On Doppler Tracking in Cosmological Spacetimes

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Heidelberg, March 7th 2007

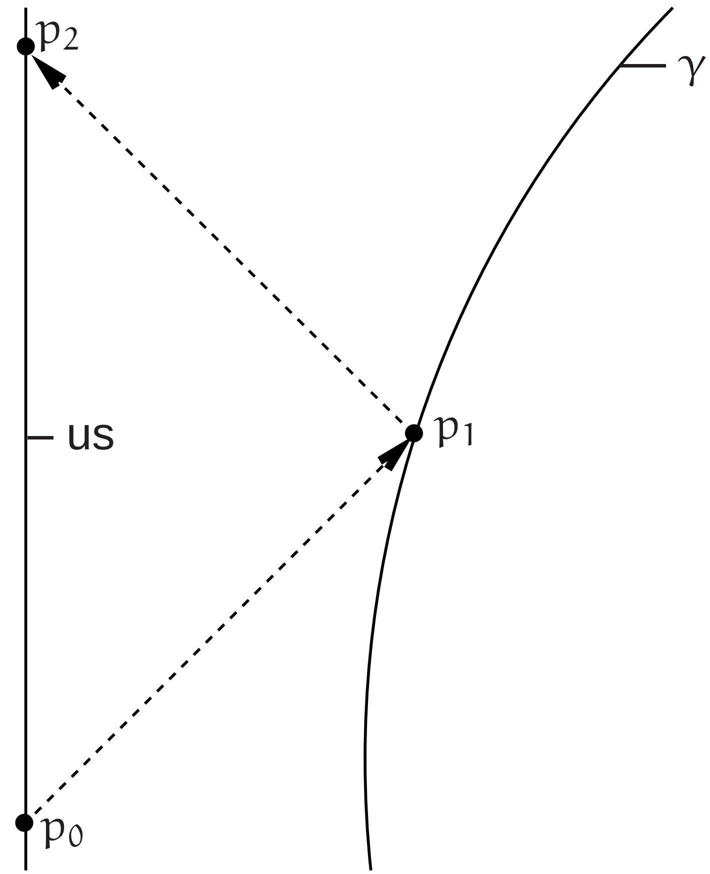
Introduction

- **Doppler Tracking** is a common method of tracking the position of vehicles in space. It involves measuring the Doppler shift of a radio signal sent from a spacecraft to a tracking station on Earth, this signal either coming from an onboard oscillator or being one that the spacecraft has coherently transponded in response to a signal received from the ground station. The second of these modes is more useful for navigation because the returning signal is measured against the same frequency reference as that of the originally transmitted signal. The Earth-based frequency reference is also more stable than the oscillator onboard the spacecraft.

- This talk is based on *Matteo Carrera & D.G, CQG* ***26*** *(2006) 7483-7492* and partly on an ESA-Study *gr-qc/0602098*.

Elementary Theory 1

- At the event $p_0 = (t_0, \vec{x}_0)$ a radio signal of frequency ω_0 is emitted towards the spacecraft and received by it at event $p_1 = (t_1, \vec{x}_1)$ with frequency ω_1.
- In case of simple **reflection**, a returning radio signal is emitted at $p_1 = (t_1, \vec{x}_1)$ with frequency ω_1 and received by us at event $p_2 = (t_2, \vec{x}_2)$ with frequency ω_2.
- Note: All frequencies refer to those measured by observers that are locally co-moving with the given world lines ($\gamma =$ world line of spacecraft)

