The Principle of Equivalence - a very brief introduction -

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Three types of masses

In Newtonian physics we have to distinguish between three types of masses:

1. The **inertial mass** Determines the force, that acts against an imposed acceleration:

$$\vec{F}_{\text{inertial}} = -m_{in}\vec{a}\,. \tag{1}$$

2. The **passive gravitational mass** determines the force, by which a body is acted upon in an external gravitational field $\vec{g}$:

$$\vec{F}_{\text{gravitational}} = m_{pg}\vec{g}\,. \tag{2}$$

3. The **active gravitational mass** determines the gravitational field produced by a body; e.g. outside a spherical mass distribution centred at $\vec{x}\,'$

$$\vec{g}(\vec{x}) = -\,G\,m_{ag}\,\frac{\vec{x}-\vec{x}\,'}{\|\vec{x}-\vec{x}\,'\|^3}\,. \tag{3}$$

The impact of *actio* $=$ *reactio*

$$\vec{F}_{12} = m_{pg}^{(1)}\,\vec{g}_2(\vec{x}_1) = \left(m_{pg}^{(1)}\,m_{ag}^{(2)}\right)\,G\,\frac{\vec{x}_2-\vec{x}_1}{\|\vec{x}_2-\vec{x}_1\|^3} \tag{4}$$

$$\vec{F}_{21} = m_{pg}^{(2)}\,\vec{g}_1(\vec{x}_2) = \left(m_{pg}^{(2)}\,m_{ag}^{(1)}\right)\,G\,\frac{\vec{x}_1-\vec{x}_2}{\|\vec{x}_1-\vec{x}_2\|^3} \tag{5}$$

$$\vec{F}_{12} = -\vec{F}_{21} \Leftrightarrow \frac{m_{pg}^{(1)}}{m_{ag}^{(1)}} = \frac{m_{pg}^{(2)}}{m_{ag}^{(2)}} = \text{universal constant} = 1 \tag{6}$$

- What remains unexplained is the equality of inertial and gravitational mass, i.e. why should

$$\frac{m_i}{m_g} = \text{universal constant?} \tag{7}$$

Weak equivalence principle for pointlike test masses

- The motion of a **pointlike test masses** is determined through by

$$\vec{F}_i + \vec{F}_g = 0 \iff -m_i\,\ddot{x}(t) + m_g\,\vec{g}\big(\vec{x}(t)\big) = 0 \tag{8}$$

- If $m_i = m_g$ this is equivalent to

$$\ddot{\vec{x}}(t) = \vec{g}\big(\vec{x}(t)\big) = -\vec{\nabla}\phi\big(\vec{x}(t)\big) \tag{9}$$

- **Weak equivalence principle:** The motion of a *pointlike test mass* in an external gravitational field depends only on the initial position and velocity

Q1 How small is pointlike?

A1 Much smaller than typical length over which $\vec{g}$ varies appreciably.

Q2 What is a test mass?

A2 No higher multipoles in mass distribution, no charge, no spin, no significant gravitational self-energy (not too small).

Principles of Equivalence

Three formulations of the equivalence principle should be clearly distinguished

1. The **weak equivalence principle (WEP)** states the universality of free fall **(UFF)**, as seen above.
2. The **strong equivalence principle (SEP)** states the universality of free fall also for bodies whose gravitational self-energy is not negligible.
3. The **Einstein equivalence principle (EEP)** states that for all non-gravitational interactions, which do not couple to tidal gravitational fields, the usual laws (special relativistic) hold in a local inertial (freely falling and non rotating) reference frame.

$\Rightarrow$ Geometrisation of gravitational interaction and universal coupling-scheme for interaction between gravity and matter.

$$\eta_{\mu\nu} \mapsto g_{\mu\nu} \qquad \partial_\mu \mapsto \nabla_\mu := \partial_\mu + D_*(\Gamma_\mu) \tag{10}$$

We see EEP as the foundation of the statements that space-time is curved and that gravity and inertia are merely attributes of space-time's geometry.

The EEP canonised

The Einstein Equivalence Principle is usually canonised in the following form (cf. C. Will: Living Reviews 2006):
EEP is equivalent to

- WEP is valid.
- The outcome of any local non-gravitational experiment is independent of the velocity of the freely-falling reference frame in which it is performed.
  $\Rightarrow$ **Local Lorentz invariance (LLI)**.
- The outcome of any local non-gravitational experiment is independent of where and when in the universe it is performed.
  $\Rightarrow$ **Local position invariance (LPI)**.

