7 th lecture

LAGRANGIANS AND HAMILTONIANS

- first, reminder of basic analytical mechanics
- second, application to relativistic mechanics
- third, generalization to field theories

an improvement on parts of Lectures 2 & 3

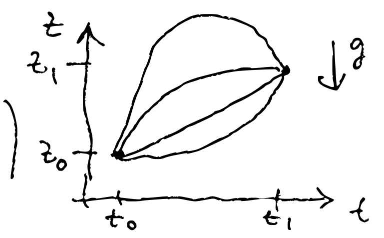
Nonrelativistic mechanics

- simple example: vertical throw in homog. grav. field

stone trajectory $z(t)$ with $z(t_0) = z_0$ & $z(t_1) = z_1$ fixed

consider action integral

$$S[z] = \int_{t_0}^{t_1} dt \, (T - V) = \int_{t_0}^{t_1} dt \left(\frac{1}{2} m \dot{z}^2 - m g z \right)$$



- principle of least action:

"actual physical trajectory minimizes the action"

- general mechanical system with n variables q_i (general'd coordinates)

$$S[q] = \int_{t_0}^{t_1} dt\, L(q_i, \dot{q}_i)$$

Lagrangian

functional of trajectory $q = \{q_i(t)\}$

more precisely $S[q \,|\, q_i^{(0)}, q_i^{(1)}, t_0, t_1]$

most general under 3 assumptions:
- local functional (L = function)
- no $\ddot{q}_i, \dddot{q}_i$ etc. ("higher der's")
- no explicit t dependence ("conservative")

task: find $q_i(t)$ with boundary value $q_i(t_0) = q_i^{(0)}$, $q_i(t_1) = q_i^{(1)}$ which minimizes the action integral

necessary condition:

$$0 = \delta S = \int_{t_0}^{t_1} dt \left\{ \frac{\partial L}{\partial q_i} \delta q_i(t) + \frac{\partial L}{\partial \dot{q}_i} \underbrace{\delta \dot{q}_i(t)}_{= \frac{d}{dt}\delta q_i(t)} \right\}$$

$$= \frac{\partial L}{\partial \dot{q}_i} \delta q_i(t) \Big|_{t_0}^{t_1} + \int_{t_0}^{t_1} dt \left\{ \frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \right\} \delta q_i(t)$$

vanishes for $\delta q_i(t_0) = 0 = \delta q_i(t_1)$

vanishes for all $\delta q_i(t)$ provided $\{\ldots\} = 0$

$\rightsquigarrow$ Euler-Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \qquad i = 1, 2, 3, \ldots, n$$

simple example:
$m\ddot{z} = -mg$

- no explicit t dependence $\rightarrow$ inv. under time translation

$t \mapsto t' = t + \Delta t \rightsquigarrow q_i \mapsto q_i' : \; q_i'(t') = q_i(t)$

$$\rightsquigarrow L(q_i, \dot{q}_i) = L(q_i', \dot{q}_i') + \Delta t \frac{dL}{dt} \rightsquigarrow S[q] = S[q']$$

Noether's theorem: energy is conserved, with

$$E = \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L \; , \quad \text{i.e.} \; \frac{dE}{dt} = 0 \text{ on physical trajectory}$$

- canonical momenta & Hamiltonian

$$p_i := \frac{\partial L}{\partial \dot{q}_i} \rightsquigarrow \dot{p}_i \overset{EL}{=} \frac{\partial L}{\partial q_i} \rightsquigarrow p_i \text{ conserved if } q_i \text{ cyclic}$$

invert $p_i(q, \dot{q})$ to $\dot{q}_i(q, p)$ & substitute into $E \rightsquigarrow$

$$H := \left(\dot{q}_i p_i - L(q, \dot{q})\right)\Big|_{\dot{q}(q,p)} \text{ is a function of } \{q_i, p_i\}$$

"phase space"

$$\leadsto \quad dH = \dot{q}_i\, dp_i + p_i\, d\dot{q}_i - \frac{\partial L}{\partial q_i}\, dq_i - \frac{\partial L}{\partial \dot{q}_i}\, d\dot{q}_i$$

$$\overset{EL}{=} \quad \dot{q}_i\, dp_i - \dot{p}_i\, dq_i$$

also $$dH = \frac{\partial H}{\partial p_i}\, dp_i + \frac{\partial H}{\partial q_i}\, dq_i$$

$$\Longrightarrow \begin{cases} \dot{q}_i = \dfrac{\partial H}{\partial p_i} \\[2mm] \dot{p}_i = -\dfrac{\partial H}{\partial q_i} \end{cases}$$

Hamilton's eqs.

