2^{nd} lecture

THE EDIFICE OF PHYSICAL THEORIES

Classical Mechanics

Newton's laws:

- physical laws are invariant under Galilei trsfms relating different inertial frames:
  space & time translations, rotations, Galilei boosts:
  $t' = t \quad , \quad \vec{r}\,' = \vec{r} - \vec{v}t$
- $\vec{F} = m\ddot{\vec{r}}$
  tautology for a single body $(\vec{r}(t) \to \vec{F}(\vec{r}(t)))$
  nontrivial if a force field $\vec{F}(\vec{r},t)$ is given $\rightsquigarrow$ ODE for $\vec{r}(t)$

historical importance: applies also to celestial mechanics
e.g. discovery of Neptune (1846) by Le Verrier
after prediction from computation of Uranus perturbations

philosophical impact: clockwork universe (Laplace)

Relativistic Mechanics   add c

two principles by Einstein (1905):

- Galilean relativity extends also to electrodynamics (optics)
- the vacuum velocity of light is observer-independent

$\rightsquigarrow$ inertial frames related by Lorentz transformations:
space & ~~time~~ translations, rotations, Lorentz boosts ($\vec{v}$):
decompose $\vec{r} = \vec{r}_{\parallel} + \vec{r}_{\perp}$ w.r.t. $\vec{v}$, name $r_0 := ct$

$$ct' = \gamma\left(ct - \tfrac{v}{c} r_{\parallel}\right), \quad r'_{\parallel} = \gamma\left(r_{\parallel} - \tfrac{v}{c} ct\right), \quad \vec{r}_{\perp}{}' = \vec{r}_{\perp}$$

with $\gamma = \left(1 - v^2/c^2\right)^{-1/2} =: \cosh\theta \rightsquigarrow \frac{v}{c} \equiv \beta = \tanh\theta, \ \gamma\beta = \sinh\theta$

$$\rightsquigarrow \begin{pmatrix} r_0 \\ r_{\parallel} \end{pmatrix}' = \begin{pmatrix} \cosh\theta & -\sinh\theta \\ -\sinh\theta & \cosh\theta \end{pmatrix} \begin{pmatrix} r_0 \\ r_{\parallel} \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} r_0 + r_{\parallel} \\ r_0 - r_{\parallel} \end{pmatrix}' = \begin{pmatrix} e^{-\theta} & 0 \\ 0 & e^{\theta} \end{pmatrix} \begin{pmatrix} r_0 + r_{\parallel} \\ r_0 - r_{\parallel} \end{pmatrix}$$

"light coordinates"
"emission & reception times"

- Newtonian dynamics gets modified:

$$\frac{d}{dt} \longrightarrow \frac{d}{d\tau} = \gamma \frac{d}{dt}$$

$$\vec{v} \longrightarrow \vec{u} = \gamma \vec{v}$$

$$\vec{p} = m\vec{v} \longrightarrow \vec{p} = m\vec{u}$$

$$\vec{F} = \dot{\vec{p}} \longrightarrow \vec{F} = \frac{d}{d\tau}(m\gamma\vec{v})$$

under boosts mixes with novel temporal components

$u_0 = \gamma c$, $p_0 = E/c$, $F_0 = \text{power}/c$

not independent, e.g. $u_0^2 - \vec{u}^2 = \gamma^2 c^2 - \gamma^2 \vec{v}^2 = c^2$ fixed

likewise $\left(E/c\right)^2 - \vec{p}^2 = m^2 c^2$

energy-momentum relation for relativistic mechanics.

Quantum mechanics adds $\hbar$

- Heisenberg uncertainty principle: $\Delta x \, \Delta p \gtrsim \hbar/2$

$\rightsquigarrow$ classical particle trajectory not measurable $\to$ does not exist

"looking" with photons transfers

energy $E = pc = \hbar\omega$ & momentum $\vec{p} = \hbar\vec{k}$

- measurable is (position) probability density

$$\rho(\vec{r},t) = |\Psi(\vec{r},t)|^2 \,, \quad \int d^3x \, \rho(\vec{r},t) = 1 \,,$$

for wave function $\Psi \in \mathbb{C}$.

