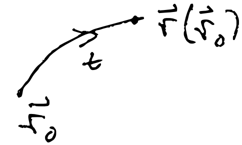
3rd lecture:

Nonrelativistic Field Theory (classical)

macroscopic matter (solids, liquids, gases, plasmas)
contain huge amount of particles $\longrightarrow$ continuum description

- example liquid
  - Lagrange description: $\vec{r}(\vec{r}_0, t)$ 

    dyn. var. $\vec{r}$ depends on cont. param. $\vec{r}_0$
    $\hookrightarrow$ position field $\vec{r}(\vec{r}_0)$
  - Euler description: $\vec{v}(\vec{r}, t)$ 
    dyn. var. $\vec{v}$ depends on cont. param. $\vec{r}$
    $\hookrightarrow$ velocity field $\vec{v}(\vec{r})$

  add pressure field $P(\vec{r}, t)$

$\curvearrowright$ Navier-Stokes eq.:

$$\underbrace{\rho}_{\text{density}} \left(\partial_t + \vec{v} \cdot \vec{\nabla} \right) \vec{v} = - \vec{\nabla} P + \underbrace{\eta}_{\text{viscosity}} \Delta \vec{v}$$

very nonlinear & very complicated

new mathem. complexity: ODE $\longrightarrow$ PDE

- other example: elastic deformations of a solid body

Lagrange description: deformation $\vec{u}(\vec{r}_0, t) = \vec{r}(\vec{r}_0, t) - \vec{r}_0$

assume $|\vec{u}| \ll |\vec{r}_0|$ $\rightsquigarrow$ excludes global transformations

keep only terms linear in $\vec{u}$, assume homogeneity & isotropy

$\rightsquigarrow$ sound wave equation:

$$\ddot{\vec{u}} = c_t^2 \Delta \vec{u} + (c_\ell^2 - c_t^2) \vec{\nabla}(\vec{\nabla} \cdot \vec{u})$$

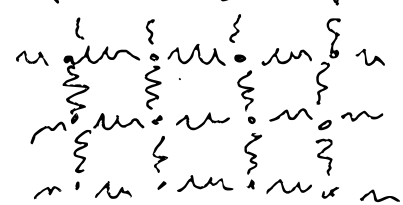
c_t, c_ℓ are sound velocities

c_t $\downarrow$ shear waves

c_ℓ $\downarrow$ compression/decompression waves

efficient approximation of lattice deformations

„spring mattress" picture



Relativistic Field Theory

- consider a complex classical scalar field subject to the KFG equation

$$\left(\Box + \frac{m^2c^2}{\hbar^2}\right)\phi(\vec{r},t) = 0$$

note: ϕ is <u>not</u> a quantum wave function!

preserves its form under Lorentz transformations

- solutions are plane waves

$$\phi_{\vec{k}}(\vec{r},t) \sim e^{i(\vec{k}\cdot\vec{r}-\omega t)}, \qquad \omega = \omega(\vec{k})$$

with dispersion relation $\omega^2 - c^2\vec{k}^2 = \left(\frac{mc^2}{\hbar}\right)^2$

phase velocity $\frac{\omega(\vec{k})}{k}$, group velocity $\frac{d\omega(\vec{k})}{dk}$, frequ.-dependent

- can generalize by adding a cubic term $\sim \phi^2\phi^* = |\phi|^2\phi$ on left-hand side $\rightsquigarrow$ interacting nonlinear waves

- more widely known: Maxwell field theory

Maxwell eqs. for $\vec{E}$ & $\vec{B}$ given ρ & $\vec{j}$ (in Heaviside units):

$$\vec{\nabla}\cdot\vec{E} = \rho \qquad\qquad \vec{\nabla}\times\vec{E} = -\frac{1}{c}\partial_t\vec{B}$$

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad\qquad \vec{\nabla}\times\vec{B} = \frac{1}{c}\vec{j} + \frac{1}{c}\partial_t\vec{E}$$

