On the influence of global expansion on the dynamics and kinematics of local systems

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Newtonian theory

- Suppose local inertial systems mutually expand from each other according to Hubble's law:

$$\dot{R}(t) = H(t)R(t), \quad \text{where} \quad H(t) := \dot{a}(t)/a(t) \tag{1}$$

- Taking into account $\dot{H} = (\ddot{a}/a) - H^2$, the Hubble acceleration is

$$\ddot{R} = \dot{H}R + H\dot{R} = (\ddot{a}/a)R = -qH^2 R \tag{2}$$

  where $q := \ddot{a}a/\dot{a}^2$ is the usual deceleration parameter

- The Newtonian force is proportional to the acceleration relative to these local inertial frames, so that the Newtonian equations of motion in an expanding universe is obtained from the usual one by replacing

$$\ddot{\vec{x}} \mapsto \ddot{\vec{x}} - (\ddot{a}/a)\vec{x} \tag{3}$$

Newtonian theory (contd.)

- Applied to the motion of a single particle in a radial $1/r^2$ force-field gives (setting $(\ddot{a}/a)_{\mathrm{now}} = -q_0 H_0^2 =: A$)

$$\tfrac{1}{2}\dot{r}^2 + U(r) = E \qquad r^2\dot{\varphi} = L \tag{4}$$

with

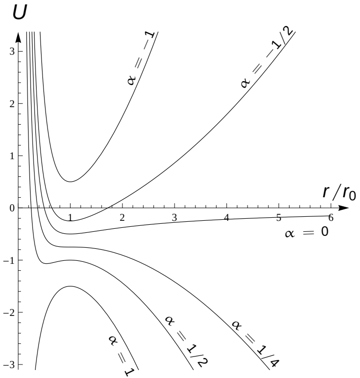
$$U(r) = \frac{L^2}{2r^2} - \frac{C}{r} - \frac{A}{2}r^2 \tag{5}$$

- The critical radius where the attracting force ($C > 0$) is balanced by the cosmological repulsion (for $A > 0$) is

$$r_c = \left[\frac{C}{A}\right]^{1/3} = \begin{cases} \left[R_s R_H^2/(2|q_0)\right]^{1/3} & \approx \; (M/M_\odot)^{1/3}\,400\,\mathrm{ly} \\ \left[R_q R_H^2/(2|q_0)\right]^{1/3} & \approx \; (q/e)^{2/3}\,30\,\mathrm{AU} \end{cases} \tag{6}$$

Here the upper and lower equality holds in case of gravitational and electrical attraction, where $R_s = 2GM/c^2$ and $R_q = 2q^2/mc^2$ respectively.

Effective potential



$$\alpha := (r_0/r_c)^3 = 1 - (L^2/Cr_0)$$

Effective potential for circular orbits for some values of α. The potential has an extremum, which for $\alpha < 1/4$ is a local minimum corresponding to stable circular orbits of radius $r < (1/4)^{1/3}\, r_c$.

Newtonian theory (contd.)

- The effect of cosmic expansion on the circular orbits is as naively anticipated: For accelerated/decelerated expansion the angular frequency at fixed radius is smaller/larger than Keplerian value:

$$\omega = \omega_K \sqrt{1 - \mathrm{sign}(q)(r/r_c)^3} \qquad (7)$$

- Alternatively, for fixed Keplerian angular frequency ω_k, the radius of circular orbits is

$$r = r_k \left(1 + \mathrm{sign}(q)\,(r/r_c)^3 + \cdots\right) \qquad (8)$$

  which is about 1mm for 100 AU orbits around the sun.

Spherical symmetry

Def. A spacetime M is **spherically symmetric** if the isometry group $\mathrm{Isom}(M) \subset \mathrm{Diff}(M)$ contains a subgroup $\cong SO(3)$ whose generic orbits are spacelike 2-spheres.

- In neighbourhoods $U \subset M$ of generic points $U \cong B \times F$ (Base $\times$ Fibre) with $F = S^2$ and projections $\pi_B : U \to B$, $\pi_F : U \to F$.
- The Metric is of **warped product** form

$$g = g_B - (R \circ \pi_B)^2 \, g_F \, , \tag{9}$$

where g_F is the round metric on the unit S^2 and $R : B \to \mathbb{R}_+$ is the **areal radius**.