- phase-space functions $f(q,p)$

consider change of f along classical trajectory $(q_i(t), p_i(t))$

$$\frac{df}{dt} = \frac{\partial f}{\partial q_i}\dot{q}_i + \frac{\partial f}{\partial p_i}\dot{p}_i \overset{\text{Ham. eqs}}{=} \frac{\partial f}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial H}{\partial q_i} =: \{f, H\}$$

conserved quantity ≡ integral of motion ≡ Poisson bracket
conserved charge ≡ a phase-space function constant along trajectory

$$\frac{df}{dt} = 0 \iff \{f, H\} = 0$$

- canonical transition to quantum mechanics

$$p_i \longmapsto \hat{p}_i = \frac{\hbar}{i}\frac{\partial}{\partial q_i}, \qquad \{f, g\} \longmapsto -\frac{i}{\hbar}[\hat{f}, \hat{g}]$$

works well for $H = T(p) + V(q)$ but ambiguous otherwise, e.g. $H = p^2 q^2$

Lorentz force

application to relativistic mechanics: Maxwell theory
Lorentz force usually postulated but can be derived

$$\frac{d\vec{p}}{dt} = \vec{F} = q\left(\vec{E} + \frac{1}{c}\vec{v}\times\vec{B}\right)$$

$\times\gamma$

$$\frac{d\vec{p}}{d\tau} = \gamma\vec{F} =: \vec{F}_{rel} = \frac{q}{c}\left(u_0\vec{E} + \vec{u}\times\vec{B}\right)$$

note: $\vec{p} = m\gamma\vec{v} = m\vec{u}$

$u = (u_0, \vec{u}) = \gamma(c, \vec{v})$

$\frac{d}{d\tau} = \gamma\frac{d}{dt}$, $x^0 = ct$

- action for a free relativistic particle $(q=0)$

$$S_0 = -mc\int_0^1 ds \quad \text{for motion between events '0' \& '1'}$$

$$ds = \sqrt{dx\cdot dx} = \sqrt{c^2dt^2 - d\vec{r}^{\,2}} = c\,dt\sqrt{1-\frac{1}{c^2}\left(\frac{d\vec{r}}{dt}\right)^2} = \underbrace{c\,dt\sqrt{1-\vec{v}^2/c^2}}_{d\tau}$$

$$L_0 = -mc^2\sqrt{1-\vec{v}^2/c^2} = -\frac{mc^2}{\gamma} = -mc^2\left(1-\frac{1}{2}\frac{v^2}{c^2}+\dots\right) = -mc^2 + \frac{1}{2}m\vec{v}^2 + \dots$$

$$\vec{p} = \frac{\partial L}{\partial\vec{v}} = \frac{m\vec{v}}{\sqrt{\quad}} = \gamma m\vec{v} \rightsquigarrow 0 = \frac{d}{dt}\frac{\partial L}{\partial\dot{\vec{r}}} = \frac{d}{dt}\vec{p} = \frac{1}{\gamma}\frac{d}{d\tau}\vec{p}$$

$$E = \vec{p}\cdot\vec{v} - L_0 = \frac{mc^2}{\sqrt{\quad}} = \gamma mc^2 = \sqrt{m^2c^4 + \vec{p}^{\,2}c^2} \leftarrow \left(\frac{\vec{v}}{c}\right)^2 = \frac{\vec{p}^{\,2}}{m^2c^2+\vec{p}^{\,2}} \rightsquigarrow \gamma = \sqrt{1+\frac{\vec{p}^{\,2}}{m^2c^2}}$$

