin{aligned} U_G &= -G\sum_{\alpha=1}^{N} m_\alpha \int d^3\vec{c}' \int d^3\vec{r}' \frac{|\psi(t;\vec{c}')|^2 \rho_c(\vec{r}')}{\|\vec{c}-\vec{c}'+\vec{r}_\alpha-\vec{r}'\|} \\ &\approx -G\int d^3\vec{c}' \int d^3\vec{r}' \int d^3\vec{r}'' \frac{|\psi(t;\vec{c}')|^2 \rho_c(\vec{r}')\rho_c(\vec{r}'')}{\|\vec{c}-\vec{c}'-\vec{r}'+\vec{r}''\|} \\ &= U_G(t;\vec{c}) \end{aligned} \tag{42}$$

$\Rightarrow$ Both cases result in SNE for c.o.m in the form (32) with $\phi = U_G(t;\vec{c})$.

Motivation
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- Schrödinger
- Penrose
What's known
- COW & Co.
- EEP
- uff a theorem
SNE
- as non-rel. limit
- dimensionless
- symmetries
- collapse
- stationary states
- generalisation
- multi particle
- separation
- approximation
- consequences
Summary
Supplementary
- UFF and UGR
- KG inertial
- KG accelerating
- collapses

Consequences

- For wide c.o.m - wave functions SNE leads to inhibitions of qm-dispersion, as discussed before. Typical collapse times for widths of $500\,\mathrm{nm}$ and masses about $10^10\,\mathrm{amu}$ are of the order of hours. However, by scaling law (24), this reduces by factor 10^5 for tenfold mass and 10^{-3} fold width.
- For narrow c.o.m. - wave functions in Born-Oppenheimer scheme one obtains an effective self-interaction in c.o.m. SNE of

$$U_G(t;\vec{c}) \approx I_{\rho_c}(\vec{0}) + \tfrac{1}{2} I''_{\rho_c}(\vec{0}) \cdot \Big(\vec{c}\otimes\vec{c} - 2\,\vec{c}\otimes\langle\vec{c}\rangle + \langle\vec{c}\otimes\vec{c}\rangle\Big)\,. \qquad (43)$$

where $I_{\rho_c}(\vec{b})$ is the gravitational interaction energy between ρ_c and $T_{\vec{d}}\rho_c$.
- In one dimension and with external harmonic potential this gives rise to modified Schrödinger evolution:

$$i\hbar\partial_t\psi(t;c) = \left(-\frac{\hbar^2}{2M}\frac{\partial^2}{\partial c^2} + \tfrac{1}{2}M\omega_c^2 c^2 + \tfrac{1}{2}M\omega_{\mathrm{SN}}^2\big(c - \langle c\rangle\big)^2\right)\psi(t;c)\,, \qquad (44)$$

As a consequence covariance ellipse of the Gaussian state rotates at frequency $\omega_q := (\omega_c^2 + \omega_{\mathrm{SN}}^2)^{(1/2)}$ whereas the centre of the ellipse orbits the origin in phase with frequency ω_c. This asynchrony is a genuine effect of self-gravity. It has been suggested that it may be observable on optomechanical systems (Yang.et al. 2013).

Motivation
- Carlip 2006
- Schrödinger
- Penrose

What's known
- COW & Co.
- EEP
- uff a theorem

SNE
- as non-rel. limit
- dimensionless
- symmetries
- collapse
- stationary states
- generalisation
- multi particle
- separation
- approximation
- consequences

Summary

Supplementary
- UFF and UGR
- KG inertial
- KG accelerating
- collapses

Motivation
- Carlip 2006
- Schrödinger
- Penrose

What's known
- COW & Co.
- EEP
- uff a theorem

SNE
- as non-rel. limit
- dimensionless
- symmetries
- collapse
- stationary states
- generalisation
- multi particle
- separation
- approximation
- consequences

Summary

Supplementary
- UFF and UGR
- KG inertial
- KG accelerating
- collapses

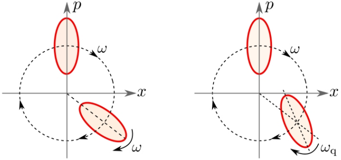
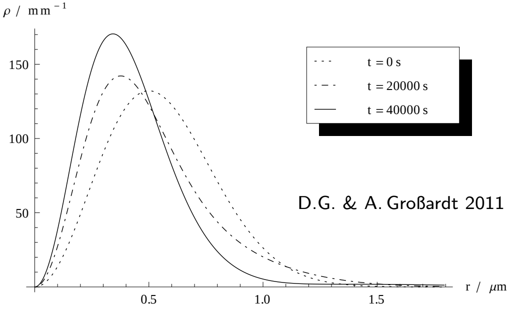