Hawking mass and Misner-Sharp energy

- In case of spherical symmetry, the **Hawking mass** (defined for any closed spacelike 2D submanifold S)

$$M_H(S) := \sqrt{\frac{\text{Area}(S)}{16\pi}}\left(1 + \frac{1}{2\pi}\int_S \theta^+ \theta^- \, d\mu_S\right) \tag{10}$$

  equals the Misner-Sharp energy

$$E := -\tfrac{1}{2} R^3 \, \mathbf{Sec}(g)\big|_{TS} = \tfrac{1}{2} R\big(1 + g(dR, dR)\big) \, , \tag{11}$$

  if S is a $SO(3)$ orbit of spherical symmetry.
- E can be considered as positive real-valued function on B, i.e. of time and radius.
- Corresponding to $\mathbf{Riem} = \mathbf{Ricci} + \mathbf{Weyl}$, E decomposes as

$$E = E_R + E_W \, . \tag{12}$$

  E_W corresponds to the mass/energy of a black-hole.

Joined Solutions

- Given two spacetimes $(M_\pm, g_\pm)$ with boundaries $\Gamma_\pm$ and diffeomorphism $\phi : \Gamma_+ \rightarrow \Gamma_-$, we can form $M := M_+ \cup_\phi M_-$ and require **Darmois' junction conditions**

$$g_+\big|_{T\Gamma_+} = \phi^* g_-\big|_{T\Gamma_-}\,, \quad K_+ = \phi^* K_-\,. \tag{13}$$

  which state the continuity of the $(\perp, \|)$ and $(\perp, \perp)$ components of the Einstein tensor across Γ and hence, via Einstein's equation, the continuity across Γ of the matter's current densities $\perp S$ for energy and momentum.

- Matching spherically symmetric spacetimes along a curve $\gamma \subset B$ of S^2 orbits requires continuity of:
  1. γ's arc length and curvature;
  2. the areal radius R;
  3. the Misner-Sharp energy.

The Swiss-Cheese Universe

- Join Schwarzschild-DeSitter spacetime

$$g_{SdS} = V(R)\,dT^2 - \big(V(R)\big)^{-1} dR^2 - R^2 g_F \qquad V(R) = 1 - (2m/R) - \tfrac{1}{3}\Lambda R^2 \tag{14}$$

to FLRW Universe

$$g_{FLRW} = dt^2 - a(t)^2 \left(\frac{dr^2}{1 - kr^2} + r^2 g_F \right) . \tag{15}$$

- Have $R_{SdS} = R$ and $R_{FLRW} = a(t)r$. At glueing radius have $R = a(t)r$ and

$$E_{SdS} = m + \tfrac{4\pi}{3} R^3_{SdS}\,\rho_\Lambda = E_{FLRW} = \tfrac{4\pi}{3} R^3_{FLRW}\,(\rho + \rho_\Lambda) \tag{16}$$

$$\Leftrightarrow \quad m = \tfrac{4\pi}{3} R^3 \rho \quad \Leftrightarrow \quad \boxed{r(t) = \left(\frac{m}{(4\pi/3)a^3(t)\rho(t)} \right)^{1/3}} \tag{17}$$

Einstein-Straus

- Einstein & Straus 1945, Schücking 1954:

$$\boxed{R_V(t) = \left(\frac{3m}{4\pi\rho(t)}\right)^{1/3}} \tag{18}$$

Note: R_V is an areal radius, whose definition is that the proper surface-area of the $SO(3)$ orbit is $4\pi R_V^2$. Hence $(4\pi/3)R_V^3$ is the proper volume if $g^{(3)}$ is flat, but smaller/larger if $g^{(3)}$ is of positive/negative curvature.

- Applied to spatially flat universe of critical density $\rho_{\text{crit}} := \frac{3H_0}{8\pi G}$, get

$$R_V = (R_S R_H^2)^{1/3} \approx (m/m_\odot)^{1/3}\, 400\,\mathrm{ly}\,, \tag{19a}$$
$$R_S := (2Gm/c^2) \approx (m/m_\odot)\, 3\,\mathrm{km}\,, \tag{19b}$$
$$R_H := (c/H_0) \approx 4\,\mathrm{Gpc} \approx 1.3\times 10^{23}\,\mathrm{km}\,. \tag{19c}$$

R_S and R_H are the **Schwarzschild radius** for the mass M and the **Hubble radius** respectively.