- the two homogeneous eqs. solved via potentials:

$$\vec{E} = -\vec{\nabla}\varphi - \frac{1}{c}\partial_t\vec{A} \quad \& \quad \vec{B} = \vec{\nabla}\times\vec{A}$$

the two inhomogeneous eqs. become 2^{nd} order for $(\varphi, \vec{A})$:

$$\Box\varphi = \rho \quad \& \quad \Box\vec{A} = \frac{1}{c}\vec{j} \qquad \text{in Lorenz gauge } \frac{1}{c}\partial_t\varphi + \vec{\nabla}\cdot\vec{A} = 0$$

gauge freedom: $\varphi \mapsto \varphi - \frac{1}{c}\partial_t\lambda$, $\vec{A} \mapsto \vec{A} + \vec{\nabla}\lambda$

- looks more complicated than KFG eq.

not obviously Lorentz invariant ($\to$ Lorentz, Einstein, Poincaré, Minkowski)

we shall derive them in 2 lines via a Lorentz-inv. Lagrangian

- in vacuo ($\rho = \vec{j} = 0$) $\exists$ wave solutions also for $\vec{E}$ & $\vec{B}$ with dispersion $\omega^2 = k^2 c^2$ and transversal $\vec{E} \perp \vec{B} \perp \vec{k}$
- NOT a continuum approximation of some discrete system
  NO non relativistic limit exists!

Quantum Field Theory (relativistic)

this is the main subject of the course

QFT $\div$ CFT as quantum mechanics $\div$ classical mechanics

QFT = quantum theory for continuously many (∞) degrees of freedom

for the moment, only some basic ideas, for free complex scalar field $\phi(\vec{r},t)$

analogy: $\vec{r}(t) \rightsquigarrow \Psi(\vec{r},t)$

"wave functional"

$\phi(\vec{r},t) \rightsquigarrow \Psi[\{\phi(\vec{r})\},t] = \Psi[\phi,t]$

momentum operator in "field representation" conjugate to $\phi(\vec{r})$

$$\hat{p} = \frac{\hbar}{i}\int d^3r \, \frac{\delta}{\delta\phi(\vec{r})}$$

Hamiltonian in this representation:

$$\hat{H} = -\frac{\hbar^2}{2m}\int d^3r \, \frac{\delta^2}{\delta\phi(\vec{r})^2} + \ldots$$

we shall derive the exact expression later ...

- most convenient way to solve such "functional Schrödinger equations" is:
put system in a box of size L & impose periodic boundary conditions!

$\rightsquigarrow$ Fourier series

$$\phi(\vec{r},t) = \sum_{\vec{n}} c_{\vec{n}}(t)\, e^{2\pi i \vec{n}\cdot\vec{r}/L} \quad , \quad \vec{n} \in \mathbb{Z}^3$$

change dynamical variables from $\{\phi(\vec{r})\}$ to $\{c_{\vec{n}} \in \mathbb{C}\}$

$\uparrow$ discrete

- in free case for KFG field ($\hbar = c = 1$)
eq. of motion $\left(\partial_t^2 - \Delta + m^2\right)\phi(\vec{r},t) = 0$
follows from a classical Hamilton function

$$H = \int d^3r \left\{ \pi^* \pi + \phi^*(-\Delta + m^2)\phi \right\} \quad \Big| \; \pi = \text{momentum density}$$

check: $\dot{\phi}(\vec{r}) = \frac{\delta H}{\delta \pi^*(\vec{r})} = \pi(\vec{r})$, $\ddot{\phi}(\vec{r}) = \dot{\pi}(\vec{r}) = -\frac{\delta H}{\delta \phi^*(\vec{r})} = (\Delta - m^2)\phi(\vec{r})$ ✓

- canonical quantization in position = field representation:

$\hat{\phi} \to \phi$ (multiplicative), $\hat{\pi} \to -i \dfrac{\delta}{\delta\phi}$

