Remarks on ‘General Covariance’ and/or ‘Background Independence’

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Introduction and Motivation

- It is a widely shared opinion that *the* most outstanding and characteristic feature of General Relativity (GR) is its manifest *background independence*. Consequently, those pursuing the canonical quantisation programme for GR see the fundamental virtue of their approach in precisely this preservation of 'background independence'.
- Some workers on string theory also hope to make progress by getting rid of background structures:

$\Rightarrow$ *Seek to make progress by identifying the background structure in our theories and removing it, replacing it with relations which evolve subject to dynamical laws.*
(Lee Smolin: "The case for background independence, hep-th/0507235)

What is a dynamical law ?

The General Structure of Equations of Motion in Classical Physics

- Dynamical laws usually contain two types of objects:

  1. **Background structures**, Σ, which are externally specified, and
  2. **Dynamical structures**, Φ, like 'particles' ($\gamma : \mathbb{R} \to M$) and 'fields' ($F : M \to V$), which are solved for.

- Equations of motion then establish a relation between these structures:

$$E M[\Phi, \Sigma] = 0 \,. \tag{1}$$

- The set of all possible (by declaration) Φ is called the space of **kinematically possible trajectories**, denoted by $\mathcal{K}$. Equation (1) is then read as selecting a subset $\mathcal{D} \subset \mathcal{K}$ of **dynamically possible trajectories** for *given* Σ. This terminology is due to James Anderson (1967).

Symmetries

- Suppose a group G acts on $\mathcal{K}$:

$$G \times \mathcal{K} \to \mathcal{K}\,, \quad (g, \Phi) \mapsto g \cdot \Phi\,. \tag{2}$$

  We call G a group of **symmetries** for the given set of equations, if it leaves $\mathcal{D} \subset \mathcal{K}$ invariant (as a subset). In other words, if for all $g \in G$:

$$EM[\Phi\,,\,\Sigma] = 0 \iff EM[g \cdot \Phi\,,\,\Sigma] = 0\,. \tag{3}$$

- This is to be distinguished from mere **covariance**, which just states the invariance of the relation established by EM (assuming also an action of G on the set of Σ's):

$$EM[\Phi\,,\,\Sigma] = 0 \iff EM[g \cdot \Phi\,,\,g \cdot \Sigma] = 0\,. \tag{4}$$

- A symmetry is automatically a covariance. Conversely, a covariance is a symmetry iff it stabilises the background structure: $g \cdot \Sigma = \Sigma$.

A Note on 'Symmetries'

- The notion of **symmetries** used here is still formal, in the sense that it makes no distinction between **proper physical symmetries**, which map physical states or histories (i.e. dpts) to other, physically **distinguishable** states or histories, and **gauge symmetries**, which map one state or history to a physically **indistinguishable** one (redundancies of description).
- In field theory, these two notions of 'symmetry' often appear in a combined form: a proper normal subgroup $\mathsf{Gau} \subset \mathsf{Sym}$ represents gauge symmetries, whereas the quotient, $\mathsf{Phys} := \mathsf{Sym}/\mathsf{Gau}$, corresponds to proper physical symmetries:

$$1 \longrightarrow \mathsf{Gau} \longrightarrow \mathsf{Sym} \longrightarrow \mathsf{Phys} \longrightarrow 1 \tag{5}$$

- The existence of Phys may be forced upon us even without the explicit mention of individuating fields (which would manifestly render the 'motions' in Phys operationally meaningful) if the field is long-ranging and its space of states is taken to contain globally charged ones. As an example consider the non-zero angular momentum solutions to Einstein's field equations (Kerr family of BHs); here a global spatial rotation cannot be a gauge transformation.