Amalgamated solutions

- Seek solutions that represent a spherically symmetric inhomogeneity in a FLRW universe. What does that mean?
- Close to inhomogeneity solutions should approximate exterior Schwarzschild metric (possibly filled in by some interior solution with matter)

$$g^{(4)} = \left[\frac{1 - m/2r}{1 + m/2r}\right]^2 dt^2 - \left[1 + \frac{m}{2r}\right]^4 g^{(3)}_{\mathrm{flat}} \tag{20}$$

and far out it should approximate FLRW universe

$$g^{(4)} = dt^2 - a^2(t)\, g^{(3)}_{\mathrm{cc}} \tag{21}$$

- The question is: how to combine these two exact solutions in order to get new exact solution describing a compact object 'immersed' in cosmological background?

McVittie's Ansatz

- In 1933, Mc Vittie suggested the following simple Ansatz in his paper: 'The mass-particle in an expanding universe':

$$g^{(4)} = \left[\frac{1-m(t)/2r}{1+m(t)/2r}\right]^2 dt^2 - a^2(t)\left[1+\frac{m(t)}{2r}\right]^4 g^{(3)}_{\mathrm{cc}} \qquad (22)$$

  where $m(t)$ and $a(t)$ are two positive functions of time t.
- In recent years, solutions within the reach of this Ansatz have often been used to study the impact of cosmological expansion on local inhomogeneities.
- In the following we shall investigate some consequences of this Ansatz. For simplicity we will set

$$g^{(3)}_{\mathrm{cc}} = g^{(3)}_{\mathrm{flat}} = dr^2 + r^2 g^{(2)}_{S^2_1} \qquad (23)$$

- This is based on recent joint work with Matteo Carrera (arXiv: 0908.3101).

Mc Vittie's Ansatz and its implications

- Mc Vittie metrics (parametrised by $a(t)$ and $m(t)$) are conformally Schwarzschild only in the obvious case where $m(t) = m_0 = \text{const.}$.
- Mc Vittie metrics are spatially Ricci-isotropic, that is

$$\mathbf{Ricci}\big|_{dT=0} \propto g\big|_{dT=0}\,, \tag{24}$$

  which renders them special in the class of spherically symmetric metrics.
- The Misner-Sharp energy is given by

$$E = E_R + E_W = \tfrac{1}{6}R^3\mathbf{Ein}(e_0, e_0) + am \tag{25}$$

- $r \to 0$ is a curvature singularity. $r \to m/2$ is also a curvature (Ricci part) singularity except when (1) $m = 0$ (FLRW), (2) m and a constant (Schwarzschild), and (3) $(am)^{\cdot} = (\dot{a}/a)^{\cdot} = 0$ (Schwarzschild-DeSitter).
- For $R_S = 2E_W$ and $R_H = c/H = ca/\dot{a}$ with $R_S \ll R_H$, there are two apparent horizons with radii $R_1 \geq R_S$ and $R_2 \leq R_H$.

Mc Vittie's Ansatz and Einstein's Equations

- Solution strategy: (1) specify a matter model ($T^{\mu\nu}$); (2) specify the function $a(t)$; (3) let Einstein's equation relate matter variables (density, pressure, velocity etc.) and $m(t)$
- This corresponds to the 'poor man's way' to solve E's equation, since it does not start from an equation of state but rather specifies $a(t)$. For example, for ideal fluid moving along integral lines of ∂_t we get

$$(a\,m)^{\dot{}} = 0 \tag{26a}$$

$$8\pi\rho = 3\left(\frac{\dot{a}}{a}\right)^2 \tag{26b}$$

$$8\pi p = -3\left(\frac{\dot{a}}{a}\right)^2 - 2\left(\frac{\dot{a}}{a}\right)^{\dot{}}\left(\frac{1+m/2r}{1-m/2r}\right) \tag{26c}$$

- From (26a) get $m(t) = m_0/a(t)$, where m_0 is the Weyl part of the Misner-Sharp energy. Its constancy is interpreted as saying that no energy flows into the central body.
- It was believed that the curvature singularity at $r = m/2$ was due to this non-accretion condition that seemed to necessitate a diverging pressure. But, as we have seen, it is built into the Ansatz.