- go to Fourier modes:

$$\phi(\vec{r}) = \sum_{\vec{n}} c_{\vec{n}}\, e^{2\pi i \vec{n}\cdot\vec{r}/L} \,, \quad \frac{\delta}{\delta\phi(\vec{r})} = \frac{1}{L^3} \sum_{\vec{n}} e^{-2\pi i \vec{n}\cdot\vec{r}/L} \frac{\partial}{\partial c_{\vec{n}}}$$

check: $\dfrac{\delta\phi(\vec{r})}{\delta\phi(\vec{r}\,')} \stackrel{!}{=} \delta^{(3)}(\vec{r}-\vec{r}\,')$

insert into $\hat{H}(\hat{\pi},\hat{\phi}) = H\left(-i\dfrac{\delta}{\delta\phi}, \phi\right)$ yields $(L^3 = V)$

$$\hat{H} = \sum_{\vec{n}} \left\{ -\frac{1}{V} \frac{\partial^2}{\partial c_{\vec{n}} \partial c_{\vec{n}}^*} + V\left[m^2 + \left(\frac{2\pi\vec{n}}{L}\right)^2\right] c_{\vec{n}} c_{\vec{n}}^* \right\}$$

$$= \sum_{\vec{n}} \hat{H}_{\vec{n}}$$

Pauli-Weisskopf Hamiltonian (1934)

for every $\vec{n}$ one has a complex harmonic oscillator! (two real)

• more explicitly, write $c_{\vec{n}} = \frac{1}{\sqrt{2}}(x_{\vec{n}} + i y_{\vec{n}})$ $\rightsquigarrow$

$$H_{\vec{n}} = -\frac{1}{2V}\left(\frac{\partial^2}{\partial x_{\vec{n}}^2} + \frac{\partial^2}{\partial y_{\vec{n}}^2}\right) + \frac{V}{2}\left(m^2 + \frac{4\pi^2 \vec{n}^2}{L^2}\right)(x_{\vec{n}}^2 + y_{\vec{n}}^2)$$

2d HO, with mass $= V$, frequency^2 $\omega_{\vec{n}}^2 = m^2 + \frac{4\pi^2 n^2}{L^2}$

not oscillations in coord. space, but in Fourier space (functional Hilbert space)

acts on $\langle \{c_{\vec{n}}\} | \Psi(t)\rangle = \Psi(\{c_{\vec{n}}\}, t)$

• Spectrum: occupation numbers $l_{\vec{n},x}$ & $l_{\vec{n},y} = 0, 1, 2, 3, \ldots$

$$\rightarrow E_{\{l_{\vec{n},x}, l_{\vec{n},y}\}} = \sum_{\vec{n}} E_{l_{\vec{n},x}, l_{\vec{n},y}} = \sum_{\vec{n}} (l_{\vec{n},x} + l_{\vec{n},y} + 1)\,\omega_{\vec{n}}$$

• ground-state energy $E_0 = \sum_{\vec{n}} \omega_{\vec{n}} = \sum_{\vec{n}} \sqrt{m^2 + \frac{4\pi^2 \vec{n}^2}{L^2}} = \infty$

this infinite constant is irrelevant unless we consider gravity

vacuum energy density = cosmological constant / 8π ???

with a Planck-scale cut-off, $\Lambda \sim \frac{m_{pl}}{l_{pl}^3} \sim m_{pl}^4$

is ~ 120 orders larger than exp. value for "dark energy"

$\hookrightarrow \sim 10\ J/km^3$

one hope is supersymmetry:

fermions contribute to vacuum energy equally but with opposite sign

$\rightsquigarrow$ a bose-fermi symmetry might cancel E_0^{total}

but supersymmetry must be broken at scales $\gtrsim 10\,TeV$

$\rightsquigarrow$ too large $\lightning$

cosmological constant problem is one of the biggest enigmas in today's physics ...