LPI and redshift -1

- Let there be a static gravitational field $\vec{g} = -g\vec{e}_z$. Assume validity of $EEP - LPI = WEP + LLI$. Then WEP guarantees local existence of freely-falling frame F^3 with coordinates $\{x_f^\mu\}$ whose acceleration is the same as that of test particles:

$$\begin{aligned} ct_f &= (z_s + c^2/g)\,\sinh(gt_s/c)\,, \\ x_f &= x_s\,, \\ y_f &= y_s\,, \\ z_f &= (z_s + c^2/g)\,\coth(gt_s/c)\,. \end{aligned}$$

- LLI guarantees that, *locally*, time measured by, e.g., an atomic clock is proportional to Minkowskian proper length in F^3. If we consider violations of LPI, the constant of proportionality might depend on the space-time point, i.e. via dependence on gravitational potential ϕ:

$$c^2\,d\tau^2 = F^2(\phi)\left[c^2 dt_f^2 - dx_f^2 - dy_f^2 - dz_f^2\right] \quad (11)$$

$$= F^2(\phi)\left[\left(1 + \frac{gz_s}{c^2}\right)^2 c^2 dt_s^2 - dx_s^2 - dy_s^2 - dz_s^2\right]\,. \quad (12)$$

LPI and redshift -2

- The redshift ζ between two identical clocks placed at rest wrt. $\{x^\mu_s\}$ at different heights z_s in the static gravitational field is then given by (all coordinates are x^μ_s now, so we drop the subscript s):

$$\begin{aligned}\zeta := \frac{\nu_{em}-\nu_{rec}}{\nu_{rec}} = \frac{\nu_{em}}{\nu_{rec}} - 1 = \frac{\Delta\tau_{rec}}{\Delta\tau_{em}} - 1 \\ = \frac{F(\phi_{rec})(1+gz_{rec}/c^2)}{F(\phi_{em})(1+gz_{em}/c^2)} - 1\end{aligned}$$

- For small $\Delta z = z_{rec} - z_{em}$ this gives to first order in Δz

$$\boxed{\Delta\zeta = (1+\alpha)\Delta\phi/c^2} \tag{13}$$

where

$$\alpha = \frac{c^2}{g}\left(\vec{e}_z \cdot \vec{\nabla}\ln(F)\right) \tag{14}$$

parametrises the deviation from GR result. α may depend on position, gravitational potential, and the type of clock one is using.

Gravitational binding energy

$$\frac{E_g}{mc^2} \approx \frac{Gm^2/R}{mc^2} = \frac{Gm/c^2}{R} = \frac{\text{Schwarzschild radius}}{\text{geometric radius}}$$

$\approx$	10^{-39}	atomic nucleus
$\approx$	10^{-27}	lab. mass $(10kg,\ 0.1m)$
$\approx$	$2 \cdot 10^{-11}$	Moon
$\approx$	$5 \cdot 10^{-11}$	Earth
$\approx$	10^{-8}	Jupiter
$\approx$	10^{-5}	Sun
$\approx$	0.2	neutron star

Nordtvedt parameter

- Let $\rho(\vec{x})$ be the density of some mass distribution. The Nordtvedt parameter η, measures the violation of $m_g = m_i$ due to gravitational self-energy contributions:

$$\frac{m_g}{m_i} = 1 + \eta \quad \frac{E_g}{mc^2}$$
$$= 1 + \eta \quad \frac{\frac{1}{2}\int d^3x \int d^3y\, G\, \frac{\rho(\vec{x})\rho(\vec{y})}{\|\vec{x}-\vec{y}\|}}{c^2 \int \rho(\vec{x})\, d^3x}$$

- In Post-Newtonian parametrised theories we have

$$\eta = 2\beta - \gamma - 3 - \tfrac{10}{3}\xi - \alpha_1 + \tfrac{2}{3}\alpha_2 - \tfrac{2}{3}\zeta_1 - \tfrac{1}{3}\zeta_3$$

- General Relativity is characterised by

$$\beta = \gamma = 1\,, \text{all other parameters } = 0 \Rightarrow \eta = 0$$

- For comparison: In Brans-Dicke theory we have the non-zero parameters

$$\beta = 1\,, \quad \gamma = \frac{1+\omega}{2+\omega} \quad \Rightarrow \eta = 1 - \gamma = \frac{1}{2+\omega}$$

Equivalence principle(s) and QM

- According to EEP, a homogeneous gravitational field cannot be distinguished from uniform acceleration wrt. an inertial system. The single-particle Schrödinger equation in a homogeneous gravitational field $\vec{g} = -g\vec{e}_z$ is given by

$$i\hbar\partial_t\Psi = \left(-\frac{\hbar^2}{2m_i}\Delta + m_g gz\right)\Psi \tag{15}$$

- Let K be an inertial reference frame without gravitational field. Let K' be constantly accelerated by $\vec{a} = g\vec{e}_z$ relativ to K. Then

$$\vec{x}', t' = \vec{x} - \tfrac{1}{2}gt^2\,, \quad t' = t \tag{16}$$

In terms of $(\vec{x}', t')$ the free one-particle Schrödinger equation is equivalent to

$$i\hbar\partial_{t'}\Psi' = \left(-\frac{\hbar^2}{2m_i}\Delta' + m_i gz'\right)\Psi' \tag{17}$$

where

$$\Psi'(\vec{x}', t') = \Psi(\vec{x}, t)\,\exp\left(-i\,\frac{m_i g}{\hbar}\left(z't' - \tfrac{1}{6}gt'^3\right)\right) \tag{18}$$

$\Rightarrow$ If $m_i = m_g$, evolution of *rays* is identical to (15).