FIG. 1 (color online). Left: according to standard quantum mechanics, both the vector $(\langle x\rangle, \langle p\rangle)$ and the uncertainty ellipse of a Gaussian state for the c.m. of a macroscopic object rotate clockwise in phase space, at the same frequency $\omega = \omega_{\mathrm{c.m.}}$. Right: according to the c.m. Schrödinger-Newton equation (2), $(\langle x\rangle, \langle p\rangle)$ still rotates at $\omega_{\mathrm{c.m.}}$, but the uncertainty ellipse rotates at $\omega_{\mathrm{q}} \equiv (\omega^2_{\mathrm{c.m.}} + \omega^2_{\mathrm{SN}})^{1/2} > \omega_{\mathrm{c.m.}}$.

Yang-Miao-Lee-Helou-Chen, PRL 110 (2013)

The time-dependent SNE



- Time evolution of rotationally symmetric Gauß packet of initial width $500\,\mathrm{nm}$. Collapse sets in for masses $m > 4 \times 10^9\,\mathrm{u}$, but collapse times are of many hours (recall scaling laws, though).
- This is a 10^6 correction to earlier simulations by Carlip and Salzman (2006).

Motivation
- Carlip 2006
- Schrödinger
- Penrose
What's known
- COW & Co.
- EEP
- uff a theorem
SNE
- as non-rel. limit
- dimensionless
- symmetries
- collapse
- stationary states
- generalisation
- multi particle
- separation
- approximation
- consequences
Summary
Supplementary
- UFF and UGR
- KG inertial
- KG accelerating
- collapses

Summary

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- Statements concerning *Quantum Tests of the Equivalence Principle* need qualification.
- How does the Schrödinger function couple to all components of the gravitational field; e.g., a gravitational wave? Give *first-principles* derivation!
- What if gravity stays classical?
- How, then, do systems in non-classical states source gravity?
- Schrödinger-Newton equation as limit of semi-classical Einstein equation.
- Inhibitions of dispersion at, e.g., $500\,\mathrm{nm}$ scale for masses above $10^10\,\mathrm{u}$.
- Potentially interesting consequences from gravity-induced non-linearities in the Schrödinger equation of many particle systems can be derived, e.g., concerning the centre-of-mass motion.
- There is an army of arguments against fundamental semi-classical gravity; but how conclusive are they really?

THANKS!

Summary

- There is no obvious way to translate EP = UFF + LLI + UGR to non-classical systems.
- Statements concerning *Quantum Tests of the Equivalence Principle* need qualification.
- How does the Schrödinger function couple to all components of the gravitational field; e.g., a gravitational wave? Give *first-principles* derivation!
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Motivation
- Carlip 2006
- Schrödinger
- Penrose

What's known
- COW & Co.
- EEP
- uff a theorem

SNE
- as non-rel. limit
- dimensionless
- symmetries
- collapse
- stationary states
- generalisation
- multi particle
- separation
- approximation
- consequences