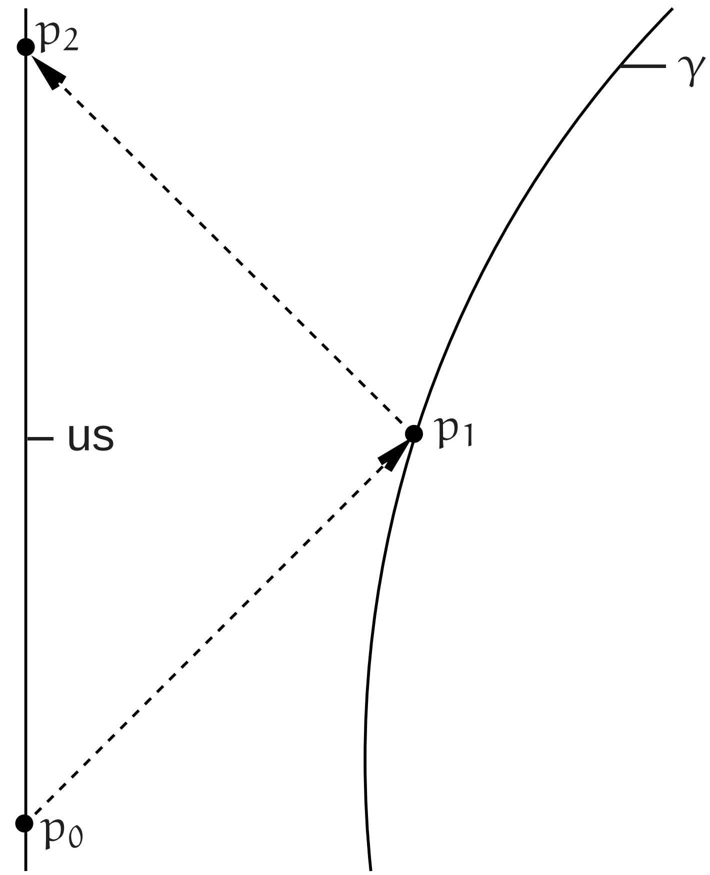


Elementary Theory 2

- For given world-lines (of us and of the spacecraft) p_0 and p_1 are determined by p_2 (for example). Hence t_1 and t_0 are determined by t_2.
- We are interested in the ratio ω_2/ω_0 as function of t_2, which is determined once the world lines are given.
- In SR we get for purely radial motion

$$\frac{\omega_2(t_2)}{\omega_0(t_0)} = \frac{1 - \beta(t_1)}{1 + \beta(t_1)}$$

where $\beta(t_1) = v(t_1)/c$ and $v(t_1)$ is the velocity of the spacecraft with respect to our (global) inertial system at the time t_1 of signal reflection.



Elementary Theory 3

- We wish to take the differential quotient of $\omega(t_2)/\omega(t_0)$ with respect to t_2, assuming a constant function ω_0. We get

$$\frac{\dot{\omega}(t_2)}{\omega_0} = \frac{-2\dot{\beta}(t_1)}{\big(1+\beta(t_1)\big)^2}\frac{dt_1}{dt_2} \tag{1}$$

- If we are resting at the origin and r is the radial coordinate of the spacecraft, we have $t_2 - t_1 = r(t_1)/c$ and therefore

$$1 - \frac{dt_1}{dt_2} = \frac{1}{c}\frac{dr(t_1)}{dt_1}\frac{dt_1}{dt_2} \iff \frac{dt_1}{dt_2} = \frac{1}{1+\beta(t_1)} \tag{2}$$

- Hence 1 becomes ($\dot{\beta} \equiv \alpha$)

$$\frac{\dot{\omega}(t_2)}{\omega_0} = \frac{-2\dot{\beta}(t_1)}{\big(1+\beta(t_1)\big)^3} \approx -2\alpha(t_1)\Big(1 - 3\beta(t_1) + \cdots\Big) \tag{3}$$

Geometric Theory 1

- In a general spacetime (M, g) [we use signature $(+, -, -, -)$ for g] there is no privileged (e.g. inertial) global reference frame by means of which we may introduce kinematical variables that characterize world lines (different ones collectively). Hence "appropriate" fiducial observer-fields need to be introduced.
- An **observer** at the event p is a future pointing unit timelike vector. An **observer field** is a field of observers. Any observer u at p gives rise to an orthogonal split of the Tangent space at p, $T_p(M) = T_p^{\|}(M) \oplus T_p^{\perp}(M)$, where

$$T_p^{\|}(M) := \text{Span}\{u\}\,, \qquad T_p^{\perp}(M) := \{v \in T_p(M) \mid g(v, u) = 0\} \tag{4}$$

