6th lecture

GROUPS AND ALGEBRAS

- any mechanical or field-theoretical system is characterized by a Lagrangian (density)
- guiding principle to construct & select:

"the symmetry principle"

- mathematical language to describe symmetries:

group theory

symmetry transformations form a group
physical quantities transform under them
$\rightsquigarrow$ scalars, vectors, tensors, spinors, ...
need basic notions of vector & tensor algebra

Scalars - Vectors - Tensors

- first in n-dimensional Euclidean space $\mathbb{R}^n$

Cartesian coordinate system are related by orthogonal transformations $x_i' = \sum_{j=1}^{n} O_{ij} x_j$, $O^T O = \mathbb{1}$

$\rightsquigarrow$ orthogonal group $O(n) = \underset{\substack{\downarrow \\ \det = +1}}{SO(n)} \cup \underbrace{R \cdot SO(n)}_{\det = -1}$ $\quad R \to$ some reflection

- scalars are invariant under $O(n)$

pseudoscalars are inv. under $SO(n)$, but odd under R

- vectors are invariant under $O(n)$: $\vec{V} = V_i \vec{e}_i$

but their components change $V_i' = \sum_{j=1}^{n} O_{ij} V_j$

| actually: polar vectors

axial vector components: $A_i' = (\det O) \sum_{j=1}^{n} O_{ij} A_j$

- tensors are invariant under $O(n)$: $\hat{T} = T_{ijk\ldots} \hat{e}_{ijk\ldots}$

but their components change as

$$T'_{ijk\ldots} = \sum_{a,b,c\ldots} O_{ia} O_{jb} O_{kc} \cdots T_{abc\ldots}$$

| scalar = tensor of rank 0
| vector = tensor of rank 1

- Einstein summation convention:
  drop summation signs, double indices always summed over
- tensors form an algebra (sums , products)

  sums ↑ same rank; products ↑ different types!

ex. $\hat{A}$ (rank 3) $\times$ $\hat{B}$ (rank 4) $\longrightarrow$ ranks 7, 5, 3, 1 :

$$C^{(7)}_{ijklmnp} = A_{ijk} B_{lmnp}$$

$$C^{(5)}_{ijlmn} = A_{ijk} B_{lmnk}$$

reduction of rank by 2 via "contraction" of an index pair ($\rightarrow$ a sum)

$$C^{(3)}_{ilm} = A_{ijk} B_{lmjk}$$

$$C^{(1)}_{l} = A_{ijk} B_{lijk}$$

— without permutation symmetry among indices, different contractions yield different tensors

exercise: show that contraction preserves tensor transformation

- 2 important tensors:
  - Kronecker delta $\delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases}$
  - Epsilon-Symbol $\varepsilon_{i_1 i_2 \dots i_n}$:
    $\varepsilon_{123\dots n} = +1$ & totally antisymmetric

$\underline{d=3}$

$$\varepsilon_{ijk}\varepsilon_{mnp} = \begin{vmatrix} \delta_{im} & \delta_{in} & \delta_{ip} \\ \delta_{jm} & \delta_{jn} & \delta_{jp} \\ \delta_{km} & \delta_{kn} & \delta_{kp} \end{vmatrix} =: \delta^{mnp}_{ijk}$$

$$\varepsilon_{ijk}\varepsilon_{mnk} = \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm} =: \delta^{mn}_{ij}$$

$$\varepsilon_{ijk}\varepsilon_{mjk} = 2\delta_{im}\,, \quad \varepsilon_{ijk}\varepsilon_{ijk} = 6$$

- second, in (3+1)-dim'l Minkowski space $\mathbb{R}^{1,3}$

inertial frames are related by Lorentz transformations $\mu = 0,1,2,3$

$$x'^\mu = O^\mu{}_\nu x^\nu \quad , \quad \eta_{\mu\nu} O^\mu{}_\rho O^\nu{}_\lambda = \eta_{\rho\lambda} \Leftrightarrow O^T \eta O = \eta$$

where $x^\mu = (t, \vec{r})$, $\eta_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$ (analog of δ_{ij})

a Lorentz scalar is $s^2 = \eta_{\mu\nu} x^\mu x^\nu = t^2 - \vec{r}^2$ (analog of $\vec{r}^2 = x_i x_i$)