Generalising Mc Vittie's solution I

- We keep McVittie's Ansatz and wish to generalise the matter model so as to allow for arbitrary energy infall (Faraoni & Jacques 2007).
- It turns out that this is possible only if radial mass currents as well as non-vanishing heat currents are present, i.e. if

$$T^{\mu\nu} = \rho u^\mu u^\nu + p\,(u^\mu u^\nu - g^{\mu\nu}) + (u^\mu q^\nu + q^\mu u^\nu) \tag{27}$$

with

$$\begin{aligned} u &= \cosh(\chi)\, e_t + \sinh(\chi)\, e_r \\ q &= |q|\,\big(\sinh(\chi)\, e_t + \cosh(\chi)\, e_r\big) \end{aligned} \tag{28}$$

- The condition of spatial Ricci-isotropy of Mc Vittie's imposes, via Einstein's equation, the condition of spatial isotropy for the energy-momentum tensor.
- For $\chi \neq 0$ and $|q| = 0$, spatial Ricci-isotropy requires $p + \rho = 0$ (Λ-like matter). This merely results in Schwarzschild-DeSitter solution.
- For $\chi = 0$ and $|q| \neq 0$, Ricci-isotropy is trivially satisfied and non-trivial solutions exist. Interestingly, they are not the limit of solutions where $\chi \neq 0$ and $|q| \neq 0$, due to Ricci-isotropy constraint, as we shall see.

Generalising Mc Vittie's solution II

- For general χ and q^μ we have for the radial energy current

$$J = -T(e_t, e_r) = \underbrace{q\,(1 + 2\,\sinh^2\chi)}_{J_h} + \underbrace{(\rho + p)\,\sinh\chi\,\cosh\chi}_{J_m} \tag{29}$$

- The constraint of spatial Ricci-isotropy reads

$$J_h\, 2\cosh^2\chi + J_m\,(1 + 2\sinh^2\chi) = 0\,, \tag{30}$$

showing that either $J_h = J_m = 0$ or J_h **and** J_m are non-zero if $\chi \neq 0$. For

- For small rapidities (30) yields $J_m + 2J_h \approx 0$ so that

$$J = J_m + J_h \approx J_m/2 \approx -J_h \tag{31}$$

Hence an outward pointing heat flux cannot balance an inward pointing flow of matter.

- The t-rate of energy accretion by central mass is given by

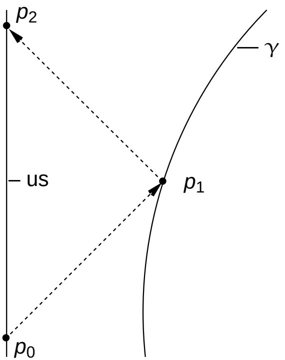
$$(am)^\cdot = (-4\pi R^2\, J)\, g_{tt} \tag{32}$$

Doppler Tracking

- **Doppler Tracking** is a common method of tracking the position of vehicles in space. It involves measuring the Doppler shift of a radio signal sent from a spacecraft to a tracking station on Earth, this signal either coming from an onboard oscillator or being one that the spacecraft has coherently transponded in response to a signal received from the ground station. The second of these modes is more useful for navigation because the returning signal is measured against the same frequency reference as that of the originally transmitted signal. The Earth-based frequency reference is also more stable than the oscillator onboard the spacecraft.
- The following is based on *Matteo Carrera & D.G, CQG **26** (2006) 7483-7492*.

Elementary theory I

- At the event $p_0 = (t_0, \vec{x}_0)$ a radio signal of frequency ω_0 is emitted towards the spacecraft and received by it at event $p_1 = (t_1, \vec{x}_1)$ with frequency ω_1.
- In case of simple **reflection**, a returning radio signal is emitted at $p_1 = (t_1, \vec{x}_1)$ with frequency ω_1 and received by us at event $p_2 = (t_2, \vec{x}_2)$ with frequency ω_2.
- Note: All frequencies refer to those measured by observers that are locally co-moving with the given worldlines ($\gamma =$ worldline of spacecraft)