The associated projection operators are given by

$$P_u^{\|} : T_p \to T_p^{\|}(M)\,, \quad v \mapsto P_u^{\|}(v) := u\, g(u, v) \tag{5a}$$

$$P_u^{\perp} : T_p \to T_p^{\perp}(M)\,, \quad v \mapsto P_u^{\perp}(v) := v - u\, g(u, v) \tag{5b}$$

Geometric Theory 2

- If two observers u and v are defined at the same point, the relative velocity (over c) of v with respect to u is given by (we write $\|v\| := \sqrt{|g(v,v)|}$)

$$\vec{\beta}_u(v) := \frac{P_u^{\perp}(v)}{\|P_u^{\|}(v)\|} \in T^{\perp}(M) \quad \text{and} \quad \beta_u(v) := \|\vec{\beta}_u(v)\| = \sqrt{1 - 1/\left[g(u,v)\right]^2} \tag{6}$$

so that

$$g(u,v) = 1/\sqrt{1 - \beta_u^2(v)} \qquad (\rightarrow \text{"gamma-factor"}) \tag{7}$$

- Note that $\beta_u(v) = \beta_v(u)$, though $\vec{\beta}_u(v)$ and $\vec{\beta}_v(u)$ are linearly independent: they lie in $P_u T(M)$ and $P_v T(M)$ respectively.
- Let $e \in P_u^{\perp} T(M)$ be a (spacelike) unit vector. We define the e-component of v's velocity relative to u by

$$\beta_u^e(v) = -g(e, \vec{\beta}_u(v)) = -\frac{g(e,v)}{g(u,v)} \tag{8}$$

Intermezzo: The One-Way Doppler Formula

- Given a lightlike vector k (wave vector) and observers u, v at the same spacetime point. The observed frequencies are

$$\omega_v(k) := g(v, k) \qquad \omega_u(k) := g(u, k) \tag{9}$$

whose ratio is given by

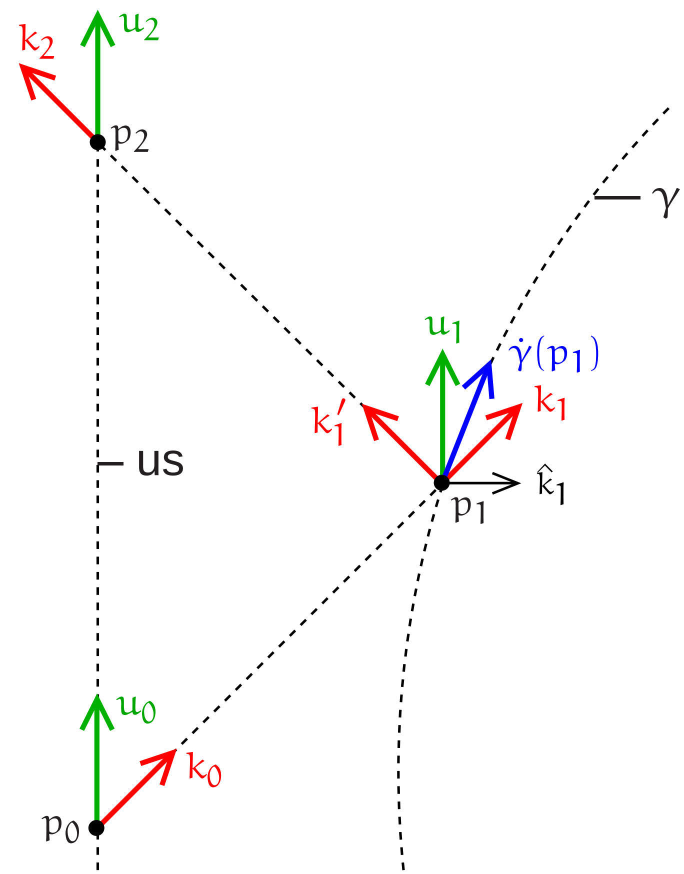
$$\begin{aligned}\frac{\omega_v(k)}{\omega_u(k)} &= \frac{g(v, k)}{g(u, k)} = \frac{g(P_u^{\|}v + P_u^{\perp}v, k)}{g(u, k)} = g(u, v)\left[1 + \frac{g(v, P_u^{\perp}k)}{g(u, v)g(u, k)}\right] \\ &= g(u, v)\left[1 - \beta_u^{\hat{k}}(v)\right]\end{aligned} \tag{10}$$

where the spacelike unit vector $\hat{k} := P_u^{\perp}k/\|P_u^{\perp}k\|$ defines the direction of k in the rest system of u.