• electromagnetic field in relativistic notation

$$A^\mu = (\varphi, \vec{A}) \quad \rightsquigarrow \quad A_\mu = (\varphi, -\vec{A})$$

$$F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu = \begin{bmatrix} 0 & * & * & * \\ \partial_1\varphi + \partial_0 A_1 & 0 & * & * \\ \partial_2\varphi + \partial_0 A_2 & -\partial_2 A_1 + \partial_1 A_2 & 0 & * \\ \partial_3\varphi + \partial_0 A_3 & -\partial_3 A_1 + \partial_1 A_3 & -\partial_3 A_2 + \partial_2 A_3 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{bmatrix} \quad \text{with} \quad \left\{ \begin{array}{l} \vec{E} = -\vec{\nabla}\varphi - \frac{1}{c}\partial_t \vec{A} \\ \vec{B} = \vec{\nabla} \times \vec{A} \end{array} \right\}$$

$$\rightsquigarrow \quad F^{\mu\nu} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix} \quad \text{via} \quad \left\{ \begin{array}{l} \vec{E} \to -\vec{E} \\ \vec{B} \to \vec{B} \end{array} \right\}$$

• contribution of $F_{\mu\nu}$ to ($q \neq 0$) particle action (in fixed EM field)

$$S = -mc \int_0^1 ds - \frac{q}{c} \int_0^1 \underbrace{A_\mu(\vec{r}(t))\, dx^\mu(t)}_{\text{one from } A|_{x(t)}} \longrightarrow \varphi\, c\,dt - \vec{A} \cdot \frac{d\vec{x}}{dt}\, dt$$

$$L = -mc^2 \sqrt{1 - \vec{v}^2/c^2} - q\varphi + \frac{q}{c}\vec{v} \cdot \vec{A} =: L_0 + L_{int} \quad \text{evaluated at } \vec{r}(t)$$

- matter (particle) current

$$L_{int} = -\frac{q}{c} A_\mu(\vec{r}(t)) \frac{dx^\mu}{dt}(t) = -q \int d^3r \; A_\mu(\vec{r})\, j^\mu(\vec{r})$$

$x^0(t) = ct$

with $j^\mu(\vec{r}) = (j^0, \vec{j}) = \left(1, \frac{\vec{v}}{c}\right) \delta^{(3)}(\vec{r} - \vec{r}(t))$ current for trajectory $\vec{r}(t)$

may check that $\partial_\mu j^\mu = \frac{1}{c}\partial_t j^0 + \vec{\nabla}\cdot\vec{j} = 0$

continuity equation $\Leftrightarrow$ charge conservation / current conservation

- canonical versus kinetic momentum

$$\vec{p}_{can} = \frac{\partial L}{\partial \vec{v}} = \frac{m\vec{v}}{\sqrt{\quad}} + \frac{q}{c}\vec{A}(\vec{r}) \qquad \text{canonical momentum}$$

$\vec{p}_{kin}$ Poisson-commuter with $\vec{r}$

- Lagrange equation

$$\frac{d}{dt}\left(p_i^{kin} + \frac{q}{c}A_i\right) = \frac{\partial L_{int}}{\partial x_i} = -q\frac{\partial \varphi}{\partial x_i} + \frac{q}{c} v_j \frac{\partial A_j}{\partial x_i}$$

$$\frac{d}{dt}\frac{q}{c}A_i = \frac{q}{c}\left(\partial_t A_i + \frac{\partial A_i}{\partial x_j}\frac{dx_j}{dt}\right) = \frac{q}{c}\left(\partial_t A_i + \frac{\partial A_i}{\partial x_j} v_j\right)$$

$$\rightsquigarrow \frac{d}{dt}p_i^{kin} = q\left(-\frac{1}{c}\partial_t A_i - \frac{\partial \varphi}{\partial x_i}\right) + \frac{q}{c}\left(\frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j}\right) v_j$$

$$= q E_i + \frac{q}{c}\varepsilon_{ijk} B_k v_j = q\left(\vec{E} + \frac{1}{c}\vec{v}\times\vec{B}\right)_i \quad \checkmark$$