Example: Maxwell's Vacuum Equations

- Consider the vacuum Maxwell equations on a fixed spacetime (Lorentzian manifold (M, g)):

$$dF = 0\,, \qquad d \star F = 0\,. \tag{6}$$

- These equations depends on the background metric g through $\star F_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}$ and also the operation of raising indices: $F^{\alpha\beta} := g^{\alpha\mu}g^{\beta\nu}F_{\mu\nu}$. Together they appear in the combination $\sqrt{\det\{g_{\kappa\lambda}\}}\,g^{\mu\alpha}g^{\nu\beta}$, which needs to be left invariant by symmetries.
- The system (6) is manifestly $\mathrm{Diff}(M)$–covariant, but the symmetries are only given by the stabiliser group of the mentioned combination, which in case of Minkowski space is the Poincaré group in all but four space-time dimensions. In four dimensions, and only there, it also includes the proper conformal transformations of Minkowski space, so that the symmetries then consist of the conformal group of Minkowski space.

Covariance Trivialised

- The requirement of $\mathrm{Diff}(M)$ covariance is rather trivial. Consider, e.g., the ordinary heat equation

$$\partial_t T = \kappa \Delta T\,. \tag{7}$$

- Rewrite it by explicitly displaying the background structures in terms of geometric objects:

$$\left\{n^\mu \nabla_\mu - \kappa(g^{\mu\nu} + n^\mu n^\nu)\nabla_\mu \nabla_\nu\right\} T = 0\,. \tag{8}$$

  where $n^\mu \partial_\mu := \partial_t$. g is the flat spacetime metric and ∇ is associated Levi-Civita covariant derivative.

- This makes $\mathrm{Diff}(M)$-covariance manifest. The symmetry group is the subgroup in $\mathrm{Diff}(M)$ that stabilises the vector field n and the metric g. It is isomorphic to $E_3 \times \mathbb{R}$ of Euclidean (spatial) motions and time translations, which is just the old symmetry group of (7).

Background Structures and Symmetries

- Given that an equation of motion is already G-covariant, we can equivalently express the condition of G being a symmetry group by

$$F[\Phi, \Sigma] = 0 \Leftrightarrow F[\Phi\,,\, g \cdot \Sigma] = 0\,, \quad \forall f \in G\,. \tag{9}$$

  That is, any solution of the equation parameterised by Σ is also a solution of the **different** equation parameterised by $g \cdot \Sigma$.

- Evidently, the more non-dynamical structures there are, the more difficult it is to satisfy (9). In generic situations it will only be satisfied if $G = \mathrm{Stab}_{\mathrm{Diff}(M)}(\Sigma)$. Hence, in distinction to the covariance group, increasing the amount of structures of the type Σ cannot enlarge the symmetry group.

Symmetries Trivialised

- An obvious strategy to turn covariances into symmetries is to formally declare the Σ's to be dynamical variables, by letting their values be determined by 'equations of motion'. Hence it is easy to dress up the Σ as Φ.
- For example, for Maxwell's equations and also for the heat-equation, we may respectively view g and (g, n) as dynamical variables, subject to the equations

$$\textbf{Riem}[g] = 0 \quad \text{and also} \quad \nabla n = 0 \,, \quad g(n, n) = c^2 \,. \tag{10}$$

- Note that (10) are autonomous equations, i.e. they do not involve the former Φ. Also, up to diffeomorphisms, their solution space is zero-dimensional (no 'real' degrees of freedom if $\mathrm{Diff}(M)$ is gauge group).
- Possible because we liberally interpret 'equation of motion' as 'any equation'. What sensible constraints characterise 'equations of motion'? Higher-dimensional solution spaces (real degrees of freedom), well-posed initial value problems, ...? This will be difficult to decide in general and therefore render impractical the requirement of background independence as heuristic guiding principle.

Anderson's Attempt Against Triviality

- In the 1960s, James Anderson, characterised what we here call **background independence** (general covariance) as the absence of what he calls **absolute structures**.