Geometric Theory 3

- Let u be an observer **field** along one integral line of which we are moving. As before, γ is the world line of the spacecraft. The field u is defined in a neighbourhood of γ.
- The wave-vector k_0 emitted at p_0 suffers three changes:
  1. propagation from p_0 to p_1: $k_0 \longrightarrow k_1$
  2. reflection at p_1: $k_1 \longrightarrow k_1'$
  3. propagation from p_1 to p_2: $k_1' \longrightarrow k_2$
- We are interested in

$$\frac{\omega_2}{\omega_0} = \frac{g(u_2, k_2)}{g(u_0, k_0)} = \left[\frac{\omega_2}{\omega_1'}\right] \left[\frac{\omega_1'}{\omega_1}\right] \left[\frac{\omega_1}{\omega_0}\right]$$



What happens at "reflection"?

- With respect to the spacecraft moving along γ with four-velocity $\dot{\gamma}$, the wave vector k_1 at p_1 splits according to

$$k_1 = P_{\dot{\gamma}}^{\parallel}(k_1) + P_{\dot{\gamma}}^{\perp}(k_1) =: k_1^{\parallel} + k_1^{\perp} \tag{11}$$

- A corner-cube reflector transported along γ will reverse $k_1^{\perp}$ while keeping $k_1^{\parallel}$ intact (i.e. neglecting a possible transponder shift):

$$\boxed{k_1 \mapsto k_1' = k_1^{\parallel} - k_1^{\perp} = 2k_1^{\parallel} - k_1} \tag{12}$$

Hence $\omega_1 := \omega_u(k_1) = g(u_1, k_1)$ and $\omega_1' := \omega_u(k_1') = g(u_1, k_1')$, the in- and out-going frequencies measured by the observer u_1 at p_1, are related by

$$\frac{\omega_1'}{\omega_1} = \frac{g(u_1, k_1')}{g(u_1, k_1)} = 2\,\frac{g(u,\dot{\gamma})g(\dot{\gamma},k')|_{p_1}}{g(u,k)|_{p_1}} - 1 = 2\,\frac{1-\beta_u^{\hat{k}}(\dot{\gamma})|_{p_1}}{1-\beta_u^2(\dot{\gamma})|_{p_1}} - 1 \tag{13}$$

Geometric Theory 4

- To account for the propagation effects, use the laws of geometric optics in (curved) spacetime to relate $\omega_0 = g(u_0, k_0)$ (at p_0) and $\omega_2 = g(u_2, k_2)$ (at p_2) to kinematical quantities of γ at p_1.
- For example, if u is Killing (like $u = \partial/\partial t$ in SR), we have $g(u_0, k_0) = g(u_1, k_1)$ and $g(u_2, k_2) = g(u_1, k_1')$. Hence we obtain

$$\frac{\omega_2}{\omega_0} = \left[\frac{\omega_2}{\omega_1'}\right]\left[\frac{\omega_1'}{\omega_1}\right]\left[\frac{\omega_1}{\omega_0}\right] = \frac{g(u_1, k_1')}{g(u_1, k_1)} = 2\,\frac{1 - \beta_u^{\hat{k}}(\dot{\gamma})|_{p_1}}{1 - \beta_u^2(\dot{\gamma})|_{p_1}} - 1 \tag{14}$$