Elementary theory II

- Given a lightlike vector k (wave vector) and observers u, v at the same spacetime point. The observed frequencies are

$$\omega_v(k) := g(v,k) \qquad \omega_u(k) := g(u,k) \tag{33}$$

whose ratio is given by

$$\begin{aligned} \frac{\omega_v(k)}{\omega_u(k)} &= \frac{g(v,k)}{g(u,k)} = \frac{g(P_u^{\parallel}v + P_u^{\perp}v, k)}{g(u,k)} \\ &= g(u,v)\left[1 + \frac{g(v, P_u^{\perp}k)}{g(u,v)g(u,k)}\right] = g(u,v)\left[1 - \beta_u^{\hat{k}}(v)\right] \end{aligned} \tag{34}$$

where the spacelike unit vector $\hat{k} := P_u^{\perp}k / \|P_u^{\perp}k\|$ defines the direction of k in the rest system of u.

Elementary theory III

- We wish to take the differential quotient of $\omega(t_2)/\omega(t_0)$ with respect to t_2, assuming a constant function ω_0. We get

$$\frac{\dot{\omega}(t_2)}{\omega_0} = \frac{-2\dot{\beta}(t_1)}{\big(1+\beta(t_1)\big)^2}\frac{dt_1}{dt_2} \tag{35}$$

- If we are resting at the origin of the coordinate system, with respect to which r is the radial distance to the spacecraft, we have $t_2 - t_1 = r(t_1)/c$ and therefore

$$1 - \frac{dt_1}{dt_2} = \frac{1}{c}\frac{dr(t_1)}{dt_1}\frac{dt_1}{dt_2} \Longleftrightarrow \frac{dt_1}{dt_2} = \frac{1}{1+\beta(t_1)} \tag{36}$$

- Hence (35) becomes ($\dot{\beta} \equiv \alpha$)

$$\frac{\dot{\omega}(t_2)}{\omega_0} = \frac{-2\dot{\beta}(t_1)}{\big(1+\beta(t_1)\big)^3} \approx -2\alpha(t_1)\Big(1 - 3\beta(t_1) + \cdots\Big) \tag{37}$$

Geometric Theory 1

- In a general spacetime (M, g) [we use signature $(+,-,-,-)$ for g] there is no privileged (e.g. inertial) global reference frame by means of which we may introduce kinematical variables that characterise worldlines (different ones collectively). Hence 'appropriate' fiducial observer-fields need to be introduced.
- An **observer** at the event p is a future pointing unit timelike vector. An **observer field** is a field of observers. Any observer u at p gives rise to an orthogonal split of the tangent space at p, $T_p(M) = T_p^{\parallel}(M) \oplus T_p^{\perp}(M)$, where

$$T_p^{\parallel}(M) := \mathrm{Span}\{u\}\,, \quad T_p^{\perp}(M) := \{v \in T_p(M) \mid g(v, u) = 0\} \tag{38}$$

The associated projection operators are given by

$$P_u^{\parallel} : T_p \to T_p^{\parallel}(M)\,, \quad v \mapsto P_u^{\parallel}(v) := u\, g(u, v) \tag{39a}$$

$$P_u^{\perp} : T_p \to T_p^{\perp}(M)\,, \quad v \mapsto P_u^{\perp}(v) := v - u\, g(u, v) \tag{39b}$$

Geometric Theory 2

- If two observers u and v are defined at the same point, the relative velocity (over c) of v with respect to u is given by (we write $\|v\| := \sqrt{|g(v,v)|}$)

$$\vec{\beta}_u(v) := \frac{P_u^\perp(v)}{\|P_u^\parallel(v)\|} \in T^\perp(M) \quad \text{and} \quad \beta_u(v) := \|\vec{\beta}_u(v)\| \tag{40}$$

where

$$g(u,v) = 1/\sqrt{1-\beta_u^2(v)} \qquad (\rightarrow \text{`gamma-factor'}) \tag{41}$$

