# Elementary considerations concerning the relation between Quantum Mechanics & Gravity


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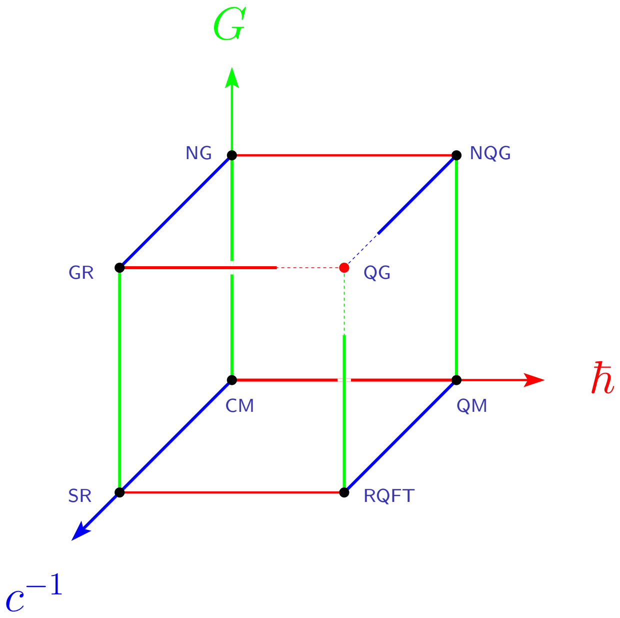
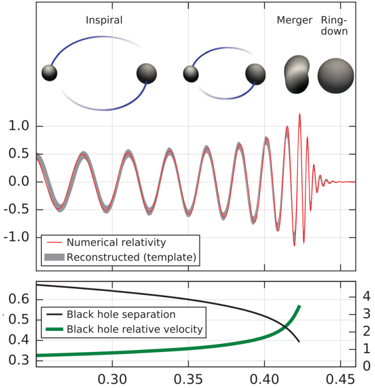
ESI Vienna, June 6th 2017

# The magic cube



# Gravitational waves




- The direct detection of gravitational waves should be comapred to Heinrich Hertz' 1888 detection of causally propagating electromagnetic waves (though he also produced them) carrying energy. Gravity is no longer a mere attribute of matter, as in Newtonian gavity. This endows the issue of "quantisation" with a proper field-theoretic meaning.

# Some history

- Einstein first mentioned gravitaional waves in his paper *Approximative Integration of the Field Equations of Gravitation*, submitted to the Prussian Academy of Science on June 22., 1916. The last sentence in that paper reads as follows:

  *"To be sure, as a consequence of its inner motion of the electrons, an atom would not only emit electromagnetic but also gravitational energy, even if only in tiny amounts. As this may in truth not apply to nature, it seems that Quantum Theory will not only modify Maxwellian electrodynamics, but also the new thoery of gravitation."*

  Einstein repeats his statement almost verbatim in his second paper on gravitational waves: "On Gravitational Waves" of January 31., 1918.

- The graviton emission-rate in hydrogen $\Gamma_{\text{grav}}(3d \to 1s)$ may easily be calculated to leading-order approximation. The lifetim is $\tau \approx 0.5 \cdot 10^{32}\,\text{yr}$.

- Averaged graviton-absorption cross-sections for gravitons by atoms have been estimated (Dyson 2012) to be $\approx 10^{-64}\,\text{cm}^2$, i.e. $10^{-41}\,\text{cm}^2$ per gramm of matter. The thermal graviton luminosity of the sun is estimated at $79\,\text{MW}$ (Weinberg 1965), corresponding to 4 gravitons absorbed by the entire mass of the earth over the lifetime (5 billion yrs.) of the sun.

# Gravitational waves versus gravitons

- A classical gravitational wave with amplitude $f$ ("strain") and angular frequency $\omega$ carries an energy density of

$$\begin{aligned} W &= \frac{c^2}{32\pi G}\,\omega^2 f^2 \\ W[\mathrm{erg}\cdot\mathrm{cm}^{-3}] &= 10^{32}\cdot\omega^2[\mathrm{kHz}]\cdot f^2 \\ &= 10^{-10} \quad \text{for kHz wave and } f = 10^{-21} \end{aligned} \tag{1}$$

- A graviton of angular frequency $\omega$ contains energy $\hbar\omega$ in volume $\lambda\!\!\!^{-}{}^3$, hence energy density

$$\begin{aligned} \hat{W} &= \frac{\hbar\omega}{\lambda\!\!\!^{-}{}^3} = \frac{\hbar\omega^4}{c^3} \\ \hat{W}[\mathrm{erg}\cdot\mathrm{cm}^{-3}] &= 3\times 10^{-47}\cdot\omega^4[\mathrm{kHz}] \end{aligned} \tag{2}$$

- The ratio is

$$\begin{aligned} \frac{W}{\hat{W}} &= 3\times 10^{78}\cdot\left(\frac{f}{\omega}\right)^2 \\ &= 3\times 10^{36}\cdot\omega^{-2}[\mathrm{kHz}] \quad \text{for } f = 10^{-21} \end{aligned} \tag{3}$$

Single graviton detection at $10^{-21}$ strain-level need $\omega \approx 10^{21}\mathrm{Hz}$.