- index position relevant (upstairs/downstairs, horizontal order)

4-momentum $(E, \vec{p}) \rightarrow p'^\mu = O^\mu{}_\nu p^\nu$ "contravariant"

4-vector potential $(\phi, \vec{A}) \rightarrow A'_\mu = A_\nu O^\nu{}_\mu = (O^T)_\mu{}^\nu A_\nu$

generic Lorentz tensor: "covariant"

$$T'^{\mu_1 \mu_2 \cdots \mu_r}{}_{\nu_1 \nu_2 \cdots \nu_s} = O^{\mu_1}{}_{\alpha_1} O^{\mu_2}{}_{\alpha_2} \cdots O^{\mu_r}{}_{\alpha_r} T^{\alpha_1 \cdots \alpha_r}{}_{\beta_1 \cdots \beta_s} O^{\beta_1}{}_{\nu_1} O^{\beta_2}{}_{\nu_2} \cdots O^{\beta_s}{}_{\nu_s}$$

raise & lower indices $V_\mu = \eta_{\mu\nu} V^\nu$, $V^\mu = \eta^{\mu\nu} V_\nu$

where $\eta^{\mu\nu}$ is inverse of $\eta_{\mu\nu}$: $\eta^{\mu\nu} \eta_{\nu\rho} = \delta^\mu{}_\rho$

invariant contractions: $\eta_{\mu\nu} p^\mu p^\nu = m^2$, $\eta^{\mu\nu} \partial_\mu \partial_\nu = \Box$

Lie groups [Sophus Lie]

groups with continuous infinitely many elements

→ have also a topological and analytical structure

→ are differentiable manifolds with a multiplication

- Orthogonal groups [index position irrelevant]

$SO(2)$: $g(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$ $\rightsquigarrow$ $g(\phi)g(\psi) = g(\phi+\psi) = g(\psi)g(\phi)$

commutative ≡ Abelian

add reflections → $O(2)$

$\phi \in [0, 2\pi)$ $\rightsquigarrow$ manifold = circle $\hookrightarrow$ $\mathbb{C}$: $\{e^{i\phi}\} = U(1)$

$SO(3)$: $g(\phi_1, \phi_2, \phi_3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_1 & -s_1 \\ 0 & s_1 & c_1 \end{pmatrix} \begin{pmatrix} c_2 & 0 & s_2 \\ 0 & 1 & 0 \\ -s_2 & 0 & c_2 \end{pmatrix} \begin{pmatrix} c_3 & -s_3 & 0 \\ s_3 & c_3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\quad s_i \equiv \sin\phi_i$, $c_i \equiv \cos\phi_i$

3 angles $\rightsquigarrow$ manifold = $S^3/\sim$ ← antipodal identification

it is noncommutative ≡ non-Abelian

almost everything about a Lie group can be inferred from the infinitesimal neighborhood at the identity! → tangent space!

- linearize group transformations (Taylor expand)

small rotations $\phi_a \ll 1 \rightsquigarrow g = \mathbb{1} + \phi_a t_a + \mathcal{O}(\phi^2)$ $\quad \phi \equiv \{\phi_a\}$

$$t_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad t_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad t_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Leftrightarrow (t_a)_{jk} = -\varepsilon_{ajk}$$

"group generators": traceless & normalized to $\mathrm{tr}(t_a t_b) = -2\delta_{ab}$

group commutator $g(\phi) g(\psi) g^{-1}(\phi) g^{-1}(\psi) \approx \mathbb{1} + \phi_a \psi_b [t_a, t_b]$

and $[t_a, t_b] = \varepsilon_{abc} t_c \rightsquigarrow g(\phi) g(\psi) \overset{\text{i.g.}}{\neq} g(\psi) g(\phi)$

[my conventions differ from Smilga's!]