- **Definition 1. [locality improved]** *Any field which is either not dynamical or whose solutions to the equations of motion are all mutually [locally] diffeomorphic is called an **absolute structure**.*
  - Note: there is a certain ambiguity here as to what structure may fall under the notion of 'field'. Does it have to be a **geometric structure** in the differential-geometric sense?
- **Definition 2. [locality improved]** *A theory is called **background independent** iff its equations are $\mathrm{Diff}(\mathrm{M})$-symmetric in the sense of equation (3) and its fields do not include absolute structures in the sense of Definition 1.*

- Let us look at this proposal in the light of some examples.

Nordström à la Einstein-Fokker I.

- In early 1914, Einstein and Fokker proposed to show that Nordströms scalar theory of gravity could be written in a form which is as satisfying from the "invariant-theoretic point of view" as the Einstein-Grossmann Entwurf-Theory.
- In this reformulation, space-time is assumed to be conformally flat

$$g_{\mu\nu} = \phi^2 \eta_{\mu\nu} \,. \tag{11}$$

- The field equation and the equation of motion for a test particle are

$$R[g] \;=\; 24\pi G\, g^{\mu\nu} T_{\mu\nu}\,, \qquad \ddot{x}^{\mu} + \Gamma^{\mu}_{\alpha\beta}\dot{x}^{\alpha}\dot{x}^{\beta} = 0\,. \tag{12}$$

- Whereas (12) clearly is $\mathrm{Diff}(M)$-symmetric, (11) is not. The flat metric, η, is an absolute structure in the sense above. This is not changed if one re-declares η to be dynamical with an additional equation $\mathbf{Riem}[\eta] = 0$. The group of symmetries remains the stabiliser group of η, which is the Poincaré group.

Nordström à la Einstein-Fokker II.

- Absolute structures are not always easy to spot. For example, we could equivalently write the Einstein-Fokker equations in the form in which η does not appear explicitly:

$$R[g] \;=\; 24\pi G\, g^{\mu\nu} T_{\mu\nu}\,, \qquad \mathbf{Weyl}[g] = 0\,, \qquad \ddot{x}^{\mu} + \Gamma^{\mu}_{\alpha\beta}\dot{x}^{\alpha}\dot{x}^{\beta} = 0\,. \tag{13}$$

- The absolute structure is now hidden in the field g. To bring it back to light, make a field redefinition, $g_{\mu\nu} \mapsto (\phi, h_{\mu\nu})$, which isolates the part determined by (13); for example:

$$\phi := \left[-\det\{g_{\mu\nu}\}\right]^{\frac{1}{8}}\,, \qquad h_{\mu\nu} := g_{\mu\nu}\left[-\det\{g_{\mu\nu}\}\right]^{-\frac{1}{4}}\,. \tag{14}$$

- Then any two solutions of (13) are such that their component fields $h_{\mu\nu}$ and $h'_{\mu\nu}$ are related by a diffeomorphism. Hence $h_{\mu\nu}$ is an absolute structure.

A Remark Concerning Action Principles I.

- Adding 'equations of motion' for absolute structures might not be easily possible from action principles without adding degrees of freedom, as the following example shows:
- Consider a real massless (for simplicity only) scalar field on flat spacetime,

$$\Box\phi := \eta^{\mu\nu}\nabla_\mu\nabla_\nu\phi = 0\,. \tag{15}$$

- According to standard strategy, the non-dynamical Minkowski metric η is eliminated by introducing the dynamical variable g, replacing η in (15), and adding the flatness condition

$$\text{Riem}[g] = 0\,. \tag{16}$$

- We ask: Is there an action principle whose Euler-Lagrange equations are (equivalent to) these equations? This seems impossible without introducing yet another field λ (a Lagrange multiplier), whose variation just yields (16).

A Remark Concerning Action Principles II.