# Gravitons and Planck scale

- Strain $f$ is relative distance $\Delta D/D$ if $D \leq \lambdabar$. The most effective absolute distance change of a graviton is

$$\Delta D = f \cdot \lambdabar \tag{4}$$

- The strain of a graviton is obtained by assuming validity of (1) for $\hat{W}$ given by (2):

$$f = \sqrt{\frac{32\pi G\hat{W}}{c^2\omega^2}} = \sqrt{32\pi} \cdot \sqrt{\frac{GM}{c^2}} \cdot \left(\frac{\omega}{c}\right) \approx 10\frac{L_p}{\lambdabar} \tag{5}$$

- Hence (4) tells us that the absolute length-change caused by a single graviton is at best

$$\boxed{\Delta D = f \cdot \lambdabar \approx 10 \cdot L_P} \tag{6}$$

which is independent of frequency.
- But can we ever meaningfully detect length changes of the ordewr of $L_P \approx 1.6 \cdot 10^{-33}\,\mathrm{cm}$ ?

# Planck scale and black holes

- Recall that, for any mass $M$, the geometric mean of its associated Compton wavelength $\lambdabar_M$ (reduced) and Schwarzschild radius $R_M$ is the universal Planck length:

$$L_P^2 = \frac{G\hbar}{c^3} = \left(\frac{\hbar}{Mc}\right) \cdot \left(\frac{GM}{c^2}\right) = \lambdabar_M \cdot R_M \qquad (7)$$

- From $\Delta p \cdot \Delta q \geq \hbar$ get with $\Delta p = M\Delta q/\Delta t$

$$(\Delta q)^2 \geq \frac{\hbar}{M} \cdot \Delta t = \lambdabar_M \cdot c\Delta t \qquad (8)$$

- Resolving $L_P$ implies $L_P \geq \Delta q$; hence (7) and (8), together with causality-requirement $c\Delta t \geq D$ imply imply

$$R_M \geq c\Delta t \geq D \qquad (9)$$

The system is a black hole!?

- Note: A black hole of mass below the Planck mass $M_P = \sqrt{\hbar c/G} \approx 2 \cdot 10^{-5}\,\mathrm{g}$ has a Schwarzschild radius below its Compton wavelength. It's not clear what "black-hole" (a genuine classical notion) is then supposed to mean.

# "On Quantization of Fields" (1963)


Léon Rosenfeld (1904-1974)

- "'It is nice to have at one's disposal such exquisite mathematical tools as the present methods of quantum field theory, but one should not forget that these methods have been elaborated in order to describe definite empirical situations, in which they find their only justification. Any question as to their range of application can only be answered by experience, not by formal argumentation. Even the legendary Chicago machine cannot deliver the sausages if it is not supplied with hogs."'

# Old hopes: Gravity as regulator

- Consider thin mass shell of Radius $R$, inertial rest-mass $M_0$, gravitational mass $M_g$, and electric charge $Q$. Its total energy is

$$E = M_0 c^2 + \frac{Q^2}{2R} - G\frac{M_g^2}{2R} \tag{10}$$

- Now use the following two principles:

$$\begin{aligned} E &= M_i c^2 \\ M_g &= M_i \end{aligned} \tag{11}$$

- Get quadratic equation for mass $M := M_i = M_g$:

$$\Rightarrow \quad M := \frac{E}{c^2} = M_0 + \frac{Q^2}{2c^2 R} - G\frac{M^2}{2c^2 R} \tag{12}$$

# Gravity as regulator (contd.)

- The solution is

$$M(R) = \frac{Rc^2}{G}\left\{-1 + \sqrt{1 + \frac{2G}{Rc^2}\left(M_0 + \frac{Q^2}{2c^2 R}\right)}\ \right\} \tag{13}$$

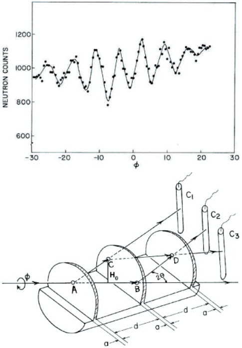
- Its $R \to 0$ limit exits

$$\lim_{R\to O} M(R) = \sqrt{\frac{2Q^2}{G}} = \sqrt{2\alpha}\cdot\frac{|Q|}{e}\cdot M_{\text{Planck}} \tag{14}$$

but its small-G approximation is not uniform in $R$ at $R = 0$ (i.e. diverges at all orders or perturbation theory in $G$):

$$\begin{aligned} M \ &= \left(m_0 + \frac{Q^2}{2c^2 R}\right) \\ &+ \sum_{n=1}^{\infty}\frac{(2n-1)!!}{(n+1)!}\cdot\left(-\frac{G}{Rc^2}\right)^n\cdot\left(m_0 + \frac{Q^2}{2c^2 R}\right)^{n+1} \end{aligned} \tag{15}$$

# QM & Gravity: Tested so far




Colella Overhauser Werner, PRL 1975

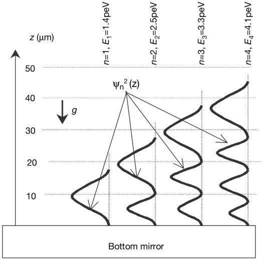



**Figure 1** Wavefunctions of the quantum states of neutrons in the potential well formed by the Earth's gravitational field and the horizontal mirror. The probability of finding neutrons at height $z$, corresponding to the $n$th quantum state, is proportional to the square of the neutron wavefunction $\psi_n^2(z)$. The vertical axis $z$ provides the length scale for this phenomenon. $E_n$ is the energy of the $n$th quantum state.

Nesvizhevsky et al., Nature 2002

$$i\hbar\dot{\Psi} = -\frac{\hbar^2}{2m_i}\Delta\Psi + V_{\text{grav}}\Psi$$

$$V_{\text{grav}} = m_g g z$$

How do you derive this from first principles?