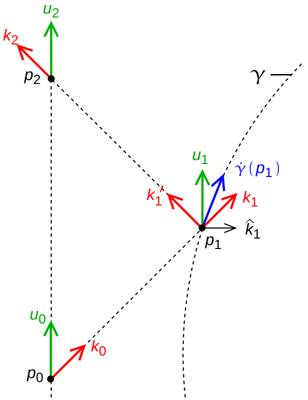
- Note that $\beta_u(v) = \beta_v(u)$, though $\vec{\beta}_u(v)$ and $\vec{\beta}_v(u)$ are linearly independent: they lie in $P_u^\perp T(M)$ and $P_v^\perp T(M)$ respectively.
- Let $e \in P_u^\perp T(M)$ be a (spacelike) unit vector. We define the e-component of v's velocity relative to u by

$$\beta_u^e(v) = -g(e, \vec{\beta}_u(v)) = -\frac{g(e,v)}{g(u,v)} \tag{42}$$

Geometric Theory III

- Let u be an observer **field** along one integral line of which we are moving. As before, γ is the worldline of the spacecraft. The field u is defined in a neighbourhood of γ.
- The wave-vector k_0 emitted at p_0 suffers three changes:
  1. propagation from p_0 to p_1:
     $k_0 \rightarrow k_1$
  2. reflection at p_1:
     $k_1 \rightarrow k_1'$
  3. propagation from p_1 to p_2:
     $k_1' \rightarrow k_2$
- We are interested in

$$\frac{\omega_2}{\omega_0} = \frac{g(u_2, k_2)}{g(u_0, k_0)} = \left[\frac{\omega_2}{\omega_1'}\right]\left[\frac{\omega_1'}{\omega_1}\right]\left[\frac{\omega_1}{\omega_0}\right]$$



Geometric Theory IV

- The exact formula in FLRW spacetime for the t_2–derivative of the frequency-shift rate is now given by

$$\frac{1}{\omega_0}\frac{d\,\omega_2(t_2)}{dt_2} = -\frac{a_0}{a_1}\Bigg\{2\big[\alpha^{\hat{k}} + \sigma(\beta, \nabla^{u}_{\dot{\gamma}}\hat{k})\big]\frac{a_1}{a_2}\big[1+\beta^{\hat{k}}\big]^{-1}\big[1-\beta^2\big]^{-1}$$

$$-4\sigma(\vec{\alpha},\vec{\beta})\frac{a_1}{a_2}\left[\frac{1-\beta^{\hat{k}}}{1+\beta^{\hat{k}}}\right]\big[1-\beta^2\big]^{-2}$$

$$+\left[\frac{\dot{a}_2}{a_2}-\frac{\dot{a}_0}{a_2}\left(\frac{1-\beta^{\hat{k}}}{1+\beta^{\hat{k}}}\right)\right]\left[\frac{1-2\beta^{\hat{k}}+\beta^2}{1-\beta^2}\right]\Bigg\}$$

FLRW

- For purely radial motion have $\beta^{\hat{k}} = \beta$ and $\alpha^{\hat{k}} = \alpha$, and we obtain the simpler expression

$$\frac{1}{\omega_0}\frac{d\,\omega_2(t_2)}{dt_2} = -\frac{a_0 a_1}{a_2^2}\left\{2\alpha(1+\beta)^{-3} + \left[\frac{\dot{a}_2}{a_1} - \frac{\dot{a}_0}{a_1}\left(\frac{1-\beta}{1+\beta}\right)\right]\left[\frac{1-\beta}{1+\beta}\right]\right\} \tag{43}$$

- Keeping only quadratic terms in β, linear terms in $H\Delta t$, and also mixed terms $\beta H\Delta t$, we get,

$$\frac{1}{\omega_0}\frac{d\,\omega_2(t_2)}{dt_2} \approx -\frac{2}{c}\left\{c\,\alpha\big(1-3\beta-3H\Delta t\big) + Hc\,\beta\right\} =: -2\,a_*/c \tag{44}$$

where a_* is the naive Doppler-tracking acceleration. Hence in this approximation there are two modifications due to cosmic expansion:

1. a downscaling of acceleration by $(1-3H\Delta t)$
   $\Rightarrow$ Pioneer: $\Delta a/a < 10^{-12}$
2. a constant contribution $Hc\,\beta$ in velocity direction
   $\Rightarrow$ Pioneer: $\Delta a/a < 10^{-7}$

Mc Vittie

- Mc Vittie's solution describes an isotropic but inhomogeneous situation approaching flat FLRW at large and Schwarzschild small radial distances from the centre of isotropy:

$$g = \left[\frac{1 - m_0/a(t)r}{1 + m_0/a(t)r}\right]^2 c^2 dt^2 - \left[1 + \frac{m_0}{2a(t)r}\right]^4 a^2(t)\left(dr^2 + r^2 d\Omega^2\right) \quad (45)$$