- angular momentum operators generate rotations:

$\hat{J}_a = \varepsilon_{abc} \hat{X}_b \hat{P}_c \Rightarrow -i \varepsilon_{abc} X_b \partial_c$ on functions $f(\vec{r})$

$\rightsquigarrow$ angular momentum algebra in QM is just $i \times$ this

- algebra $\{t = \phi_a t_a\}$ is the "Lie algebra" of $SO(3)$

(a 3-dim'l $\underset{\text{real}}{\wedge}$ vector space with basis $\{t_a\}$ & product: $[t, t'] = -[t', t]$ + Jacobi identity)

- generalize to n dimensions: $SO(n)$

elementary rotations associated with a plane (ij) $i \neq j$

$\exists\ n(n-1)/2$ angles $\rightarrow$ dim. of the $SO(n)$ manifold

generators $(t_{ij})_{mn} = -(\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm})$ obey

$$[t_{ij}, t_{kl}] = \delta_{ik} t_{jl} - \delta_{il} t_{jk} - \delta_{jk} t_{il} + \delta_{jl} t_{ik}$$

- general compact Lie group of dimension D:

generators t_a $a = 1, \dots, D$, antihermitian, obey

$$[t_a, t_b] = f_{abc} t_c$$, f_{abc} totally antisymmetric "structure constants"

- back to the group via exponential (near $\mathbb{1}$):

$$g(\phi) = \exp(\phi_a t_a) \quad \text{maps Lie } G \rightarrow G$$

$$= \mathbb{1} + \phi_a t_a + \frac{1}{2!}\phi_a\phi_b t_a t_b + \frac{1}{3!}\phi_a\phi_b\phi_c t_a t_b t_c + \dots$$

attention: can fail or become non-unique when ϕ gets large!

· Lorentz group [index position relevant]

real 4x4 matrices $\mathcal{O}$ subject to $\mathcal{O}^T \eta\, \mathcal{O} = \eta$, $\eta = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$

"pseudo-rotations" in $\mathbb{R}^{1,3}$ $\rightsquigarrow$ denoted by $O(1,3)$

decomposes into 4 disjoint components:

$$O(1,3) = \overbrace{\begin{tikzcd} \mathrm{SO}(1,3)^\uparrow \arrow[r, "P"] \arrow[d, "T"] \arrow[dr, "PT"] & P\cdot \mathrm{SO}(1,3)^\uparrow \\ \mathrm{SO}(1,3)^\downarrow & P\cdot \mathrm{SO}(1,3)^\downarrow \end{tikzcd}}^{O(1,3)^\uparrow:\ \mathcal{O}^0{}_0 > 0}$$

$\mathrm{SO}(1,3)$, $\det = +1$: $\left\{ \mathrm{SO}(1,3)^\uparrow,\ \mathrm{SO}(1,3)^\downarrow \right.$

$P = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$ $\quad T = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$

near $\mathbb{1}$: $\mathcal{O} = \mathbb{1} + \vec{\theta}\cdot\vec{J} + \vec{v}\cdot\vec{K} + \dots$

$J_1 = \begin{bmatrix} & & & \\ & & & \\ & & & -1 \\ & & 1 & \end{bmatrix}$, $J_2 = \begin{bmatrix} & & & \\ & & & 1 \\ & & & \\ & -1 & & \end{bmatrix}$, $J_3 = \begin{bmatrix} & & & \\ & & -1 & \\ & 1 & & \\ & & & \end{bmatrix}$ anti-hermitian rotations

$K_1 = \begin{bmatrix} & 1 & & \\ 1 & & & \\ & & & \\ & & & \end{bmatrix}$, $K_2 = \begin{bmatrix} & & 1 & \\ & & & \\ 1 & & & \\ & & & \end{bmatrix}$, $K_3 = \begin{bmatrix} & & & 1 \\ & & & \\ & & & \\ 1 & & & \end{bmatrix}$ hermitian boosts

generators

not compact!

Lie algebra: $[J_a, J_b] = \varepsilon_{abc} J_c$, $[J_a, K_b] = \varepsilon_{abc} K_c$, $[K_a, K_b] = -\varepsilon_{abc} J_c$

decouple: $\left\{ \begin{array}{l} \vec{M} = \frac{1}{2}(\vec{J} + i\vec{K}) \\ \vec{N} = \frac{1}{2}(\vec{J} - i\vec{K}) \end{array} \right\} \Rightarrow \begin{array}{l} [M_a, M_b] = \varepsilon_{abc} M_c \\ [N_a, N_b] = \varepsilon_{abc} N_c, \end{array}$ $[M_a, N_b] = 0$ but over $\mathbb{C}$!