- In FLRW-spacetimes $u = \partial/\partial t$ is not Killing though almost conformally Killing. One has $a_0 g(u_0, k_0) = a_1 g(u_1, k_1)$ and $a_2 g(u_2, k_2) = a_1 g(u_1, k_1')$ and gets instead of (14)

$$\boxed{\frac{\omega_2}{\omega_0} = \left[\frac{\omega_2}{\omega_1'}\right]\left[\frac{\omega_1'}{\omega_1}\right]\left[\frac{\omega_1}{\omega_0}\right] = \frac{a_0}{a_2}\,\frac{g(u_1, k_1')}{g(u_1, k_1)} = \frac{a_0}{a_2}\left\{2\,\frac{1 - \beta_u^{\hat{k}}(\dot{\gamma})|_{p_1}}{1 - \beta_u^2(\dot{\gamma})|_{p_1}} - 1\right\}} \tag{15}$$

Approximation

- In standard FLRW coordinates have geodesic field $u = \partial/\partial t$. We set $\Delta t := (t_2 - t_1)/2$ and $H := \dot{a}/a$.
- To linear order in β and $H\Delta t$ have

$$\frac{\omega_2}{\omega_1} \approx 1 - 2\left(\beta_u^{\hat{k}}(\dot{\gamma})|_{p_1} + H_*\Delta t\right) \tag{16}$$

where $H_* = H(t_*)$ with t_* anywhere in $[t_1, t_2]$.

Relative Spatial Acceleration

- Given a world-line γ and an observer field u in its neighbourhood. Spatial tensor fields along γ are those whose contractions with u vanish. For them we introduce the following covariant derivative:

$$\nabla^{u}_{\dot\gamma} := \|P^{\|}_{u}\dot\gamma\|^{-1}\, P^{\perp}_{u} \circ \nabla^{u}_{\dot\gamma} \circ P^{\perp}_{u} \tag{17}$$

which is compatible with the **spatial metric**

$$\sigma_u := -P^{\perp}_{u}\, g \tag{18}$$

- In particular, we define the **relative spatial acceleration** (over c) of γ with respect to u by

$$\vec\alpha_u(\gamma) := \nabla^{u}_{\dot\gamma}\vec\beta_u(\dot\gamma) \tag{19}$$

and its component along the spatial direction e by

$$\alpha^{e}_{u}(\gamma) := \sigma_u\big(e, \vec\alpha_u(\gamma)\big) = -g\big(e, \vec\alpha_u(\gamma)\big) \tag{20}$$

Geometric Theory 5

- In order to calculate the derivative $\dot{\omega}_2(t_2)/\omega_0(t_0)$ we need to know the derivatives dt_1/dt_2 and dt_0/dt_2. They follow from (restricting to the flat FLRW case for simplicity):

$$\int_{t_1(t_2)}^{t_2} \frac{dt}{a(t)} = -\frac{1}{c}\int_{r_1(t_1(t_2))}^{r_2} dr \quad \Rightarrow \quad \frac{dt_1}{dt_2} = \frac{a(t_1)}{a(t_2)} \left(1 + \beta_u^{\hat{k}}(\dot{\gamma})|_{p_1}\right)^{-1} \tag{21}$$

and likewise

$$\int_{t_0(t_2)}^{t_2} \frac{dt}{a(t)} = \frac{1}{c}\left\{\int_{r_0}^{r_1(t_1(t_2))} - \int_{r_1(t_1(t_2))}^{r_2}\right\} dr \quad \Rightarrow \quad \frac{dt_0}{dt_2} = \frac{a(t_0)}{a(t_2)} \frac{1 - \beta_u^{\hat{k}}(\dot{\gamma})|_{p_1}}{1 + \beta_u^{\hat{k}}(\dot{\gamma})|_{p_1}} \tag{22}$$

Geometric Theory 6

- The exact formula for the t_2–derivative of the frequency-shift rate is now given by

$$\frac{1}{\omega_0}\frac{d\omega_2(t_2)}{dt_2} = -\frac{a_0}{a_1}\Bigg\{2\left[\alpha^{\hat{k}} + \sigma(\beta, \nabla^u_{\dot{\gamma}}\hat{k})\right]\frac{a_1}{a_2}\left[1+\beta^{\hat{k}}\right]^{-1}\left[1-\beta^2\right]^{-1}$$
$$-4\sigma(\vec{\alpha},\vec{\beta})\frac{a_1}{a_2}\left[\frac{1-\beta^{\hat{k}}}{1+\beta^{\hat{k}}}\right]\left[1-\beta^2\right]^{-2}$$
$$+\left[\frac{\dot{a}_2}{a_2} - \frac{\dot{a}_0}{a_2}\left(\frac{1-\beta^{\hat{k}}}{1+\beta^{\hat{k}}}\right)\right]\left[\frac{1-2\beta^{\hat{k}}+\beta^2}{1-\beta^2}\right]\Bigg\} \tag{23}$$