# Einstein's Equivalence Principle (EEP)

- **Universality of Free Fall (UFF):** "Test bodies" determine path structure on spacetime (not necessarily of Riemannian type). UFF-violations are parametrised by the Eötvös factor

$$\eta(A,B) := 2\,\frac{|a(A)-a(B)|}{|a(A)+a(B)|} \tag{16}$$

- **Local Lorentz Invariance (LLI):** Local non-gravitational experiments exhibit no preferred directions in spacetime, neither timelike nor spacelike. Possible violations of LLI concern, e.g., variations in $\Delta c/c$.

- **Universality of Gravitational Redshift (UGR):** "Standard clocks" are universally affected by the gravitational field. UGR-violations are parametrised by the $\alpha$-factor

$$\frac{\Delta\nu}{\nu} = (1+\alpha)\,\frac{\Delta U}{c^2} \tag{17}$$

# Consequences and difficulties of the equivalence principle

- Gravity can be geometrised and hence ceases to be a force (in the Newtonian sense). This only works if *all* dynamical aspects of gravity can be encoded in space-time geometry and if *all* matter components see the *same* geometry to which they *universally* couple.
- This universal coupling scheme translates to special-relativistic (Poincaré invariant) field theories, but not in an obvious fashion to "non-relativistic" (Galilei invariant) Quantum Mechanics.
- Three approaches come to mind: Redo "Schr"odinger Quantisation" for relativistic particles in curved spacetime in a post-newtonian expansion (thus also taking account of vector- and tensor parts of Einsteinian g-field), or 2) derive post-newtonian expansions of relativistic field equations (Klein-Gordan, Dirac, etc.), or 3) start from QFT in cureved spacetime.
- Unless all this is understood much better, there is no obvious meaning to "Quantum tests of *the* equivalence principle. The many confusions in recent years on various claims concerning such "quantum-tests" reflect the variation of such meanings and the absence of hard criteria to compate them.

# UFF – UGR dependence: Energy conservation

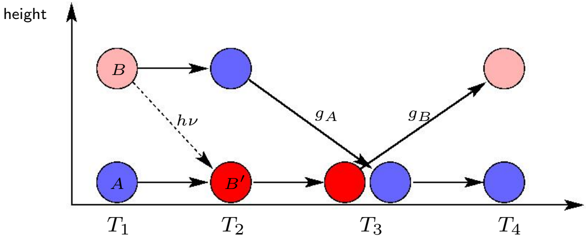
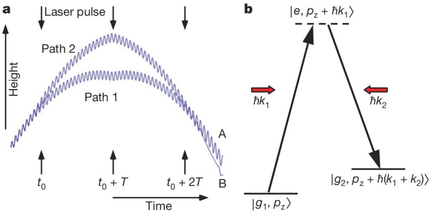



Figure: Gedankenexperiment by NORDTVEDT to show that energy conservation connects violations of UFF and UGR. Considered are two copies of a system that is capable of 3 energy states $A$, $B$, and $B'$ (blue, pink, and red), with $E_A < E_B < E_{B'}$. Initially system 2 is in state $B$ and placed a height $h$ above system 1 which is in state $A$. At time $T_1$ system 2 makes a transition $B \to A$ and sends out a photon of energy $h\nu = E_B - E_A$. At time $T_2$ system 1 absorbs this photon, which is now blue-shifted, and makes a transition $A \to B'$. At $T_3$ system 2 has been dropped from height $h$ with acceleration $g_A$, has hit system 1 inelastically, leaving one system in state $A$ and at rest, and the other system in state $B$ with an upward motion with kinetic energy $E_{\text{kin}} = M_A g_A h + (E_{B'} - E_B)$. The latter motion is decelerated by $g_B$, which may differ from $g_A$. At $T_4$ the system in state $B$ has climbed to the **same** height $h$ by energy conservation. Hence have $E_{\text{kin}} = M_B g_B h$ and therefore $M_A g_A h + M_{B'} c^2 = M_B c^2 + M_B g_B h$, from which we get

$$\frac{\delta\nu}{\nu} = \frac{(M_{B'} - M_A) - (M_B - M_A)}{M_B - M_A} = \frac{g_B h}{c^2}\left[1 + \frac{M_A}{M_B - M_A}\,\frac{g_B - g_A}{g_B}\right] \tag{18a}$$

$$\Rightarrow \alpha = \frac{M_A}{M_B - M_A}\,\frac{g_B - g_A}{g_B} =: \frac{\delta g/g}{\delta M/M} \tag{18b}$$

# An alleged $10^4$-improvement of UGR-tests: What is a clock?




(Müller *et al.*, Nature 2010)

Have (using $k := \Delta p/\hbar$)

$$\Delta\phi = k\,T^2 \cdot g^{(\mathrm{Cs})} = k\,T^2 \cdot \frac{m_g^{(\mathrm{Cs})}}{m_i^{(\mathrm{Cs})}} \cdot g^{\mathrm{Earth}}$$

$$= k\,T^2 \cdot \frac{m_g^{(\mathrm{Cs})}}{m_i^{(\mathrm{Cs})}} \cdot \frac{m_i^{(\mathrm{Ref})}}{m_g^{(\mathrm{Ref})}} \cdot g^{(\mathrm{Ref})} = \eta(\mathrm{Cs}, \mathrm{Ref}) \cdot kT^2 \cdot g^{(\mathrm{Ref})} \tag{19}$$

- Proportional to (1+Eötvös-factor) in UFF-violating theories.
- $\mathbb{Q}$ How does it depend on $\alpha$ in UGR-violating theories? Müller *et al.* argue for $\propto (1+\alpha)$ by *representation dependent* interpretation of $\Delta\phi$ as a mere redshift.