- deterministic time evolution of wave function

$$i\hbar\partial_t \Psi = \hat{H}\Psi \,, \quad \hat{H} = -\frac{\hbar^2}{2m}\Delta + V(\vec{r}) \text{ Hamiltonian}$$

Schrödinger equ., determines evolution of state

$|\Psi\rangle \in \mathcal{H}$, "components" $\langle \vec{r} | \Psi \rangle = \Psi(\vec{r},t)$ in position basis

- full knowledge of $|\Psi\rangle$ is unrealistic, instead formulate two physical problems:

① stationary state problem $\Longrightarrow$ energy spectrum

- separation ansatz $\Psi(\vec{r},t) = \psi(\vec{r})\,\chi(t)$ $\rightsquigarrow$

$i\hbar\,\partial_t \chi = E\chi$ & $\hat{H}\psi(\vec{r}) = E\,\psi(\vec{r})$ for constant E

$\chi(t) = c\,e^{-\frac{i}{\hbar}Et}$; eigenvalue problem for $\hat{H}$

- normalizability $\int d^3r\, |\Psi(\vec{r},t)|^2 = 1$ restricts possible E values

$\longrightarrow$ "quantization" of the energy (discrete spectrum of $\hat{H}$)

analogous to discrete vibrational frequencies of a string or drum

- another condition: $E \in \mathbb{R}$ $\Longleftrightarrow$ $\int d^3r\, \rho(\vec{r},t) = \text{const.}$

$\rightsquigarrow$ $V(\vec{r})$ cannot be too wild ...

- what about continuous part of spectrum?

free-particle solution ($V\equiv 0$) is $\psi(\vec{r}) = c\,e^{\frac{i}{\hbar}\vec{p}\cdot\vec{r}}$, $E = \vec{p}^{\,2}/2m$

is not normalizable $\rightsquigarrow$ put system in a ficticious box L^3:

need boundary conditions, e.g. $\psi(\vec{r}+L\vec{e}_i) = \psi(\vec{r})$, $i=1,2,3$

but then admissible $\vec{p}$ are quantized: $\vec{p} = 2\pi\hbar\vec{n}/L$, $\vec{n}\in\mathbb{Z}^3$

and normalization fixes $C = L^{-3/2}$

limit $L\to\infty$: quasi-continuum, mathem. justification for continuum

② Scattering problem $\Rightarrow$ cross sections, S-matrix

- again, $i\hbar\partial_t \Psi = \hat{H}\Psi$, $\hat{H} = \hat{\vec{p}}^2/2m + V(\vec{r})$

  with $V(|\vec{r}|\to\infty) \searrow 0$ and focus on $t\to\pm\infty$

  assume particle is far from origin for $t\to\pm\infty$, i.e. free

  $\rightsquigarrow$ incoming plane wave with $\vec{p}_{in}$ at $t=-\infty$

  outgoing plane wave with $\vec{p}_{out}$ at $t=+\infty$

- particle may pass near origin, affected by potential

  $\rightsquigarrow$ $\vec{p}_{out} \neq \vec{p}_{in}$ (not stationary)

  what is the probability (amplitude) for scattering?

  computation of this "scattering amplitude" & cross section is best performed in quantum field theory (more later...)

- classical limit:

characteristic $\vec{p}\cdot\vec{r} \gg \hbar \;\rightarrow\; e^{\frac{i}{\hbar}\vec{p}\cdot\vec{r}}$ oscillates very rapidly.

can consider a wave packet (ψ localized in a small region)

& finds the trajectory $\vec{r}_0(t)$ of its center given by Newton's law

- correspondence principle:

quantum $\rightarrow$ classical when characteristic action $\gg \hbar$

compare with

wave optics $\longrightarrow$ geometric optics (light rays)

Relativistic Quantum Mechanics

add $c, \hbar$

- try to merge the relativistic & quantum descriptions

Schrödinger eq. is Galilei-invariant $\rightsquigarrow$ modify to Lorentz inv.

$\Delta \to \quad \Box = \frac{1}{c^2}\partial_t^2 - \Delta \quad , \quad V(\vec{r}) \to m^2c^2/\hbar^2$ invariant

try

- $\left(\Box + \frac{m^2c^2}{\hbar^2}\right)\Psi(\vec{r},t) = 0$ Klein-Fock-Gordon equ. (KFG) (1926)

is of 2nd order in time derivative,

1st-order equivalent:

$$i\hbar\partial_t\Psi = c\sqrt{m^2c^2 - \hbar^2\Delta}\ \Psi \qquad \left(= mc^2\sqrt{1-\kappa}\ \Psi\right)$$

two problems with $\sqrt{\ }$: why $+\sqrt{\ }$, not $-\sqrt{\ }$

branch point of $\sqrt{\ }$

- nonrelativistic limit: expand in $\kappa = \hbar^2\Delta/m^2c^2$

$$\rightsquigarrow i\hbar\partial_t\Psi = mc^2\left(1 - \frac{\hbar^2\Delta}{2m^2c^2} + \dots\right)\Psi \approx \left(mc^2 - \frac{\hbar^2}{2m}\Delta\right)\Psi$$

- can add potential (naturally to $\partial_t\Psi$):

$$\left[\frac{1}{c^2}\left(\partial_t + \frac{i}{\hbar}V(\vec{r})\right)^2 - \Delta + m^2c^2/\hbar^2\right]\Psi(\vec{r},t) = 0$$