Geometric Theory 7

- For purely radial motion have $\beta^{\hat{k}} = \beta$ and $\alpha^{\hat{k}} = \alpha$, and we obtain the simpler expression

$$\frac{1}{\omega_0}\frac{d\omega_2(t_2)}{dt_2} = -\frac{a_0 a_1}{a_2^2}\left\{2\alpha(1+\beta)^{-3} + \left[\frac{\dot{a}_2}{a_1} - \frac{\dot{a}_0}{a_1}\left(\frac{1-\beta}{1+\beta}\right)\right]\left[\frac{1-\beta}{1+\beta}\right]\right\} \tag{24}$$

- Keeping only quadratic terms in β, linear terms in $H\Delta t$, and also mixed terms $\beta H\Delta t$, we get,

$$\frac{1}{\omega_0}\frac{d\omega_2(t_2)}{dt_2} \approx -\frac{2}{c}\left\{c\alpha\left(1 - 3\beta - 3H\Delta t\right) + Hc\beta\right\} =: -2\, a_*/c \tag{25}$$

where a_* is the naive Doppler-tracking acceleration. Hence in this approximation there are two modifications due to cosmic expansion:

1. a downscaling of acceleration by $(1 - 3H\Delta t)$ $\Rightarrow$ Pioneer: $\Delta a/a < 10^{-12}$
2. a constant contribution $Hc\beta$ in velocity direction $\Rightarrow$ Pioneer: $\Delta a/a < 10^{-7}$

Generalization to McVittie Spacetime

- McVittie's solution describes an isotropic but inhomogeneous situation approaching flat FLRW at large and Schwarzschild small radial distances from the centre of isotropy:

$$g = \left[\frac{1 - m_0/a(t)r}{1 + m_0/a(t)r}\right]^2 c^2 dt^2 - \left[1 + \frac{m_0}{2a(t)r}\right]^4 a^2(t)\left(dr^2 + r^2\, d\Omega^2\right) \qquad (26)$$

- Taking the observer field u parallel to $\partial/\partial t$ (which is not geodesic) we obtain in the same approximation

$$\frac{1}{\omega_0}\frac{d\omega_2(t_2)}{dt_2} \approx -\frac{2}{c}\left\{c\alpha\big(1 - 3\beta - 3H\Delta\tau + (m_0 c/R^2)\Delta\tau\big) + Hc\beta\right\} \qquad (27)$$

Other Coordinates 1

- Instead of the standard co-moving radial coordinate r in FLRW models on may employ the cosmologically simultaneous geodesic distance r_* (here flat case):

$$(t\,,\,r) \mapsto (t_*\,,\,r_*) := (t\,,a(t)r) \tag{28}$$

so that the new field of geodesically equidistant observers $r_* = const.$ is

$$u_* = \frac{1}{\|\partial/\partial t_*\|}\frac{\partial}{\partial t_*} \quad \text{where} \quad \frac{\partial}{\partial t_*} = \frac{\partial}{\partial t} - H(t)r\frac{\partial}{\partial r} \tag{29}$$

- Since u_* is not geodesic (inward accelerated) get additional cosmological acceleration $(\ddot{a}/a)r_*$ in radial direction in Newtonian equation of motion. More general, for geodesics in McVittie spacetime, we obtain to leading order

$$\alpha_{u_*}(\gamma) \approx \left(\frac{\ddot{a}}{a}r_* - \frac{m_0}{r_*^2}\right)\,\vec{e}_r \circ \gamma \tag{30}$$