# The "clocks-from-rocks" dispute

- A clock ticking at frequency $\omega$ suffers gravitational phase-shift in Kasevich-Chu situation of

$$\begin{aligned}\Delta\phi &= \Delta\omega T \\ &= \omega\frac{\Delta U}{c^2}T \\ &= \omega\frac{g\,\Delta h}{c^2}T \\ &= \omega\frac{g\,\Delta p}{mc^2}\,T^2 \\ &= \left(\frac{\omega}{mc^2/\hbar}\right)\,g\,T^2\,\frac{\Delta p}{\hbar}\end{aligned} \tag{20}$$

This equals (19) if

$$\omega = mc^2/\hbar \tag{21}$$

- Objection!

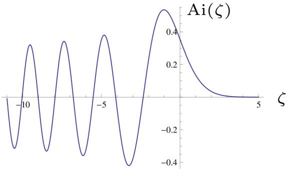
# Homogeneous static gravitational field: Bound states

- Time independent Schrödinger equation in linear potential $V(z) = m_g g z$ is equivalent to:

$$\left(\frac{d^2}{d\zeta^2} - \zeta\right)\psi = 0\,, \quad \zeta := \kappa z - \varepsilon \tag{22}$$

where

$$\kappa := \left[\frac{2m_i\, m_g\, g}{\hbar^2}\right]^{\frac{1}{3}}\,, \quad \varepsilon := E\cdot\left[\frac{2m_i}{m_g^2\, g^2\, \hbar^2}\right]^{\frac{1}{3}} \tag{23}$$



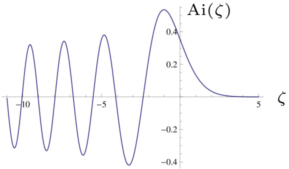

- Complement by hard (horizontal) wall $V(z) = \infty$ for $z \leq 0$ get energy eigenstates from boundary condition $\psi(z=0) = 0$, hence $\varepsilon = -z_n$:

$$E(n) = -z_n\left[\frac{m_g^2}{m_i}\cdot\frac{g^2\hbar^2}{2}\right]^{\frac{1}{3}} \tag{24}$$

# Two different masses: Homogeneous static gravitational field
# Equivalence principle and free-fall time

- Classical turning point $z_{\text{turn}}$

$$m_g g\, z_{\text{turn}} = E \Leftrightarrow z_{\text{turn}} = \frac{E}{m_g g} = \frac{\varepsilon}{\kappa} \Leftrightarrow \zeta = 0\,. \tag{25}$$




- Large $(-\zeta)$ - expansion of Airy function gives decomposition of ingoing and outgoing waves with phase delay of

$$\Delta\theta(z) = \frac{4}{3}\Big[\kappa\big(E/m_g g - z\big)\Big]^{3/2} - \pi/2 \tag{26}$$

corresponding to a "Peres time of flight" (Davies 2004)

$$T(z) := \hbar\frac{\partial\Delta\theta}{\partial E} = 2\,\frac{\hbar\kappa^{\frac{3}{2}}}{m_g g}\sqrt{z_{\text{turn}} - z} = 2\sqrt{\frac{m_i}{m_g}}\cdot\sqrt{2\cdot\frac{z_{\text{turn}} - z}{g}} \tag{27}$$

- For other than linear potential we will *not* get *classical* return time.

# UFF in QM

The following proposition states precisely the extent to which UFF is valid within QM.

- We consider a particle of mass $m$ in spatially homogeneous force field $\vec{F}(t)$. The classical trajectories solve

$$\ddot{\vec{\xi}}(t) = \vec{F}(t)/m \tag{28}$$

  Let $\xi(t)$ denote a solution with $\vec{\xi}(0) = \vec{0}$ and some initial velocity. Its flow-map $\Phi : \mathbb{R}^4 \to \mathbb{R}^4$ defines a *freely-falling frame*:

$$\Phi(t, \vec{x}) = \bigl(t, \vec{x} + \xi(t)\bigr)\,. \tag{29}$$

- **Proposition:** $\psi$ solves the forced Schrödinger equation

$$i\hbar\partial_t\psi = \left(-\frac{\hbar^2}{2m_i}\Delta - \vec{F}(t)\cdot\vec{x}\right)\psi \tag{30}$$

  iff

$$\psi = \bigl(\exp(i\alpha)\,\psi'\bigr)\circ\Phi^{-1}\,, \tag{31}$$

  where $\psi'$ solves the free Schrödinger equation and

$$\alpha(t,\vec{x}) = \frac{m_i}{\hbar}\left\{\dot{\vec{\xi}}(t)\cdot\bigl(\vec{x}+\vec{\xi}(t)\bigr) - \frac{1}{2}\int^t dt'\,\|\dot{\vec{\xi}}(t')\|^2\right\}\,. \tag{32}$$

- Where does the phase come from?

# Schrödinger-Newton equation

- Consider Einstein – Klein-Gordon system

$$R_{ab} - \tfrac{1}{2} g_{ab} R = \tfrac{8\pi G}{c^4}\, T^{KG}_{ab}(\phi)\,, \qquad (\Box_g + m^2)\phi = 0 \tag{33}$$

- Make WKB-like ansatz

$$\phi(\vec{x},t) = \exp\left(\frac{ic^2}{\hbar} S(\vec{x},t)\right) \sum_{n=0}^{\infty} \left(\frac{\sqrt{\hbar}}{c}\right)^n a_n(\vec{x},t), \tag{34}$$

  and perform $1/c$ expansion (D.G. & A. Großardt 2012).