- excited states

simplest case: excite one oscillator $(\underline{\vec{n}}, x)$
to first level, e.g. $\ell_{\underline{\vec{n}}, x} = 1$, others $= 0$

$\rightsquigarrow E - E_0 = 1 \cdot \omega_{\underline{\vec{n}}} = \sqrt{m^2 + \frac{4\pi^2 \underline{n}^2}{L^2}} = \sqrt{m^2 + \vec{p}^{\,2}}$

$\rightsquigarrow$ relativistic particle with mass m
and momentum $\vec{p} = \pm \frac{2\pi\hbar \underline{\vec{n}}}{L}$

"one-particle state", standard terminology:

$\Psi_{vac} \sim \prod_{\vec{n}} e^{-V \omega_{\vec{n}} c_{\vec{n}} c^*_{\vec{n}}}$

$\Psi_{1\text{-particle}} \sim c^*_{\underline{\vec{n}}} \Psi_{vac}$, $\Psi_{1\text{-antiparticle}} \sim c_{\underline{\vec{n}}} \Psi_{vac}$

with $\vec{p} = 2\pi\hbar\underline{\vec{n}}/L$ with $\vec{p} = -2\pi\hbar\underline{\vec{n}}/L$

higher excited states given by Fock basis, e.g.

$\Psi_{4p\,3\bar{p}} \sim c^*_1 c^*_2 c^*_3 c^*_4 c_5 c_6 c_7 \cdot \Psi_{vac}$ w/ various momenta (distinct)

momenta $\vec{p}_1, \dots, \vec{p}_4$ momenta $\vec{p}_5, \dots, \vec{p}_7$ $[c_i \equiv c_{\vec{n}_i}]$

- negative-energy trouble is resolved:
  - ϕ is not a probability amplitude, $|\phi|^2$ has no physical meaning
  - true Hamiltonian is not $c\sqrt{m^2c^2 - \hbar^2\Delta}$ but $\sum_{\vec{n}} \hat{H}_{\vec{n}}$

    the latter is positive definite (with $E_0 \to 0$),

    energies of particles & antiparticles coincide ($V \equiv 0$)
- add an external potential $V(\vec{r})$ $\rightsquigarrow$

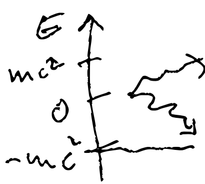
  $\omega_{\vec{n}} \longrightarrow \sqrt{m^2 + 4\pi^2 n^2 / L^2} + V(\vec{r})$

  when $V(\vec{r}) < -mc^2$, some modes turn negative

  $\rightsquigarrow$ one-particle states with negative energy

  when $V(\vec{r}) < -2mc^2$, spontaneous pair creation occurs

vacuum instability!



e^+ escapes   ex. heavy atom, $Z \gtrsim 175$

e^- hover near nucleus $\longrightarrow$ screen nuclear charge Z

$e^- p \to n\, \nu_e$ lower Z below ~ 175 ☺

- how are field-field interactions described?

add nonlinear term in Hamiltonian, e.g.

$$H_{int} = \frac{\lambda}{4} \int d^3r \, (\phi^* \phi)^2 (\vec{r})$$ self-interaction

adds a cubic term $\sim -\frac{\lambda}{2} (\phi^* \phi) \phi$ to KFG eq.

Fourier expand ⇝ $[c_i \equiv c_{\vec{n}_i}]$

$$\hat{H}_{int} = \frac{\lambda V}{4} \sum_{\vec{n}_1, \vec{n}_2, \vec{n}_3, \vec{n}_4} c_3^* c_4^* c_1 c_2 \, \delta(\vec{n}_1 + \vec{n}_2 - \vec{n}_3 - \vec{n}_4)$$

⇝ infinite system of coupled oscillators

- interaction can only be treated perturbatively (λ small)

perturbation theory: leading energy shift is

$\langle \text{free}_{out} | \hat{H}_{int} | \text{free}_{in} \rangle$ for our $\frac{\lambda}{4} |\phi|^4$ interaction $\left| \vec{p}_i = \frac{2\pi\hbar \vec{n}_i}{L} \right.$

$$\langle \vec{p}_3, \vec{p}_4 | \hat{H}_{int} | \vec{p}_1, \vec{p}_2 \rangle \sim \int \prod_{n} dc_{\vec{n}}^* dc_{\vec{n}}^* \, \Psi_{3,4}^* \hat{H}_{int} \Psi_{1,2} \sim \lambda \delta^{(3)}(\vec{p}_1 + \vec{p}_2 - \vec{p}_3 - \vec{p}_4)$$