Other Coordinates 2

- In (t_*, r_*) coordinates, the flat FLRW metric assumes the form

$$g = c^2\left\{1-\left(Hr_*/c\right)^2\right\}\Big\{\underbrace{dt_* + \frac{Hr_*/c^2}{1-\left(Hr_*/c\right)^2}\,dr_*}_{\theta=\text{simultaneity 1-form}}\Big\}^2 - \Big\{\underbrace{\frac{dr_*^2}{1-\left(Hr_*/c\right)^2} + r_*^2\,d\Omega^2}_{h=\text{spatial radar metric}}\Big\} \tag{31}$$

- Radar distance (measured by h) and Einstein simultaneity ($\theta = 0$) are given by

$$l_* = (c/H)\sin^{-1}(Hr_*/c) \approx r_*\{1+\tfrac{1}{6}(Hr_*/c)^2 + \mathcal{O}(3)\} \tag{32}$$

$$\Delta t_* = (1/2H)\ln\left(1-(Hr_*/c)^2\right) \approx (r_*/c)\{-\tfrac{1}{2}(Hr_*/c) + \mathcal{O}(2)\} \tag{33}$$

- Mapping out a trajectory $l_*(t_*)$ in terms of **radar distance of Einstein-simultaneous events** hence means to write

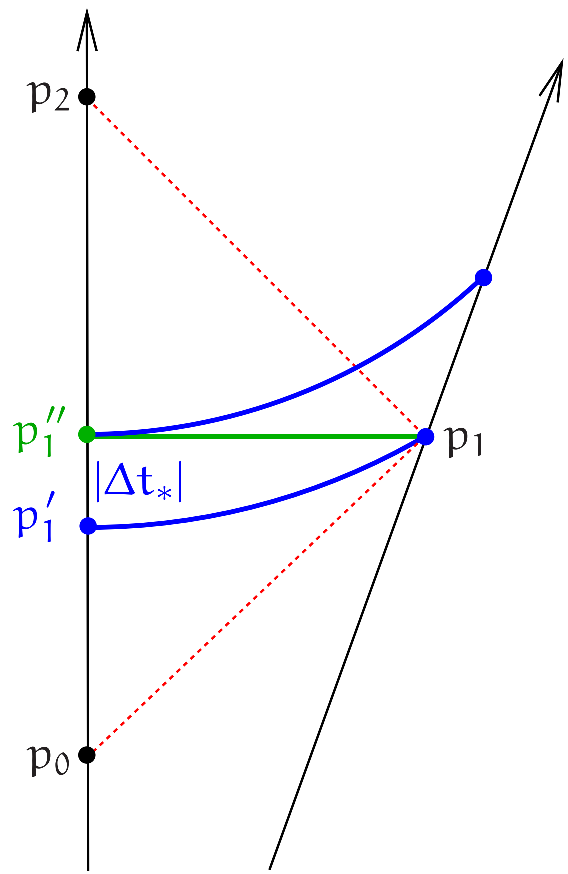
$$l_*(t_*) := (c/H)\sin^{-1}\left(r_*(t_*+\Delta t_*)H/c\right) \approx r_* - \tfrac{1}{2}(v/c)(Hc)(r_*/c)^2 + \cdots \tag{34}$$

which in leading order leads to

$$\ddot{l}_* \approx \ddot{r}_* - (Hc)(v/c)^3 + \cdots \tag{35}$$

Cosmological vs. Einstein Simultaneity

- The reflection-event p_1 lies on the same hypersurface of constant cosmological time $t = t_*$ as the event p_1' on our worldline. However, our eigentime at p_1' is not the mean of our eigentimes at p_0 and p_2. Rather, this is true for the event p_1'', which is hence Einstein-simultaneous with p_1 and which lies to the future of p_1' by the amount $|\Delta t_*|$, given by (33).



Summary

- Derivation of exact double-Doppler-formula for FLRW spacetimes.
- Derivation of approximate double-Doppler-formula for McVittie spacetime.

$\Rightarrow$ There exist no Pioneer-like anomalies due to cosmic expansion.

$\Rightarrow$ Kinematical effects consistently estimated, which e.g. lead to Hc–term at $(v/c)^3$–suppressed level.