- Obtain

$$i\hbar\partial_t\psi = \left(-\frac{\hbar^2}{2m}\Delta + mV\right)\psi \tag{35}$$

  where

$$\Delta V = 4\pi G\left(\rho + m|\psi|^2\right). \tag{36}$$

- Ignoring self-coupling, this just generalises previous results and conforms with expectations.

# Schrödinger-Newton equation

- Without external sources get **"Schrödinger-Newton equation"** (Diosi 1984, Penrose 1998):

$$\boxed{i\hbar\,\partial_t\psi(t,\vec{x}) = \left(-\frac{\hbar^2}{2m}\Delta - Gm^2\int\frac{|\psi(t,\vec{y})|^2}{\|\vec{x}-\vec{y}\|}\,d^3y\right)\psi(t,\vec{x})} \qquad (37)$$

- It can be derived from the action

$$\begin{aligned}\mathcal{S}[\psi,\psi^*] = \int dt\biggl\{ &\frac{i\hbar}{2}\int d^3x\Bigl(\psi^*(t,\vec{x})\dot{\psi}(t,\vec{x}) - \psi(t,\vec{x})\dot{\psi}^*(t,\vec{x})\Bigr)\\ &-\frac{\hbar^2}{2m}\int d^3x\,(\vec{\nabla}\psi(t,\vec{x}))\cdot(\vec{\nabla}\psi^*(t,\vec{x}))\\ &+\frac{Gm^2}{2}\iint d^3x\,d^3y\,\frac{|\psi(t,\vec{x})|^2\,|\psi(t,\vec{y})|^2}{\|\vec{x}-\vec{y}\|}\biggr\}. \end{aligned} \qquad (38)$$

# SNE: Dimensionless form

- Introducing a length-scale $\ell$ we can use dimensionless coordinates

$$\vec{x}' := \vec{x}/\ell\,, \qquad t' := t \cdot \frac{\hbar}{2m\ell}\,, \qquad \psi' = \ell^{3/2}\psi \tag{39}$$

and rewrite the SNE as

$$i\,\partial_{t'}\psi'(t',\vec{x}') = \left(-\Delta' - K\int \frac{|\psi'(t',\vec{y}')|^2}{\|\vec{x}'-\vec{y}'\|}\,d^3y'\right)\psi'(t',\vec{x}')\,, \tag{40}$$

with dimensionless coupling constant

$$K := 2\cdot\frac{Gm^3\ell}{\hbar^2} = 2\cdot\left(\frac{\ell}{\ell_P}\right)\left(\frac{m}{m_P}\right)^3 \approx 6\cdot\left(\frac{\ell}{100\,\mathrm{nm}}\right)\left(\frac{m}{10^{10}\,\mathrm{u}}\right)^3 \tag{41}$$

- Here we used Planck-length and Planck-mass

$$\ell_P := \sqrt{\frac{\hbar G}{c^3}} = 1.6\times 10^{-26}\,\mathrm{nm}\,, \quad m_P := \sqrt{\frac{\hbar c}{G}} = 1.3\times 10^{19}\,\mathrm{u}\,. \tag{42}$$

# Symmetries and scaling properties of SNE





- The SNE has the same symmetries as ordinary Schrödinger equation: Full inhomogeneous Galilei group, including parity and time reversal, and global $U(1)$ phase transformations.
- Also it has the following scaling covariance: Let

$$S_\lambda[\psi](t,\vec{x}) := \lambda^{9/2}\psi(\lambda^5 t\,,\,\lambda^3\vec{x})\,, \tag{43}$$

then $S_\lambda[\psi]$ satisfies the SNE for mass parameter $\lambda m$ iff $\psi$ satisfies SNE for mass parameter $m$

# Collapse: Naive estimate

- Free Gaussian

$$\Psi_{\text{free}}(r,t) = \left(\pi a^2\right)^{-3/4} \left(1 + \frac{i\,\hbar\,t}{m\,a^2}\right)^{-3/2} \exp\left(-\frac{r^2}{2a^2\,\left(1 + \frac{i\,\hbar\,t}{m\,a^2}\right)}\right). \tag{44}$$

- Radial probability density, $\rho(r,t) = 4\pi\, r^2\, |\Psi_{\text{free}}(r,t)|^2$, has a global maximum at

$$r_p = a\sqrt{1 + \frac{\hbar^2 t^2}{m^2 a^4}} \quad \Rightarrow \quad \ddot{r}_p = \frac{\hbar^2}{m^2\, r_p^3}\,. \tag{45}$$

- At time $t = 0$ (say) this outward acceleration due to dispersion, $\ddot{r}_p = \frac{\hbar^2}{m^2\,a^3}$, equals gravitational inward acceleration $\frac{G\,m}{r^2}$ at time $t = 0$ if (compare (41))

$$m^3 a = m_P^3 \ell_p. \tag{46}$$

- For $a = 500\,\mathrm{nm}$ this yields a naive estimate for the threshold mass for collapse of about $4 \times 10^9\mathrm{u}$.

