Aspects of Classical Physics - Tutorial 1

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Maxwell's equations with differential forms

a) Show that every exact form is closed. (Recall: a form ω is said to be exact if $\exists\rho$ such that $\omega = \mathrm{d}\rho$, and ω is said to be closed if $\mathrm{d}\omega = 0$.)

b) Writing the four-potential as a one-form, $A = (\phi/c)\mathrm{d}x^0 + A_i \mathrm{d}x^i$, compute the two-form $\mathrm{d}A$. Write its components in matrix form.

c) When we use differential forms $\mathrm{d}^2 A = 0$ is just a trivial statement (as d^2 of any form is zero). Expand the components of this expression in terms of the electromagnetic fields. What is the physical meaning of each equation?

d) How do we see in this formulation that a gauge transformation does not change the electromagnetic fields?

e) Compute the Hodge star of the following two-forms: $\mathrm{d}x^0 \wedge \mathrm{d}x^1, \mathrm{d}x^0 \wedge \mathrm{d}x^2$ and $\mathrm{d}x^1 \wedge \mathrm{d}x^3$.

f) We already know what happens with the action of the differential operator $\mathrm{d} : \Omega^p(M) \to \Omega^{p+1}(M)$ on $F = \mathrm{d}A$. Now act with the so-called codifferential operator $\delta : \Omega^p(M) \to \Omega^{p-1}(M)$ defined as $\delta = \star\mathrm{d}\star$. What do we get from this?