$\rightsquigarrow H = \frac{1}{2m}\left(\vec{P}_{can} - \frac{q}{c}\vec{A}\right)^2 + q\varphi$

Field theories

have seen classical field equations (KFG, Maxwell)
but they were postulated, not derived
will now derive them from least-action principle

- complex scalar field

$$S = \int dt\, L = \int dt \int d^3r\, \mathcal{L} \quad \Leftrightarrow \quad L = \int d^3r\, \mathcal{L}$$

remark 1: Lorentz invariance $\Rightarrow$ $\mathcal{L}$ should be Lorentz scalar (up to $\partial_\mu z^\mu$)
remark 2: $\int_{t_0}^{t_1} dt \longrightarrow \int_{-\infty}^{+\infty} dt$ plus fall-off: $\phi(\vec{r},t) \to 0$ for $|t| \to \infty$ or $|\vec{r}| \to \infty$

carry 3 assumptions on L over to field theory $\rightsquigarrow$

$\mathcal{L}(\phi, \phi^*, \partial_\mu\phi, \partial_\mu\phi^*)$ a function of $\phi(x)$ etc., no expl. x complex conjugate

$$\delta S = \int d^4x \left\{ \frac{\partial \mathcal{L}}{\partial \phi} \delta\phi(x) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta \partial_\mu \phi(x) + c.c. \right\}$$

$\delta\partial_\mu\phi = \partial_\mu \delta\phi$, p.i., drop bdy. terms

$$= \int d^4x \left\{ \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)} \right\} \delta\phi(x) + c.c.$$

$$\delta S = 0 \quad \forall\, \delta\phi(x) \iff \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} - \frac{\partial \mathcal{L}}{\partial \phi} = 0 \quad \text{and c.c.}$$

→ Euler-Lagrange (EL) eqs. for our field system

can guess the Lagrangian density $\mathcal{L}$ for this case:

$$\mathcal{L} = (\partial_\mu \phi^*)(\partial^\mu \phi) - m^2 \phi^* \phi \overset{EL}{\rightsquigarrow} (\Box + m^2)\phi^* = 0$$

for a real scalar field

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{m^2}{2}\phi^2$$

dimensional analysis: $[\mathcal{L}] = M^4$, $[\phi] = M = [A_\mu]$

- add self-interaction ⟺ higher-than-quadratic terms in $\mathcal{L}$

$$\mathcal{L} = (\partial_\mu \phi^*)(\partial^\mu \phi) - m^2 \phi^* \phi - \frac{\lambda}{4}(\phi^*\phi)^2 \qquad \lambda > 0$$

EL^*: $(\Box + m^2)\phi + \frac{\lambda}{2}\phi^2\phi^* = 0$ (→ lecture 3)

- canonical momentum (density):

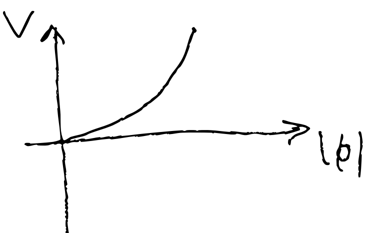
$$\Pi(x) = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} = \partial_0 \phi^*(x) \qquad \& \text{ c.c.}$$

Hamiltonian density

$$\begin{aligned}\mathcal{H} &= \Pi^* \partial_0 \phi^* + \Pi \, \partial_0 \phi - \mathcal{L} \\ &= \Pi^* \Pi + (\vec{\nabla}\phi^*)(\vec{\nabla}\phi) + m^2 \phi^* \phi + \frac{\lambda}{4}(\phi^*\phi)^2 > 0 \\ &= |\Pi|^2 + |\vec{\nabla}\phi|^2 + V(\phi, \phi^*)\end{aligned}$$

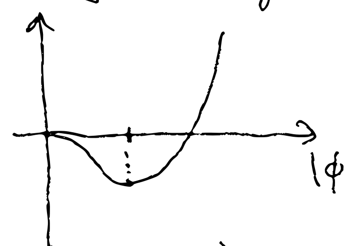
- field potential $\quad V(\phi, \phi) = m^2 |\phi|^2 + \frac{\lambda}{4} |\phi|^4$