- Taking the observer field u parallel to $\partial/\partial t$ (which is not geodesic) we obtain in the same approximation

$$\frac{1}{\omega_0}\frac{d\omega_2(t_2)}{dt_2} \approx -\frac{2}{c}\left\{c\alpha\left(1 - 3\beta - 3H\Delta\tau + (m_0 c/R^2)\Delta\tau\right) + Hc\beta\right\} \quad (46)$$

Other Coordinates 1

- Instead of the standard co-moving radial coordinate r in FLRW models on may employ the cosmologically simultaneous geodesic distance r_* (here flat case):

$$(t\,,\,r)\mapsto(t_*\,,\,r_*):=(t\,,a(t)r) \tag{47}$$

so that the new field of geodesically equidistant observers $r_* = const.$ is

$$u_* = \frac{1}{\|\partial/\partial t_*\|}\,\frac{\partial}{\partial t_*} \quad \text{where} \quad \frac{\partial}{\partial t_*} = \frac{\partial}{\partial t} - H(t)r\,\frac{\partial}{\partial r} \tag{48}$$

- Since u_* is not geodesic (inward accelerated) get additional cosmological acceleration $(\ddot{a}/a)r_*$ in radial direction in Newtonian equation of motion. More general, for geodesics in McVittie spacetime, we obtain to leading order

$$\alpha_{u_*}(\gamma) \approx \left(\frac{\ddot{a}}{a}r_* - \frac{m_0}{r_*^2}\right)\vec{e}_r \circ \gamma \tag{49}$$

Other Coordinates 2

In (t_*, r_*) coordinates, the flat FLRW metric assumes the form

$$g = c^2\Big\{1-(Hr_*/c)^2\Big\}\Big\{\underbrace{dt_* + \frac{Hr_*/c^2}{1-(Hr_*/c)^2}\,dr_*}_{\theta=\text{simultaneity 1-form}}\Big\}^2 - \Big\{\underbrace{\frac{dr_*^2}{1-(Hr_*/c)^2} + r_*^2\,d\Omega^2}_{h=\text{spatial radar metric}}\Big\}$$

- Radar distance (measured by h) and Einstein simultaneity ($\theta = 0$) are given by

$$l_* = (c/H)\sin^{-1}(Hr_*/c) \approx r_*\{1 + \tfrac{1}{6}(Hr_*/c)^2 + \mathcal{O}(3)\} \quad (50)\ (52)$$

$$\Delta t_* = (1/2H)\ln\big(1-(Hr_*/c)^2\big) \approx (r_*/c)\{-\tfrac{1}{2}(Hr_*/c) + \mathcal{O}(2)\} \quad (51)$$

- Mapping out a trajectory $l_*(t_*)$ in terms of **radar distance of Einstein-simultaneous events** hence means to write

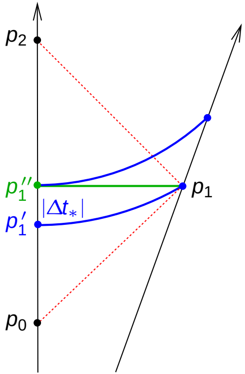
$$l_*(t_*) := (c/H)\sin^{-1}\big(r_*(t_* + \Delta t_*)H/c\big) \approx r_* - \tfrac{1}{2}(v/c)(Hc)(r_*/c)^2 + \cdots$$

which in leading order leads to

$$\ddot{l}_* \approx \ddot{r}_* - (Hc)(v/c)^3 + \cdots \tag{53}$$

Cosmological versus Einstein Simultaneity

- The reflection-event p_1 lies on the same hypersurface of constant cosmological time $t = t_*$ as the event p_1' on our worldline. However, our eigentime at p_1' is not the mean of our eigentimes at p_0 and p_2. Rather, this is true for the event p_1'', which is hence Einstein-simultaneous with p_1 and which lies to the future of p_1' by the amount $|\Delta t_*|$, given by (51).



Summary

- Derivation of exact double-Doppler-formula for FLRW spacetimes.
- Derivation of approximate double-Doppler-formula for Mc Vittie spacetime.

⇒ There exist no Pioneer-like anomalies due to cosmic expansion.

⇒ Kinematical effects consistently estimated, which e.g. lead to Hc–term at $(v/c)^3$–suppressed level.

THANK YOU!