- Adding a field $\lambda^{\alpha\beta\mu\nu} = -\lambda^{\beta\alpha\mu\nu} = -\lambda^{\alpha\beta\nu\mu} = +\lambda^{\mu\nu\alpha\beta}$, the action could be

$$S = \frac{1}{2}\int dV\, g^{\mu\nu}\nabla_\mu\phi\,\nabla_\nu\phi + \frac{1}{4}\int dV\,\lambda^{\alpha\beta\mu\nu}R_{\alpha\beta\mu\nu}\,. \tag{17}$$

- Variation with respect to ϕ and λ yield (15) and (16) respectively, and variation with respect to g gives in addition (where $T^{\alpha\beta}$ is the em-tensor for ϕ)

$$\nabla_\mu\nabla_\nu\lambda^{\alpha\mu\beta\nu} = T^{\alpha\beta}\,. \tag{18}$$

- The integrability condition $\nabla_\alpha T^{\alpha\beta} = 0$ is satisfied due to (15). Hence solutions λ always exist (locally).
- We note that g is still an absolute structure, since any two solutions have diffeomorphic g's. But this is not likewise true for the field λ.

A Remark Concerning Action Principles III.

- In fact many solutions exist, since any solution $\lambda^{[\alpha\mu][\beta\nu]}$ is mapped to a (new ?) solution by
$$\lambda^{[\alpha\mu][\beta\nu]} \mapsto \lambda^{[\alpha\mu][\beta\nu]} + \nabla_\sigma \lambda^{[\alpha\mu][\beta\nu\sigma]} \tag{19}$$
for any $\lambda^{[\alpha\mu][\beta\nu\sigma]}$.
- One has the possibility to regard it as gauge transformation (for sufficiently rapid fall-off of the λ field), since motions (19) cost no action due to 2nd Bianchi Identity:
$$R_{\alpha\mu[\beta\nu;\sigma]} \equiv 0\,. \tag{20}$$
- Only if interpretation as gauge-symmetry is adopted is this formulation equivalent to original one given by (15) and (16).
- A detailed Hamiltonian analysis (constraint-structure) would be necessary to unambiguously work out degrees of freedom. This has not been done.

HOWEVER ...

- If $\mathrm{Diff}(M)$ acts as a gauge group, then there is a four-function worth of redundant labellings of physical states among the Φ. Hence one expects to be able to find four (one-component) functions among the Φ which can be set to some given four functions by suitable application of $\mathrm{Diff}(M)$, for any solution of the equations of motion.
- For example, in pure gravity, for any given space-time coordinate system, $\{x^\mu\}$, we may (locally) always achieve $g_{00} = -1$ and $g_{0k} = 0$ ($k = 1, 2, 3$) in *that* system.
- Another function of the $g_{\mu\nu}$ with more geometric appeal is $\det\{g_{\mu\nu}\}$, which we may (locally) always set equal to -1 (Moser's theorem).
- Alternatively, if our model contains dust matter with four-velocity vector field $v = v^\mu \partial_\mu$, we may (locally) always achieve $v^\mu = (1, 0, 0, 0)$. Note that this is an *exclusive* alternative to the choices above.
- This is not true for one-forms $\alpha = \alpha_\mu dx^\mu$. For example, closedness ($d\alpha = 0$) or exactness ($\alpha = df$) are $\mathrm{Diff}(M)$-invariant properties. But this should not be overemphasised.

Conclusion

- The existence of non-dynamical fields in gauge theories is a tautology. They should be exempt from the equation *non-dynamical = absolute*.
- In our previous examples (Einstein-Fokker, Heat equation) **more** than the expected four functions could be set to fixed values. There were ten components of $\eta_{\mu\nu}$ and four components of n^μ. **This** was the remarkable feature that showed their non-dynamical character.
- I propose to maintain the equality *background = absolute structures* but to refine the notion of absolute structure, so as to not apply to fields whose non-dynamical character is merely due to gauge freedoms.
- Mathematically this may turn out to be difficult, as it presupposes a solution to the problem of local (and, eventually, also global) gauge fixation. Degrees of freedom are faithfully labelled by points in a **quotient space** of the space of Φ's, whose representation as subspace is highly non-unique.

Thanks!