$\rightarrow$ three cases depending on signs of m^2 & λ



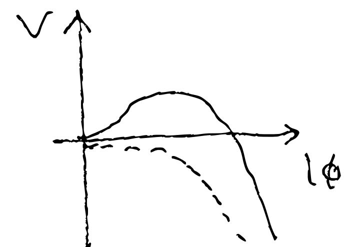
$\lambda > 0, \ m^2 > 0$

$\phi_{min} = 0$



$\lambda > 0, \ m^2 < 0$

$\phi_{min} = \sqrt{\frac{2|m^2|}{\lambda}} \, e^{i\alpha}$



$\lambda < 0$

unstable ($\phi \to \infty$)

- canonical quantization (in field representation)

$$\Pi(x) \to -i \frac{\delta}{\delta\phi(x)} \quad , \quad \Pi^*(x) \longrightarrow -i \frac{\delta}{\delta\phi^*(x)}$$

acting on wave functional $\Psi[\phi, \phi^*, t]$ ($\to$ lecture 3)

- in a spatial box $L \times L \times L$ $(V = L^3)$

reminder: $\phi(\vec{r}, t) = \sum_{\vec{n}} c_{\vec{n}}(t)\, e^{2\pi i \vec{n}\cdot\vec{r}/L}$ $\quad \vec{n} \in \mathbb{Z}^3$

$$\frac{\delta}{\delta\phi(\vec{r}, t)} = \frac{1}{V} \sum_{\vec{n}} e^{-2\pi i\, \vec{n}\cdot\vec{r}/L} \frac{\partial}{\partial c_{\vec{n}}(t)}$$

$$\hat{H} = \int d^3r\, \hat{\mathcal{H}} = \sum_{\vec{n}} \hat{H}_{\vec{n}} + \hat{H}_{int}$$

Pauli-Weisskopf

$$= \sum_{\vec{n}} \left\{ -\frac{1}{V} \frac{\partial^2}{\partial c_{\vec{n}}\, \partial c^*_{\vec{n}}} + V \overbrace{\left[m^2 + \left(\frac{2\pi \vec{n}}{L}\right)^2 \right]}^{\omega^2_{\vec{n}}} c_{\vec{n}} c^*_{\vec{n}} \right\} + \hat{H}_{int}$$

conservative system $\rightsquigarrow$ energy conserved $\rightsquigarrow$ $\hat{H}$-eigenstates $\sim e^{-\frac{i}{\hbar}Et}$, $E = \text{const.}$

also no explicit $\vec{r}$ in $\mathcal{L}$ $\rightsquigarrow$ momentum conserved $\rightsquigarrow$ $[\hat{P}, \hat{H}] = 0$

Noether: $\vec{\underline{P}} = \int d^3r \left(\vec{\nabla}\phi \cdot \Pi + c.c.\right)$ classical

$$\downarrow$$

$\hat{\vec{\underline{P}}} = -i \int d^3r \left(\vec{\nabla}\phi \frac{\delta}{\delta\phi} + \vec{\nabla}\phi^* \frac{\delta}{\delta\phi^*} \right)$ quantum theoretical

in the box: $\hat{\vec{\underline{P}}} = \frac{2\pi}{L} \sum_{\vec{n}} \vec{n} \left(c^*_{\vec{n}} \frac{\delta}{\delta c^*_{\vec{n}}} - c_{\vec{n}} \frac{\delta}{\delta c_{\vec{n}}} \right)$

$\rightsquigarrow$ eigenvalues = possible momenta carried by field state

$$\hat{\vec{\underline{P}}}\, \underline{\Psi} = \vec{k}\, \underline{\Psi}$$

remember lecture 3:

$\underline{\Psi}_{vac} \sim \prod_{\vec{n}} e^{-V\omega_{\vec{n}} c_{\vec{n}} c^*_{\vec{n}}}$ $(H_{int} = 0)$