Summary

Supplementary
- UFF and UGR
- KG inertial
- KG accelerating
- collapses

UFF – UGR dependence: Energy conservation

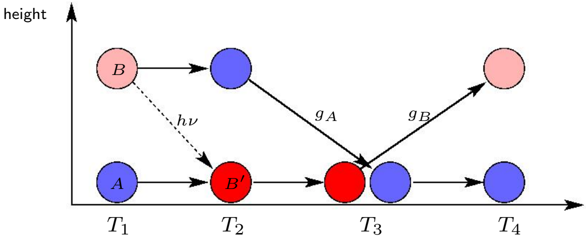

Figure: Gedankenexperiment by NORDTVEDT to show that energy conservation connects violations of UFF and UGR. Considered are two copies of a system that is capable of 3 energy states A, B, and B' (blue, pink, and red), with $E_A < E_B < E_{B'}$. Initially system 2 is in state B and placed a height h above system 1 which is in state A. At time T_1 system 2 makes a transition $B \rightarrow A$ and sends out a photon of energy $h\nu = E_B - E_A$. At time T_2 system 1 absorbs this photon, which is now blue-shifted, and makes a transition $A \rightarrow B'$. At T_3 system 2 has been dropped from height h with acceleration g_A, has hit system 1 inelastically, leaving one system in state A and at rest, and the other system in state B with an upward motion with kinetic energy $E_{\text{kin}} = M_A g_A h + (E_{B'} - E_B)$. The latter motion is decelerated by g_B, which may differ from g_A. At T_4 the system in state B has climbed to the **same** height h by energy conservation. Hence have $E_{\text{kin}} = M_B g_B h$ and therefore $M_A g_A h + M_{B'} c^2 = M_B c^2 + M_B g_B h$, from which we get

$$\frac{\delta\nu}{\nu} = \frac{(M_{B'} - M_A) - (M_B - M_A)}{M_B - M_A} = \frac{g_B h}{c^2}\left[1 + \frac{M_A}{M_B - M_A}\,\frac{g_B - g_A}{g_B}\right] \tag{45a}$$

$$\Rightarrow \alpha = \frac{M_A}{M_B - M_A}\,\frac{g_B - g_A}{g_B} =: \frac{\delta g/g}{\delta M/M} \tag{45b}$$

S & KG: Inertial motion

- Galilei symmetry is a suitable $1/c \to 0$ limit (contraction) of Poincaré symmetry. Likewise, the Schrödinger equation for ψ is a suitable $1/c \to 0$ limit of the Klein-Gordon equation for ϕ if we set

$$\phi(t,\vec{x}) = \exp\{-imc^2\, t/\hbar\}\, \psi(t,\vec{x})\,. \tag{46}$$

- The Klein-Gordon field transforms as scalar

$$\phi'(t',\vec{x}') = \phi(t,\vec{x})\,. \tag{47}$$

  Hence (46) implies

$$\psi'(t',\vec{x}') = \exp\{-imc^2\,(t-t')/\hbar\}\, \psi(t,\vec{x})\,. \tag{48}$$

- Using

$$t = \frac{t' + \vec{x}'\cdot\vec{v}/c^2}{\sqrt{1-v^2/c^2}} = t' + c^{-2}\bigl(\vec{x}'\cdot\vec{v} + t'v^2/2\bigr) + \mathcal{O}(1/c^4)\,, \tag{49}$$

  The $1/c \to 0$ limit of Poincaré symmetry by proper representations turns into Galilei symmetry by non-trivial ray representations

$$\psi'(t',\vec{x}') = \exp\{-im(\vec{x}'\cdot\vec{v} + t'v^2/2)/\hbar\}\ \psi(t,\vec{x})\,. \tag{50}$$

Motivation
- Carlip 2006
- Schrödinger
- Penrose

What's known
- COW & Co.
- EEP
- uff a theorem

SNE
- as non-rel. limit
- dimensionless
- symmetries
- collapse
- stationary states
- generalisation
- multi particle
- separation
- approximation
- consequences

Summary

Supplementary
- UFF and UGR
- KG inertial
- KG accelerating
- collapses

S & KG: Rigid accelerations

- In Minkowski space, rigid motions in x-direction and of arbitrary acceleration of a body parametrised by ξ are given by family of timelike lines $\tau \mapsto \big(ct(\tau,\xi)\,,\,x(\tau,\xi)\big)$, where

$$ct(\tau,\xi) = c\int^{\tau} d\tau' \cosh\chi(\tau') + \xi\,\sinh\chi(\tau) \tag{51a}$$

$$x(\tau,\xi) = c\int^{\tau} d\tau' \sinh\chi(\tau') + \xi\,\cosh\chi(\tau) \tag{51b}$$

  Here τ is eigentime of body element $\xi = 0$ and $\chi(\tau) = \tanh^{-1}(v/c)$ is rapidity of all body elements at τ.