- Unitary groups

unitary trsfms leave inv. norm & angles in $\mathbb{C}^n$

$$|z|^2 = (z_j)^* z_j = z^{*j} z_j = z^\dagger z \,, \quad z^\dagger w = z^{*j} w_j$$

$$\Leftrightarrow \quad U U^\dagger = U^\dagger U = \mathbb{1} \rightsquigarrow \begin{cases} \text{complex } n\times n \text{ matrices, } \det U = e^{i\theta} \\ n^2 \text{ real conditions on } 2n^2 \text{ parameters} \end{cases}$$

$U(n)$ group is a real manifold of dimension n^2

$U(n)$ is non-Abelian for $n>1$ $\quad (U(1) = SO(2) = S^1)$

$$U(n) = \underbrace{U(1) \times SU(n)}_{} \Leftrightarrow U = (\det U) \cdot \tilde{U} \,, \quad \det \tilde{U} = 1$$

$\hookrightarrow \dim = n^2 - 1$

$SU(n)$ leaves invariant also the volume $\varepsilon^{j_1 j_2 \cdots j_n} z_{j_1} z_{j_2} \cdots z_{j_n}$

- $SU(2)$ $\quad \dim = 3$ $\quad$ Pauli matrices $\sigma_a = \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$

generators $\quad t_a = -\frac{i}{2} \sigma_a = \left\{ \frac{1}{2}\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \right\}$

$$\operatorname{tr} t_a t_b = -\frac{1}{2} \delta_{ab}$$

form basis of $SU(2)$ Lie algebra, with $[t_a, t_b] = \varepsilon_{abc} t_c$

$\rightsquigarrow$ Lie algebras of $SU(2)$ and $SO(3)$ coincide !

• 1:2 relation between $SO(3)$ & $SU(2)$

$U \in SU(2) \Rightarrow O_{ab} = -2\,\mathrm{tr}\,(U t_a U^+ t_b) \in SO(3)$

reason: map $\vec{r} \in \mathbb{R}^3 \longmapsto \hat{r} = \vec{r}\cdot\vec{\sigma} = \begin{pmatrix} x_3 & x_1 - ix_2 \\ x_1 + ix_2 & -x_3 \end{pmatrix}$ hermitian 2×2

unitary action $\hat{r} \longmapsto U \hat{r} U^+$ preserves $\vec{r}_1 \cdot \vec{r}_2 = \frac{1}{2}\mathrm{tr}(\hat{r}_1 \hat{r}_2)$

$\rightsquigarrow$ defines a rotation O in $\mathbb{R}^3$ $\rightsquigarrow$ homomorphism $SU(2) \to SO(3)$

but not bijective: $\{U, -U\} \to$ same O

$\rightsquigarrow$ $SU(2)$ is a "double cover" of $SO(3)$, $SO(3) = \dfrac{SU(2)}{\mathbb{Z}_2}$

topologically: $SU(2) \simeq S^3$

parametrize $U = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$ with $|a|^2 + |b|^2 = 1 \Longleftrightarrow S^3$ ✓

$a = x_1 + ix_2$
$b = x_3 + ix_4$

$\rightsquigarrow$ $SO(3) \simeq S^3/\text{antipode}$

• $SU(3)$ $\dim = 8$

a basis of generators (Gell-Mann matrices)

$t_1 = \frac{-i}{2}\begin{pmatrix} & 1 & \\ 1 & & \\ & & \end{pmatrix}$, $t_2 = -\frac{i}{2}\begin{pmatrix} & -i & \\ i & & \\ & & \end{pmatrix}$, $t_3 = -\frac{i}{2}\begin{pmatrix} 1 & & \\ & -1 & \\ & & \end{pmatrix}$

$t_4 = -\frac{i}{2}\begin{pmatrix} & & 1 \\ & & \\ 1 & & \end{pmatrix}$, $t_5 = -\frac{i}{2}\begin{pmatrix} & & -i \\ & & \\ i & & \end{pmatrix}$,