# Stationary states: Analytical existence and numerical values

- Note that outward acceleration due to dispersion is $\propto r^{-3}$ and inward acceleration due to gravity $\propto r^{-2}$. Hence there will be no collapse to a $\delta$-singularity.
- An analytic proof for the existence of a stable ground state has been given by E. Lieb in 1977 in the context of the Choquard equation for one-component plasmas, which is, however, formally identical.
- Tod et al. investigated bound states numerically and found the (unique) stable ground state at Energy $E_0$ and width $a_0$, given by

$$E_0 = -0.163\,\frac{G^2 m^5}{\hbar^2} = -0.163 \cdot mc^2 \cdot \left(\frac{m}{m_P}\right)^4 \approx -mc^2 \cdot 10^{-36}\, m^4[10^{10}\,u] \tag{47a}$$

$$a_0 = \frac{2\hbar^2}{Gm^3} = 6 \cdot 10^6\,\mathrm{ly} \cdot m^{-3}[u] \approx 10^{-6}\,\mathrm{cm} \cdot m^{-3}[10^{10}\,u] \tag{47b}$$

# Stationary states: Rough estimates

- A rough energy-estimate for the ground state is obtained, as usual, by setting
$$E \approx \frac{\hbar^2}{2ma^2} - \frac{Gm^2}{2a}\,. \tag{48}$$
- Minimising in $a$ then gives rough estimates for ground state
$$a_0 = \frac{2\hbar^2}{Gm^3} = 2\ell_P \cdot \left(\frac{m_p}{m}\right)^3\,, \qquad E_0 = -\frac{1}{8}\frac{G^2m^5}{\hbar^2} \tag{49}$$
- Sanity check for applicability of Newtonian gravity (weak field approximation) is that diameter of mass distribution is much larger than its Schwarzschild radius
$$a_0 = \frac{2\hbar^2}{Gm^3} \gg \frac{2Gm}{c^2} \quad \Leftrightarrow \quad \left(\frac{m}{m_p}\right)^4 \ll 1 \tag{50}$$

# General SNE

- SNE is of form

$$i\hbar\partial_t\psi = \left(-\frac{\hbar^2}{2m}\Delta + \left(\phi\star|\psi|^2(t,\vec{x})\right)\right)\psi(t,\vec{x}) \qquad (51)$$

where

$$\phi\star|\psi|^2(t,\vec{x}) = -Gm^2\int\frac{|\psi(t,\vec{x})|^2}{\|\vec{x}-\vec{y}\|}\,d^3y \qquad (52)$$

i.e.

$$\phi(\vec{x}) = -\frac{Gm^2}{r}\,. \qquad (53)$$

- Equation (51) is still valid with modified $\phi$ for separated centre-of-mass wave-function. For example, for homogeneous spherically-symmetric matter distribution get

$$\phi(r) = \begin{cases} -\frac{Gm^2}{R}\left(\frac{3}{2}-\frac{r^2}{2R^2}\right) & \text{for } r < R \\ -\frac{Gm^2}{r} & \text{for } r \geq R \end{cases} \qquad (54)$$

- **This equation can be derived for the centre-of-mass wavefunction of an $N$-particle system obeying the original n-particle SNE of Diósi (1984).**

# The $N$-particle SNE

Principle: *Each particle is under the influence of the Newtonian gravitational potential that is sourced by an active gravitational mass-density to which each particle contributes proportional to its probability density in position space as given by the marginal distribution of the total wave function.*

- Hence

$$\rho(t;\vec{x}) = \sum_{j=0}^{N} m_j P_j(t;\vec{x}) = \sum_{j=0}^{N} m_j \int |\Psi_N(t;\vec{y}_1,\cdots,\vec{y}_N)|^2\,\delta^{(3)}(\vec{y}_j-\vec{x})\,d^{3N}y \tag{55}$$

giving rise to the gravitational potential

$$\begin{aligned} U_G(t;\vec{y}_1,\cdots,\vec{y}_N) &= -G\sum_{i=0}^{N}\int \frac{m_i\rho(t;\vec{x})}{\|\vec{y}_i-\vec{x}\|}\,d^3x \\ &= -G\sum_{i=0}^{N}\sum_{j=0}^{N}\int \frac{m_i m_j P_j(t;\vec{x})}{\|\vec{y}_i-\vec{x}\|}\,d^3x \end{aligned} \tag{56}$$

- Note that the mutual gravitational interaction is not local and includes self interaction, in contrast to what we usually assume in electrodynamics. It is this difference that implies modifications of the dynamics for the centre-of-mass wavefunction. These modifiations are like for the 1-particle SNE if the width of the wave function is large compared to the support of the matter distribution (D.G. & A. Großardt 2014).

# Separation

- Using instead of $\{\vec{x}_i \mid i = 0, 1, \cdots N\}$ centre-of-mass $\vec{c}$ and relative coordinates $\{\vec{r}_\alpha \mid \alpha = 1, \cdots N\}$ (thereby distinguishing the 0-th particle),

$$\vec{c} := \frac{1}{M}\sum_{a=0}^{N} m_a\,\vec{x}_a = \frac{m_0}{M}\vec{x}_0 + \sum_{\beta=1}^{N}\frac{m_\beta}{M}\vec{x}_\beta\,, \tag{57a}$$

$$\vec{r}_\alpha := \vec{x}_\alpha - \vec{c} = -\frac{m_0}{M}\,\vec{x}_0 + \sum_{\beta=1}^{N}\left(\delta_{\alpha\beta} - \frac{m_\beta}{M}\right)\vec{x}_\beta \tag{57b}$$