$\hat{\vec{\underline{P}}}\, \underline{\Psi}_{vac} = 0$

$$\hat{\vec{\underline{P}}}\, c^*_{\vec{n}}\, \underline{\Psi}_{vac} = \frac{2\pi}{L} \underline{\vec{n}} \cdot c^*_{\vec{n}}\, \underline{\Psi}_{vac}$$

$\to$ mom. $\vec{k}$ for particle

but $$\hat{\vec{\underline{P}}}\, c_{\vec{n}}\, \underline{\Psi}_{vac} = -\frac{2\pi}{L} \underline{\vec{n}} \cdot c_{\vec{n}}\, \underline{\Psi}_{vac}$$

$\to$ mom. $\vec{k}$ for <u>anti</u>particle

- action of quantum field on Fock states

reminder of harmonic oscillator $\hat{H}_{osc} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2}\hat{q}^2$

define $a = \frac{m\omega\hat{q} + i\hat{p}}{\sqrt{2m\omega}}$ & $a^\dagger = \frac{m\omega\hat{q} - i\hat{p}}{\sqrt{2m\omega}}$ s.t. $[a, a^\dagger] = 1$

Fock states: $a|0\rangle = 0$, $a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle$, $a|n\rangle = \sqrt{n}\,|n-1\rangle$

position operator $\hat{q} = \frac{a + a^\dagger}{\sqrt{2m\omega}} \rightsquigarrow \langle n+1|\hat{q}|n\rangle = \langle n|\hat{q}|n+1\rangle = \sqrt{\frac{n+1}{2m\omega}}$

$\rightsquigarrow \hat{q}|n\rangle = \sqrt{\frac{n+1}{2m\omega}}\,|n+1\rangle + \sqrt{\frac{n}{2m\omega}}\,|n-1\rangle$

now, $\phi(x) \leftrightarrow \{c_{\vec{n}}, c^*_{\vec{n}}\} \leftrightarrow \{X_{\vec{n}}, Y_{\vec{n}}\}$ oscillators (frequency $\omega_{\vec{n}}$)

therefore on Fock states $|\Psi\rangle$

if present in $|\Psi\rangle$

$c_{\vec{n}}|\Psi\rangle = |\Psi + \text{antiparticle with } \vec{k} = \frac{2\pi}{L}\vec{n}\rangle + |\Psi - \text{particle with } \vec{k} = \frac{2\pi}{L}\vec{n}\rangle$

$c^*_{\vec{n}}|\Psi\rangle = |\Psi + \text{particle with } \vec{k} = \frac{2\pi}{L}\vec{n}\rangle + |\Psi - \text{antiparticle with } \vec{k} = \frac{2\pi}{L}\vec{n}\rangle$

$\rightsquigarrow$ $\phi(x)$ destroys a particle or creates an antiparticle $\Delta Q = -1$

$\phi^*(x)$ destroys an antiparticle or create a particle $\Delta Q = +1$

(in plane-wave superpositions of momentum eigenstates)

· back to Maxwell

$$L = L_0 + L_{int} = -mc^2\sqrt{1-\vec{v}^2/c^2} - q\int d^3r\, \underbrace{A_\mu(x)\, j^\mu(x)}_{\mathcal{L}_{int}}$$

good for dynamics of particle in fixed field (→ Lorentz force)

want to make Maxwell field dynamical (→ Maxwell eqs.)

need to add an action for electromagnetic field

∃ only two Lorentz invariants quadratic & 2^{nd} order:

$$\mathcal{L}_{Max} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}(\vec{E}^2 - \vec{B}^2)$$

$$\mathcal{L}_{top} = \frac{1}{8}\varepsilon^{\mu\nu\rho\lambda}F_{\mu\nu}F_{\rho\lambda} = \vec{E}\cdot\vec{B} = \frac{1}{2}\partial_\mu\left(\varepsilon^{\mu\nu\rho\lambda}A_\nu\partial_\rho A_\lambda\right)$$

· gauge invariance $A_\mu \mapsto A_\mu + \partial_\mu\lambda(x)$, $F_{\mu\nu} \mapsto F_{\mu\nu}$

"local symmetry" is <u>not</u> a symmetry but a <u>redundancy</u>!