$t_6 = -\frac{i}{2}\begin{pmatrix} & & \\ & & 1 \\ & 1 & \end{pmatrix}$, $t_7 = -\frac{i}{2}\begin{pmatrix} & & \\ & & -i \\ & i & \end{pmatrix}$, $t_8 = -\frac{i}{2}\frac{1}{\sqrt{3}}\begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix}$

$\rightsquigarrow$ totally antisym. structure constants f_{abc}, $a,b,c = 1,\dots,8$

Representations

have seen two incarnations of $SO(3)$, as

- orthog. 3×3 matrices of $\det = 1$, acting on $\mathbb{R}^3$
- pairs of unitary 2×2 matrices of $\det = 1$, acting on $\mathbb{C}^2$

these are two "representations" of the same abstract group

general definition for a group G & a vector space V^n

a (linear matrix) representation is a map

$$R: \quad G \longrightarrow \{\text{certain } n\times n \text{ matrices}\}$$
$$g \longmapsto R(g): \quad V^n \longrightarrow V^n$$
$$|\psi\rangle \longmapsto R(g)|\psi\rangle$$

with the property $R(g_1 \cdot g_2) = R(g_1)\cdot R(g_2)$

$\rightsquigarrow$ $R(\mathrm{id}) = \mathbb{1}$, $\quad R(g^{-1}) = R(g)^{-1}$

- attributes of representations
  - equivalent: $R_2(g) \sim R_1(g)$ if $\exists M$ s.t. $R_2(g) = M R_1(g) M^{-1}$
  - real/complex: $R(g)$ = real / complex matrix
  - orthog./unitary: $R(g) \in O(n) \,/\, U(n)$
  - faithful: $g_1 \neq g_2 \rightsquigarrow R(g_1) \neq R(g_2)$
  - reducible: $\exists$ invariant subspace $V_+^m \subset V^n$

    s.t. $R(g): V_+^m \to V_+^m \quad \forall g$

    $$\rightsquigarrow R(g) = \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \begin{bmatrix} V_+ \\ * \end{bmatrix}$$

  - fully reducible: $V^n = V_+^m \oplus V_-^{n-m}$

    s.t. $R(g) = R_+(g) \oplus R_-(g)$ direct sum

    $$\rightsquigarrow R(g) = \left[\begin{array}{c|c} * & 0 \\ \hline 0 & * \end{array}\right] \begin{bmatrix} V_+ \\ V_- \end{bmatrix}$$

irreducible = cannot be reduced

- examples of reps. for $SU(2)$

irreducible rep's are given by "spin" $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$
& have dimension $n = 2j+1$

- $j=0$: $V = \mathbb{C}$ scalar
- $j=\frac{1}{2}$: $V = \mathbb{C}^2 \ni |\psi\rangle = \psi_+ |{+\tfrac{1}{2}}\rangle + \psi_- |{-\tfrac{1}{2}}\rangle$ spinor $\psi_{\alpha = \pm}$
- $j=1$: $V = \mathbb{C}^3 \ni |\psi\rangle = \psi_+ |{+1}\rangle + \psi_0 |0\rangle + \psi_- |{-1}\rangle$ vector

$$= -\frac{1}{\sqrt{2}}(V_x + iV_y)|{+1}\rangle + V_z|0\rangle + \frac{1}{\sqrt{2}}(V_x - iV_y)|{-1}\rangle$$

$\rightsquigarrow$ (real) vector $\vec{V}$ in $\mathbb{R}^3$ basis change

$$= V_x|x\rangle + V_y|y\rangle + V_z|z\rangle$$ manifestly real

- $j=3/2$: $V = \mathbb{C}^4 \rightsquigarrow j_z \equiv t_3 = 3/2, 1/2, -1/2, -3/2 \rightsquigarrow R(g)$ is 4×4
- $j=2$: $V = \mathbb{C}^5 \rightsquigarrow |\psi\rangle \sim$ traceless sym. tensor in $\mathbb{R}^3$

etc. [look complex but are pseudo-real: $R_j^* \sim R_j$]

integral j (n odd): not faithful but true real rep. of $SO(3)$
half-int. j (n even): faithful but "double-valued" rep. of $SO(3)$

