Aspects of Classical Physics - Tutorial 4

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The Bogomolny bound

The term proportional to $\frac{1}{2}\mathrm{tr}\, F_{\mu\nu}F^{\mu\nu} = \mathrm{tr}\, F \wedge *F$ is not the only Lorentz and gauge invariant term that is quadratic in A and of second order in derivatives, there is one other such term, namely the four-form $\mathrm{tr}\, F \wedge F$. But this term does not contribute to the equations of motion, so it is often not considered. Nevertheless it has some use as we are going to discuss here.

Let us consider an SU(2) gauge theory, but in a 4-dimensional Euclidean space. We normalize the generators (in the adjoint representation) to $\mathrm{tr}(t_a t_b) = 2\,\delta_{ab}$. The usual SU(2) Yang–Mills action is given by

$$S_{\mathrm{YM}} \;=\; -\frac{1}{8\,g^2}\int \mathrm{d}^4x\ \mathrm{tr}\, F_{\mu\nu}F^{\mu\nu} \;=\; -\frac{1}{4\,g^2}\int \mathrm{tr}\, F\wedge *F \tag{1}$$

where

$$\begin{aligned} &F = \tfrac{1}{2}\, F_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu\ , \qquad *F = \tfrac{1}{2}\big(\tfrac{1}{2}\epsilon_{\mu\nu\rho\lambda}F^{\rho\lambda}\big)\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu\ , \\ &F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \qquad \Leftrightarrow \qquad F = \mathrm{d}A + A\wedge A. \end{aligned} \tag{2}$$

S_{YM} can also be interpreted as the energy of a static Yang–Mills field in 4+1 dimensions.

a) Rewrite (1) as

$$\begin{aligned} S_{\mathrm{YM}} \;&=\; -\frac{1}{16\,g^2}\int \mathrm{d}^4x\ \mathrm{tr}\,(F_{\mu\nu}\mp *F_{\mu\nu})(F^{\mu\nu}\mp *F^{\mu\nu}) \;\mp\; \frac{8\pi^2}{g^2}\int C_2 \\ &=\; -\frac{1}{8\,g^2}\int \mathrm{tr}\,(F\mp *F)\wedge(*F\mp F) \;\mp\; \frac{8\pi^2}{g^2}\int C_2\ , \end{aligned} \tag{3}$$

and find the second Chern form C_2.

b) Show that $\mathrm{d}C_2 = 0$ and also that $C_2 = \mathrm{d}Y_3$, with the so-called Chern–Simons term

$$Y_3 \;=\; \frac{1}{32\pi^2}\,\mathrm{tr}\big(A\wedge\mathrm{d}A + \tfrac{2}{3}A\wedge A\wedge A\big)\ . \tag{4}$$

c) The requirement that $S_{\mathrm{YM}} < \infty$ enforces

$$A_\infty \;:=\; A(r\to\infty) \;=\; U^{-1}\mathrm{d}U \qquad \text{with} \quad U \in \mathrm{SU}(2)\ . \tag{5}$$

Show that this implies that $F_\infty = 0$.

d) Given this, express $c_2 = \int_{\mathbb{R}^4} C_2$ in terms of U. The result is going to be the degree $k = \deg(U)$ of a map from where to where? Hint: use Stokes' theorem and express $\mathrm{d}A$ through F.

e) Use the above to derive a bound for the action S_{YM} in the sector $c_2 = k$. Choose the sign in (3) according to the sign of k.