⤳ $A_\mu(x)$ is ambiguous, not physical (e.g. gauge $A_0 = 0$)

$$\mathcal{L}_{max} = -\frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu + \frac{1}{2}\partial_\mu A_\nu \partial^\nu A^\mu = \frac{1}{2}A_\nu \Box A^\nu + \frac{1}{2}(\partial\cdot A)^2 + \partial_\mu(\ldots)^\mu$$

can gauge $\partial\cdot A = 0$ ⤳ $\mathcal{L}_{max} = \frac{1}{2}A\cdot\Box A$

- Maxwell interactions

$$L_{int} = -q\int d^3r\, A_\mu(x)\, j^\mu(x)$$ changes under gauge trsfm.

by $\delta L_{int} = -q\int d^3r\, \partial_\mu\lambda(x)\, j^\mu(x) \overset{p.i.}{=} q\int d^3r\, \lambda(x)\, \partial_\mu j^\mu(x)$

$\rightsquigarrow$ $\partial_\mu j^\mu = 0$ is essential for gauge invariance!

- Maxwell mass term?

$$L_{mass} = \frac{1}{2}M^2 A_\mu A^\mu = \frac{1}{2}M^2(A_0^2 - \vec{A}^2)$$

is <u>not</u> gauge-invariant $\longrightarrow$ not renormalizable!

- Maxwell equations

$$\partial_\mu \frac{\partial \mathcal{L}_{max+int}}{\partial(\partial_\mu A_\nu)} - \frac{\partial \mathcal{L}_{max+int}}{\partial A_\nu} = 0 \overset{\text{check!}}{\Longrightarrow} \partial_\mu F^{\mu\nu} = q\, j^\nu$$

inhomog. Maxwell eqs.

contract with ∂_ν $\rightsquigarrow$

$$\partial_\nu\partial_\mu F^{\mu\nu} = q\,\partial_\nu j^\nu \overset{!}{=} 0$$ $\rightsquigarrow$ $\partial_\mu j^\mu = 0$ needed for consistency

homog. Max. eqs.: $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \rightarrow \partial_\mu F_{\rho\lambda} + \partial_\rho F_{\lambda\mu} + \partial_\lambda F_{\mu\rho} = 0$

- QED is only renormalizable because of gauge inv.

$\Leftrightarrow$ QFT of vector bosons is consistent only as a gauge thy.

$\Leftrightarrow$ photon must be massless to guarantee renormalizability

$\Leftrightarrow$ physical vector bosons can only have 2 (not 4) polarizations

- Maxwell Hamiltonian

$$p_i = \frac{\partial \mathcal{L}_{max}}{\partial \dot{A}_i} = F^{0i} = -E_i \quad \text{but} \quad p_0 = \frac{\partial \mathcal{L}_{max}}{\partial \dot{A}_0} = 0$$

$\exists$ no $\dot{A}_0$ in $\mathcal{L}$! $\rightsquigarrow$ A_0 is <u>not</u> dynamical

$\rightsquigarrow$ phase-space constraint

$$H = \int d^3r \left\{ p_i \dot{A}_i - \mathcal{L}_{max} \right\} = \int d^3r \left\{ -E_i \dot{A}_i - \tfrac{1}{2}\vec{E}^2 + \tfrac{1}{2}\vec{B}^2 \right\}$$

$$= \int d^3r \left\{ E_i(E_i - \partial_i A_0) - \tfrac{1}{2}\vec{E}^2 + \tfrac{1}{2}\vec{B}^2 \right\}$$

$E_i = F_{i0} = \partial_i A_0 - \partial_0 A_i$

$$\overset{\text{p.i.}}{=} \int d^3r \left\{ \tfrac{1}{2}(\vec{E}^2 + \vec{B}^2) + A_0 \vec{\nabla}\cdot\vec{E} \right\}$$

Lagrange multiplier implementing Gauß law $\vec{\nabla}\cdot\vec{E} = 0$