- Get in large $N$ limit with $\Psi(\vec{x}_0, \cdots \vec{x}_N) = \psi(\vec{c})\chi(\vec{r}_1, \cdots \vec{r}_N)$

$$U_G(t;\vec{c},\vec{r}_1,\cdots,\vec{r}_N) = -G\sum_{\alpha=1}^{N} m_\alpha \int d^3\vec{c}\,' \int d^3\vec{r}\,' \,\frac{|\psi(t;\vec{c}\,')|^2\rho_c(\vec{r}\,')}{\|\vec{c}-\vec{c}\,'+\vec{r}_\alpha-\vec{r}\,'\|}\,, \tag{58}$$

where

$$\rho_c(t;\vec{r}) := \sum_{\beta=1}^{N} m_\beta \left\{\int \prod_{\substack{\gamma=1\\ \gamma\neq\beta}}^{N} d^3\vec{r}_\gamma\right\} |\chi(t;\vec{r}_1,\cdots,\vec{r}_{\beta-1},\vec{r},\vec{r}_{\beta+1},\cdots,\vec{r}_N)|^2\,. \tag{59}$$

# Approximation

- For a separation into centre-of-mass and relative motion we wish to get rid of $\vec{r}_\alpha$-dependence in (58).
- This can, e.g., be achieved by assuming the width of the c.o.m wave function to be much larger than diameter of mass districution. Then,

$$\begin{aligned} U_G = & - G \sum_{\alpha=1}^{N} m_\alpha \int d^3\vec{c}\,' \int d^3\vec{r}\,' \frac{|\psi(t;\vec{c}\,')|^2 \rho_c(\vec{r}\,')}{\|\vec{c}-\vec{c}\,'+\vec{r}_\alpha-\vec{r}\,'\|} \\ & \approx -GM \int d^3\vec{c}\,' \int d^3\vec{r}\,' \frac{|\psi(t;\vec{c}\,')|^2 \rho_c(\vec{r}\,')}{\|\vec{c}-\vec{c}\,'-\vec{r}\,'\|} = U_G(t;\vec{c}) \end{aligned} \tag{60}$$

- Alternatively one may apply a Born-Oppenheimer approximation that consists of replacing $U_G$ with its expectation-value in the state $\chi$ for the relative motion:

$$\begin{aligned} U_G = & - G \sum_{\alpha=1}^{N} m_\alpha \int d^3\vec{c}\,' \int d^3\vec{r}\,' \frac{|\psi(t;\vec{c}\,')|^2 \rho_c(\vec{r}\,')}{\|\vec{c}-\vec{c}\,'+\vec{r}_\alpha-\vec{r}\,'\|} \\ & \approx - G \int d^3\vec{c}\,' \int d^3\vec{r}\,' \int d^3\vec{r}\,'' \frac{|\psi(t;\vec{c}\,')|^2 \rho_c(\vec{r}\,')\rho_c(\vec{r}\,'')}{\|\vec{c}-\vec{c}\,'-\vec{r}\,'+\vec{r}\,''\|} \\ & = U_G(t;\vec{c}) \end{aligned} \tag{61}$$

$\Rightarrow$ Both cases result in SNE for c.o.m in the form (51) with $\phi = U_G(t;\vec{c})$.

# Consequences

- For wide c.o.m - wave functions SNE leads to inhibitions of qm-dispersion, as discussed before. Typical collapse times for widths of $500\,\mathrm{nm}$ and masses about $10^10\,\mathrm{amu}$ are of the order of hours. However, by scaling law (43), this reduces by factor $10^5$ for tenfold mass and $10^{-3}$ fold width.
- For narrow c.o.m. - wave functions in Born-Oppenheimer scheme one obtains an effective self-interaction in c.o.m. SNE of

$$U_G(t;\vec{c}) \approx I_{\rho_c}(\vec{0}) + \tfrac{1}{2} I''_{\rho_c}(\vec{0}) \cdot \Big(\vec{c}\otimes\vec{c} - 2\,\vec{c}\otimes\langle\vec{c}\rangle + \langle\vec{c}\otimes\vec{c}\rangle\Big)\,. \tag{62}$$

where $I_{\rho_c}(\vec{b})$ is the gravitational interaction energy between $\rho_c$ and $T_{\vec{d}}\rho_c$.
- In one dimension and with external harmonic potential this gives rise to modified Schrödinger evolution:

$$i\hbar\partial_t\psi(t;c) = \left(-\frac{\hbar^2}{2M}\frac{\partial^2}{\partial c^2} + \tfrac{1}{2}M\omega_c^2 c^2 + \tfrac{1}{2}M\omega_{\mathrm{SN}}^2\big(c-\langle c\rangle\big)^2\right)\,\psi(t;c)\,, \tag{63}$$

As a consequence covariance ellipse of the Gaussian state rotates at frequency $\omega_q := (\omega_c^2+\omega_{\mathrm{SN}}^2)^{(1/2)}$ whereas the centre of the ellipse orbits the origin in phase with frequency $\omega_c$. This asynchrony is a genuine effect of self-gravity. It has been suggested that it may be observable via the output spectra of optomechanical systems (Yang.et al. 2013).