· building up R_j from $R_{1/2}$ with tensor products

— $R_{1/2} \otimes R_{1/2}$: acts on $\mathbb{C}^2 \otimes \mathbb{C}^2 = \mathbb{C}^4 \ni \psi_\alpha \chi_\beta$ reducible

decompose

- symmetric: $\psi_+\chi_+$, $\frac{1}{\sqrt{2}}(\psi_+\chi_- + \psi_-\chi_+)$, $\psi_-\chi_-$ $\longleftarrow R_1$
- antisym.: $\frac{1}{\sqrt{2}}(\psi_+\chi_- - \psi_-\chi_+) \longleftarrow R_0$

— R_j can be obtained as $\underbrace{R_{1/2} \otimes R_{1/2} \otimes \cdots \otimes R_{1/2}}_{2j \text{ times}} \Big|_{\text{sym.}}$

acts on $\mathbb{C}^{2j+1}$

$$\mathbb{C}^{2j+1} \ni \psi_{(\alpha}\chi_\beta\phi_\gamma \cdots \omega_{\delta)} \rightsquigarrow \underbrace{\left\{ \begin{array}{c} +++\cdots++ \\ +++\cdots+- \\ +++\cdots-- \\ \vdots \\ ---\cdots-- \end{array} \right\}}_{2j \text{ slots}} \; 2j+1 \text{ options}$$

- representations of $SU(n)$ $i,j,k,=1,\dots,n$
  - defining or "fundamental" rep.: unitary $n\times n$ acting on $\mathbb{C}^n$
    $$\psi_j' = U_j{}^k \psi_k \qquad \psi_j \sim \text{column vector} \quad \text{"}R_n\text{"}$$
  - "antifundamental" rep.: by complex conjugation $\uparrow\downarrow$
    $$\psi^{*k\,\prime} = \psi^{*j} U^{\dagger}{}_j{}^k \qquad \psi^{*k} \sim \text{row vector} \quad \text{"}R_{\bar n}\text{"}$$
    $\rightsquigarrow \psi^*\psi \longmapsto \psi^* U^\dagger U \psi = \psi^*\psi$ invariant ✓
    for $n=2$ it is equivalent to fund. rep: $\psi^{*j} = \varepsilon^{jk}\psi_k$
  - adjoint rep.: $R_n \otimes R_{\bar n} = R_{adj} \oplus R_0 \leftarrow$ trivial rep. $\Leftrightarrow \chi\cdot\psi$ scalar
    $$A_j{}^k = \psi_j \chi^{*k} - \frac{1}{n}\delta_j{}^k \psi_\ell \chi^{*\ell} \quad \text{s.t. } \mathrm{tr}\,A \equiv A_j{}^j = 0$$
    $\rightsquigarrow$ $n\times n$ traceless matrix $\in \mathbb{C}^{n^2-1}$ $\rightsquigarrow$ $\dim V_{adj} = \dim SU(n)$
    $\rightsquigarrow$ can view V_{adj} as the $SU(n)$ Lie algebra, basis $\{t_a\}$
    $\rightsquigarrow A_j{}^k = A_a (t_a)_j{}^k$, $a = 1,\dots,n^2-1$ $\rightsquigarrow \hat{A} = A_a t_a$
    natural action of $SU(n)$ on its own Lie algebra $\hat{A} \mapsto U\hat{A}U^\dagger$

- many more reps. of $SU(n)$, labelled by $n-1$ "spin" or Young tableaux

ex.: rank-3 sym. $SU(3)$ tensor ψ_{ijk} $\quad i,j,k=1,2,3$

dim $=10$: $\psi_{111}, \psi_{112}, \psi_{113}, \psi_{122}, \psi_{123}, \psi_{133}, \psi_{222}, \psi_{223}, \psi_{233}, \psi_{333}$

"decuplet"

- representations of the Lorentz group

remember that $SO(1,3)$ Lie algebra decomposes (over $\mathbb{C}$) into two commuting copies of $SU(2)$ Lie algebras

$$\rightsquigarrow \underbrace{\mathcal{O}}_{SO(1,3)^{\uparrow}} = \exp\{u_a \underbrace{M_a}_{SU(2)_L}\} \exp\{u_a^* \underbrace{N_a}_{SU(2)_R}\} \Bigg|_{\substack{\text{real slice} \\ v_a = u_a^*}}$$