- The Minkowski metric in co-moving coordinates (τ,ξ) reads ($g := c\dot{\chi}$)

$$ds^2 = c^2\,dt^2 - d\vec{x}^2 = \left(1 + \frac{g(\tau)\,\xi}{c^2}\right)c^2\,d\tau^2 - d\vec{\xi}^2\,. \tag{52}$$

- Write down Klein-Gordon equation in co-moving coordinates

$$\left\{\Box_g + m^2\right\}\phi = \left\{(-\det g)^{-1/2}\;\partial_a\left[(-\det g)^{1/2}\;g^{ab}\partial_b\right] + m^2\right\}\phi = 0\,. \tag{53}$$

Motivation
- Carlip 2006
- Schrödinger
- Penrose

What's known
- COW & Co.
- EEP
- uff a theorem

SNE
- as non-rel. limit
- dimensionless
- symmetries
- collapse
- stationary states
- generalisation
- multi particle
- separation
- approximation
- consequences

Summary

Supplementary
- UFF and UGR
- KG inertial
- KG accelerating
- collapses

S & KG: Rigid accelerations

- In analogy to (46) write
$$\phi(t,\vec{x}) = \exp\{-imc^2\,\tau/\hbar\}\,\psi(t,\vec{x}) \tag{54}$$
and take $1/c^2 \to 0$ limit; get
$$i\hbar\partial_\tau\psi = \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial\vec{\xi}^2} + mg(\tau)\xi\right)\psi\,. \tag{55}$$
This corresponds to particle in homogeneous but time-dependent gravitational field pointing in negative ξ-direction.
- Note that again ϕ transformed as scalar (compare (47))
$$\phi^{\rm inert}(t,\vec{x}) = \phi^{\rm acc}(\tau,\vec{\xi}) \tag{56}$$
but that again this is not true for ψ, where (compare (46))
$$\begin{aligned}\phi^{\rm inert}(t,\vec{x}) &= \exp\{-imc^2\,t/\hbar\}\,\psi^{\rm inert}(t,\vec{x})\\ \phi^{\rm acc}(\tau,\vec{\xi}) &= \exp\{-imc^2\,\tau/\hbar\}\,\psi^{\rm acc}(\tau,\vec{\xi})\,,\end{aligned} \tag{57}$$
- Hence (compare (48))
$$\psi^{\rm acc}(\tau,\vec{\xi}) = \exp\{-imc^2\,(t-\tau)/\hbar\}\;\psi^{\rm inert}(t,\vec{x})\,. \tag{58}$$

Motivation
- Carlip 2006
- Schrödinger
- Penrose
What's known
- COW & Co.
- EEP
- uff a theorem
SNE
- as non-rel. limit
- dimensionless
- symmetries
- collapse
- stationary states
- generalisation
- multi particle
- separation
- approximation
- consequences
Summary
Supplementary
- UFF and UGR
- KG inertial
- KG accelerating
- collapses

Bound states and collapse: Naive estimates

- Using the total energy $E = T + V$, with

$$E = \frac{\hbar^2}{2m} \int d^3x \, \|\nabla\Psi(\vec{x})\|^2 - \frac{Gm^2}{2} \int d^3x \int d^3y \, \frac{|\Psi(\vec{x})|^2 \, |\Psi(\vec{y})|^2}{\|\vec{x} - \vec{y}\|} . \tag{59}$$

we can express to second time-derivative of the second moment of $|\Psi|$:

$$\ddot{Q} = \frac{d^2}{dt^2} \int \|\vec{x}\|^2 \, |\Psi(t, \vec{x})|^2 \, d^3x = \frac{1}{m} \, 2(2E - V) \, . \tag{60}$$

showing that $\ddot{Q} < 0$ implies $E < 0$ (note $V < 0$).

- A spherically symmetric Gaussian of width a:

$$\Psi(r, t = 0) = (\pi a^2)^{-3/4} \exp\left(-\frac{r^2}{2a^2} \right) \tag{61}$$

has

$$E = \frac{\hbar^2}{2ma^4} - \frac{2\,G\,m^2}{\sqrt{\pi}\,a^3} \sinh^{-1}(1) \approx \frac{\hbar^2}{2ma^4} - \frac{G\,m^2}{a^3} \, , \tag{62}$$

so that $E < 0$ is equivalent to

$$m^3 a > \frac{\hbar^2}{2G} = \frac{1}{2} \left(\frac{\hbar c}{G} \right)^{3/2} \left(\frac{\hbar G}{c^3} \right)^{1/2} = \tfrac{1}{2} m_P^3 \ell_P \, . \tag{63}$$

For $a = 500\,\mathrm{nm}$ this gives $m > 3.3 \times 10^9\,\mathrm{u}$