- hydrogen spectrum:

$$E_n = -\frac{me^4}{2\hbar^2 n^2} + \delta_n$$ relativ. corrections of order $\alpha \cdot E_{bohr}$

why Schrödinger failed to calculate δ_n correctly with KFG?

electrons are not described by KFG, but by Dirac equ.!

spin-0 particles spin-½ particles

both eqs. have same nonrel. limit ✓

- interpretation of the wave fctn. $\Psi(\vec{r}, t)$

no longer is $|\Psi|^2$ a probability density, since $\partial_t \int |\Psi|^2 \neq 0$

- this is related to a violation of a basic qu. mech. principle:

$\vec{r}$ measurement (at the expense of $\vec{p}$) can still be arbitrarily precise, of order of λ of observing photon γ, but high-E photon admits not only $\gamma e^- \to \gamma e^-$

but also $\gamma e^- \to \gamma e^- e^+ e^-$ (pair creation)

qu.: which e^- has been localized?

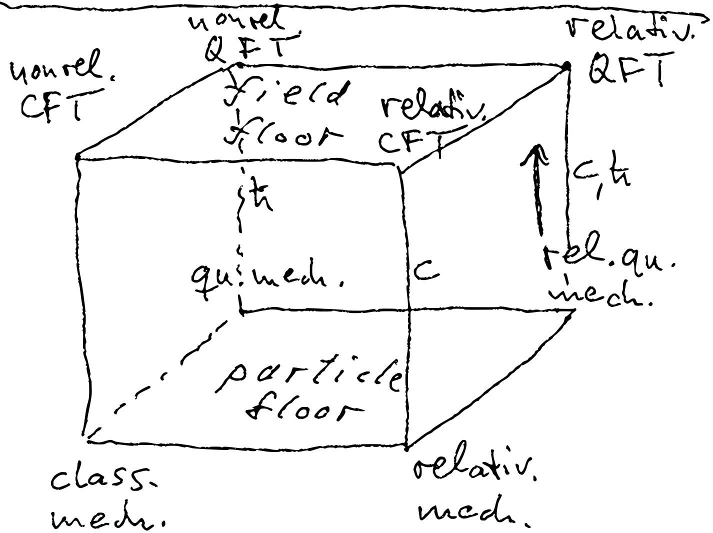
ans.: $\vec{r}$ of e^- cannot be determined better than $\lambda_c = h/mc$
(Compton wavelength)

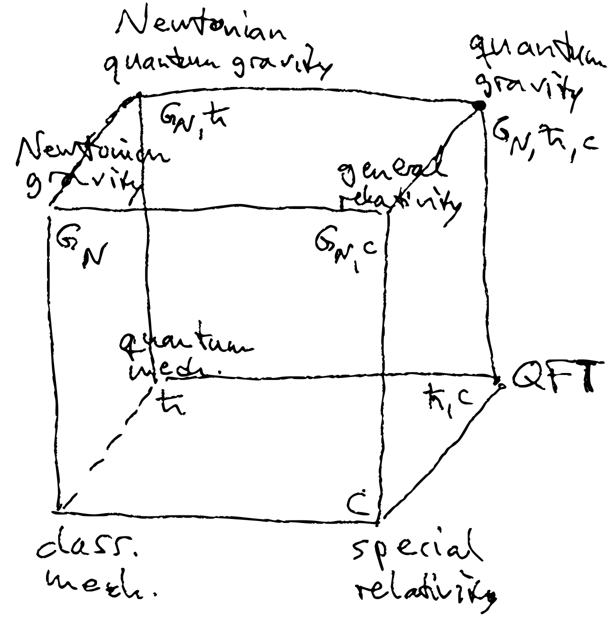
→ pair creation ruins naive single-particle quantum-mechanical description (particle number not conserved)

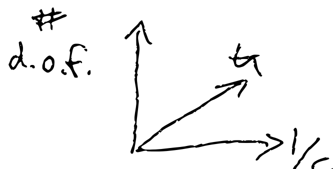
→ relativistic quantum mechanics is not self-consistent !

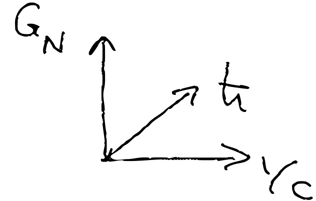
rather, a phenomenological approximation of something more fundamental (QFT), as long as energies are too small for particle creation

two cubical edifices:









nonrel. CFT : hydrodynamics
relativ. CFT : electrodynamics

particle : few degrees of freedom
field : many (∞) degrees of freedom

nonrel. QFT : ordinary quantum matter (phonons, superfluidity, ...)
relativ. QFT : elementary particle physics (the Standard Model)