$\rightarrow$ can build any (finite-dim'l) Lorentz group representation by combining spin-j_L with spin-j_R irreps of $SU(2)$

fundamental $j_L = \frac{1}{2},\ j_R = 0$: $\xi_\alpha \quad \alpha = 1,2$

fundamental $j_L = 0,\ j_R = 0$: $\eta_{\dot\alpha} \quad \dot\alpha = 1,2$

higher j_L / j_R by symmetrized tensor products

(j_L, j_R) irreps flip under complex conjugation:

$$(j_L, j_R)^* = (j_R, j_L)$$

- $(\frac{1}{2}, 0)$: left-handed (Weyl) spinor ξ_α c.c.
- $(0, \frac{1}{2})$: right-handed (Weyl) spinor $\eta_{\dot\alpha}$
- $(\frac{1}{2}, \frac{1}{2})$: $\xi_\alpha \eta_{\dot\alpha} = V_{\alpha\dot\alpha} = (v^\mu \sigma_\mu)_{\alpha\dot\alpha}$ with $\sigma_\mu = (\mathbb{1}, \vec\sigma)$

$$= \begin{pmatrix} v_0 - v_3 & -v_1 + i v_2 \\ -v_1 - i v_2 & v_0 + v_3 \end{pmatrix} \rightsquigarrow \text{vector } v^\mu$$

Why?

- $1:2$ relation between $SO(1,3)^\uparrow$ and $SL(2,\mathbb{C})$

$SL(2,\mathbb{C}) \ni g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ complex with $ad-bc=1$

$\dim_{\mathbb{C}} = 4-1 = 3 \rightarrow \dim_{\mathbb{R}} = 6 = \dim SO(1,3)$ ✓

$O^\mu{}_\nu = \frac{1}{2} \operatorname{tr}(\bar\sigma^\mu g \sigma_\nu g^\dagger)$ $\quad \Big| \quad \sigma_\mu = (\mathbb{1}, \sigma_a)$, $\quad \bar\sigma_\mu = (\mathbb{1}, -\sigma_a)$

$\in SO(1,3)^\uparrow$

Lorentz group action on (Weyl) spinors:

$\xi \longmapsto g\xi \quad , \quad \eta \longmapsto \eta g^\dagger$ $\qquad$ double-valued $SO(1,3)$ reps.

or $\xi'_\alpha = g_\alpha{}^\beta \xi_\beta \quad , \quad \eta'_{\dot\alpha} = g^*_{\dot\alpha}{}^{\dot\beta} \eta_{\dot\beta}$ $\qquad$ inequivalent

$SO(1,3) = SL(2,\mathbb{C})/\mathbb{Z}_2$

- combine $\xi_\alpha \eta_{\dot\alpha} = V_{\alpha\dot\alpha}$ & impose hermiticity

$\rightsquigarrow$ hermitian 4×4 matrix $V = V^\mu \sigma_\mu \rightarrow (\frac{1}{2},\frac{1}{2})$ rep.

transforms as $V \mapsto gVg^\dagger \iff V^\mu \mapsto O^\mu{}_\nu V^\nu$

- combine ξ_α & $\eta_{\dot\alpha}$ to $\psi = \begin{pmatrix} \xi_\alpha \\ \eta^{\dot\alpha} \end{pmatrix}$ Dirac spinor

  this is $(\frac{1}{2},0) \oplus (0,\frac{1}{2})$

  transforms as $\psi \mapsto e^{\frac{1}{2}\theta_{\mu\nu}\Sigma^{\mu\nu}} \psi$

  with $\Sigma^{\mu\nu} = -\Sigma^{\nu\mu} = \frac{1}{4}\begin{pmatrix} \sigma^\mu\bar\sigma^\nu - \sigma^\nu\bar\sigma^\mu & 0 \\ 0 & \bar\sigma^\mu\sigma^\nu - \bar\sigma^\nu\sigma^\mu \end{pmatrix}$

- 2π rotations map spinors to minus themselves

  (reflects double cover, $e^{4\pi i \frac{1}{2}\theta} = 1$ for spinors)