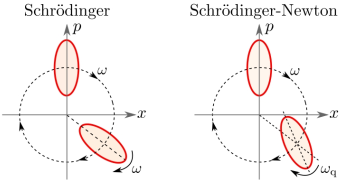
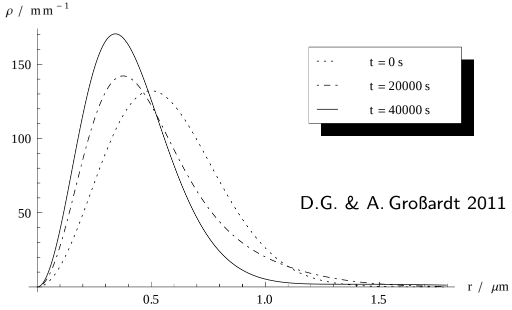



FIG. 1 (color online). Left: according to standard quantum mechanics, both the vector $(\langle x\rangle, \langle p\rangle)$ and the uncertainty ellipse of a Gaussian state for the c.m. of a macroscopic object rotate clockwise in phase space, at the same frequency $\omega = \omega_{\mathrm{c.m.}}$. Right: according to the c.m. Schrödinger-Newton equation (2), $(\langle x\rangle, \langle p\rangle)$ still rotates at $\omega_{\mathrm{c.m.}}$, but the uncertainty ellipse rotates at $\omega_{\mathrm{q}} \equiv (\omega_{\mathrm{c.m.}}^2 + \omega_{\mathrm{SN}}^2)^{1/2} > \omega_{\mathrm{c.m.}}$.

# The time-dependent SNE




- Time evolution of rotationally symmetric Gauß packet of initial width $500\,\mathrm{nm}$. Collapse sets in for masses $m > 4 \times 10^9\,\mathrm{u}$, but collapse times are of many hours (recall scaling laws, though).
- This is a $10^6$ correction to earlier simulations by Carlip and Salzman (2006).

# Summary

- Huge gap between hopes and facts.
- Notion and status of “graviton” is unclear.
- There is no obvious way to translate EP = UFF + LLI + UGR to non-classical systems.
- Statements concerning *Quantum Tests of the Equivalence Principle* need qualification.
- How does the Schrödinger function couple to all components of the gravitational field; e.g., a gravitational wave? Give *first-principles* derivation!
- What if gravity stays classical?
- How, then, do systems in non-classical states gravitate?
- There is an army of arguments against fundamental semi-classical gravity; but how conclusive are they really?
- Potentially interesting consequences from gravity-induced non-linearities in the Schrödinger equation of many particle systems can be derived, e.g., concerning the centre-of-mass motion.

THANKS!

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# Schrödinger 1927

***Der Energieimpulssatz der Materiewellen;***
***von E. Schrödinger***

. . . . . .

Fragt man sich nun, ob diese in sich geschlossene Feldtheorie — von der vorläufigen Nichtberücksichtigung des Elektronendralles abgesehen — der Wirklichkeit entspricht in der Art, wie man das früher von dergleichen Theorien erhofft hatte, so ist die Frage zu *verneinen*. Die durchgerechneten Beispiele, vor allem das H-Atom, zeigen nämlich, daß man in die Wellengleichung (1) *nicht* diejenigen Potentiale einzusetzen hat, welche sich aus den Potentialgleichungen (15′) mit dem Viererstrom (9) ergeben. Vielmehr hat man bekanntlich beim H-Atom in (1) für die $\varphi_\alpha$ die *vorgegebenen* Potentiale des Kerns und eventueller „äußerer" elektromagnetischer Felder einzutragen und die Gleichung nach $\psi$ aufzulösen. Aus (9) berechnet sich dann die von diesem $\psi$ „erzeugte" Stromverteilung, aus ihr nach (15′) die von ihr erzeugten Potentiale. Diese ergeben dann, zu den vorgegebenen Potentialen hinzugefügt, diejenigen Potentiale, mit denen das Atom als ganzes nach außen wirkt. Man

. . . . . .

Gerade die *Geschlossenheit* der Feldgleichungen erscheint somit in eigenartiger Weise durchbrochen. Man kann das heute wohl noch nicht ganz verstehen, hat es aber mit folgenden zwei Dingen in Zusammenhang zu bringen.

. . . . . .

Ob die Lösung der Schwierigkeit wirklich nur in der von einigen Seiten[2]) vorgeschlagenen bloß *statistischen* Auffassung der Feldtheorie zu suchen ist, müssen wir wohl vorläufig dahingestellt sein lassen. Mir persönlich erscheint diese Auffassung heute nicht mehr[1]) endgültig befriedigend, selbst wenn sie sich praktisch brauchbar erweist. Sie scheint mir einen allzu prinzipiellen Verzicht auf das Verständnis des Einzelvorgangs zu bedeuten.

- Schrödinger "closes" the set of Schrödinger-Maxwell equations by letting $\psi$ source the electromagnetic potentials to which $\psi$ couples, thereby introducing non-linearities, similar to radiation-reaction in the classical theory.
- He asserts that "computations" for the H-atom lead to discrepancies which refute such a self-coupling.
- He wonders why in Quantum Mechanics the closedness of the system of field equations is violated in such a peculiar fashion ("in eigenartiger Weise durchbrochen") and comments of possible impact of probability interpretation on classical concepts of local exchange of energy and momentum.

IOP PUBLISHING CLASSICAL AND QUANTUM GRAVITY

Class. Quantum Grav. **25** (2008) 154010 (6pp) doi:10.1088/0264-9381/25/15/154010

# Is quantum gravity necessary?

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Received 6 February 2008, in final form 24 March 2008
Published 21 July 2008
Online at stacks.iop.org/CQG/25/154010

**Abstract**
In view of the enormous difficulties we seem to face in quantizing general relativity, we should perhaps consider the possibility that gravity is a fundamentally classical interaction. Theoretical arguments against such mixed classical–quantum models are strong, but not conclusive, and the question is ultimately one for experiment. I review some work in progress on the possibility of experimental tests, exploiting the nonlinearity of the classical–quantum coupling, which could help settle this question.

PACS numbers: 04.60.−m, 04.80.Cc, 03.75.Dg