- seems to describe scattering of $\vec{p}_1 + \vec{p}_2 \to \vec{p}_3 + \vec{p}_4$

not quite: only momentum conservation here

better to pass from Schrödinger to Heisenberg picture of time evolution (or interaction)

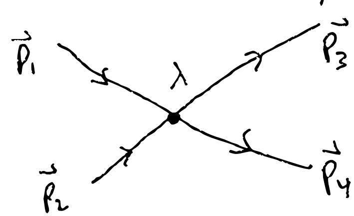
Schr.: Operator $(\vec{r})$ State $\Psi(t)$

Heis.: Operator $(\vec{r}, t)$ State Ψ ← better for relativ. description

⇝ will yield relativistic scattering amplitude (p^μ conserv.)

no accurate derivation in this course, only later a semi-heuristic derivation of the Feynman rules → analytic expressions

- at this point, only draw a simple picture:



our first Feynman diagram:

scalar $2 \to 2$ scattering amplitude to lowest order

Finally

Non-relativistic quantum field theory

- describes quantum effects in nonrelativistic many-body systems, like solids or liquids

described by class. fields $\rightsquigarrow$ sound waves   Navier-Stokes eq.

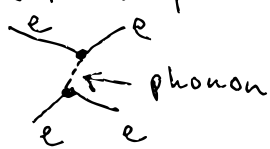
- ex. sound waves in a crystal

quantize $\to$ phonons

$\left.\begin{array}{l}\text{phonon wavelength} \gtrsim \text{lattice unit} \\ \text{phonon energy} \lesssim \hbar \times \text{Debye frequency}\end{array}\right\} \Rightarrow$ specific heat at low temperature

- ex. superconductivity

electron-phonon interaction in crystal produces effective $e^- e^-$ attraction



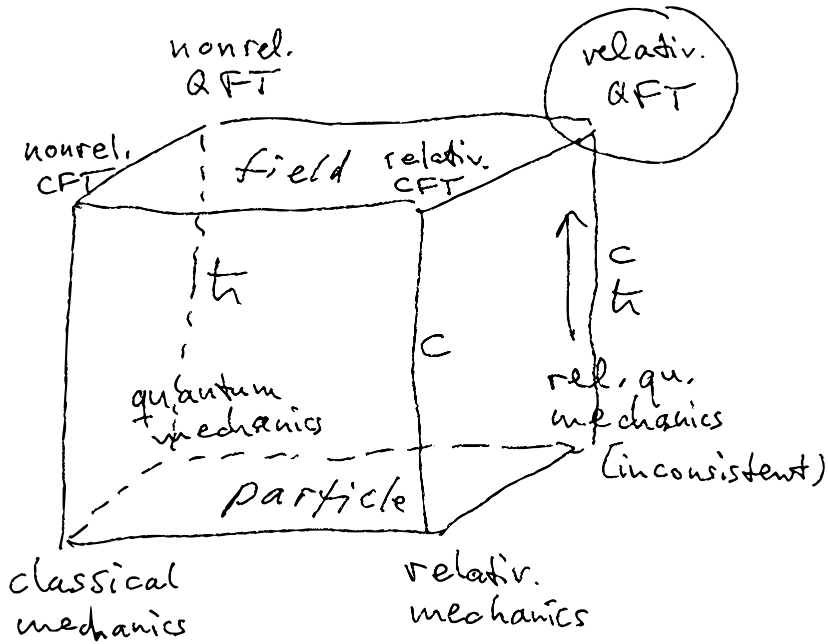
which at low temperature can overcome the Coulomb repulsion

$\rightsquigarrow$ $e^- e^-$ bound states (Cooper pairs) form, evade Pauli principle

$\rightsquigarrow$ described by an effective bosonic quantum field

- ex quantum hydrodynamics

needed to understand the properties of liquid helium



$\left.\begin{array}{l}\text{classical} \\ \text{quantum} \\ \text{non relativ.} \\ \text{relativistic}\end{array}\right\}$ wall

now back to the Standard Model

level – a bird's eye view