Tutorial 2 - Fundamental Interactions

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1 Casimir Force

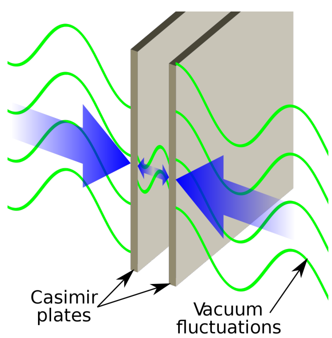


Figure 1: Source: https://en.wikipedia.org/wiki/Casimir_effect

The (static) Casimir effect is a well-known effect arising from the vacuum fluctuations of QFT. One approaches two parallel perfectly conducting and neutral metal plates and it is possible to measure a force between the plates. Let us derive the expression of the Casimir force.

One shows that the energy operator (Hamiltonian) can be written as

$$H = \sum_{\vec{k}} \hbar\omega_{\vec{k}} \left(a^{\dagger}_{\vec{k}} a_{\vec{k}} + \frac{1}{2} \right), \tag{1}$$

where $a_{\vec{k}}$ is the annihilation operator of the state labeled by $\vec{k}$, and $a^{\dagger}_{\vec{k}}$ is its creation operator. Then the "vacuum energy" (vev: vacuum expectation value) reads

$$\langle 0|H|0\rangle = \frac{1}{2} \sum_{k} \hbar\omega_{\vec{k}}, \tag{2}$$

where $\omega_{\vec{k}} = c\sqrt{k_x^2 + k_y^2 + k_z^2}$. The momenta k_x and k_y in the plane of the plates vary continuously, so we can work in polar coordinates in this plane. Let κ

be the momentum-radius. The momentum k_z is quantized by the boundary conditions imposed by the plates. Let a be the distance between them.

a) Substituting $\sum_{k_x,k_y}$ by $A\int \frac{dk_x dk_y}{(2\pi)^2}$, where A is the area of the plates ($A \gg a^2$), rewrite the VEV in terms of an integral in κ and a summation in a natural label n.

Note that both the sum and the integral diverge. Let us now use one of the methods to extract a finite physical quantity from this divergent expression.

b) Define VEV(s) changing the exponent of the argument from 1/2 (square root) to $(1-s)/2$, then calculate the integral as a function of s.
(The integral only formally converges for $\mathrm{Re}(s) > 3$, but we are going to use an analytic continuation for $s = 0$.)

c) Using one of the expressions for the Riemann zeta function, $\zeta(s) = \sum_{n>0} n^{-s}$ (in principle only valid for $\mathrm{Re}(s) > 1$) and the value of the analytical continuation of this expression at -3, $\zeta(-3) = 1/120$, compute VEV(0) $= E$. Then, using $F = -dE/da$, compute the expression for the Casimir force.

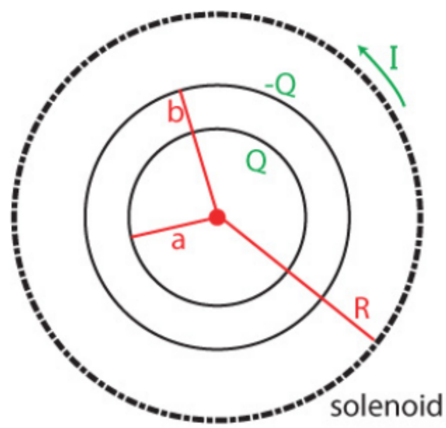
2 Angular Momentum "Paradox"

Maxwell's equations in Heaviside units:

$$\begin{aligned}
\boldsymbol{\nabla} \times \boldsymbol{E} &= -\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t}, \\
\boldsymbol{\nabla} \cdot \boldsymbol{B} &= 0, \\
\boldsymbol{E} &= \rho, \\
\boldsymbol{\nabla} \times \boldsymbol{B} &= \frac{\boldsymbol{j}}{c} + \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t}.
\end{aligned}$$

a) Find the density of momentum $\vec{\tilde{p}}$ carried by electromagnetic fields in vacuum is terms of the electric and magnetic field.

Now keep in mind that the density of angular momentum will be given by $\vec{\tilde{L}} = \vec{r} \times \vec{\tilde{p}}$. Consider the following setup: two coaxial nonconducting cylindrical shells with very long lengths l. The smaller shell, "cylinder A", has radius a and a total uniformly distributed charge Q. The bigger shell, "cylinder B", has radius b and charge Q (also uniformly distributed). These two cylinders are free to rotate around their axes. They are inside a equally long and coaxial solenoid of radius $R(R > b > a)$ which is carrying a constant current I, generating a constant magnetic field B_0 in the region of the cylinders. Both cylinders



are initially at rest. From this initial static setup, imagine the current on the solenoid is decreased to zero (without any external force applied to the system, e.g. the solenoid is a superconductor slowly heating up, and suddenly becomes a normal conductor above some critical temperature, which then starts to kill the current by resistance).

b) Find the instantaneous eletric field induced by the changing field $\vec{B}$ at radius r as a function of r, dB/dt and $\hat{\varphi}$, where $B = |\vec{B}|$ and $\hat{\varphi}$ is the counter-clockwise direction in the figure above.

c) Find the angular momentum gained by each cylinder by the end, when the solenoid magnetic field has decreased to zero. (Assume that the two cylinders are rotating slowly enough that you can completely disregard the magnetic fields generated by them).

d) Calculate the electric field in all regions of space in the initial static situation.

e) Is angular momentum conserved? If so, show it quantitatively.

f) Now imagine we repeat the experiment without"cylinder B". Discuss angular momentum conservation in this case.