Tutorial 4 - Fundamental Interactions

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Born-Infeld Electromagnetism

In Maxwell's theory for electromagnetism, a point charge generates an electric field which carries infinite energy. So the self-energy of a point charge in this theory is infinite. In 1934, Born and Infeld developed a modified theory of electromagnetism in which a point charge generates an electric field which carries finite energy. Let us start first with a warm-up on Maxwell theory. We will use conventions where the speed of light $c = 1$ and space-time signature corresponding to $\eta_{\mu\nu} = \text{diag}(+1, -1, -1, -1)$. Recall that

$$\vec{E} = -\vec{\nabla}\phi - \frac{d\vec{A}}{dt}, \qquad \vec{B} = \vec{\nabla} \times \vec{A}, \tag{1}$$

can be encoded in relativistic notation simply as $F_{mn} = \partial_m A_n - \partial_n A_m$, where $m \in \{0, 1, 2, 3\}$, and $A^m = (\phi, \vec{A})$. More precisely, we have $F_{0j} = E_j$ and $F_{jk} = -\epsilon_{jkl} B^l$ where here the indices run over space values only, $j = 1, 2, 3$. Two relativistic formulas which can easily be verified are

$$F_{mn}F^{mn} = -2(\vec{E} \cdot \vec{E} - \vec{B} \cdot \vec{B}), \qquad \epsilon_{mnpq}F^{mn}F^{pq} = 8\vec{E} \cdot \vec{B}. \tag{2}$$

Finally, when discussing matter sources we will also pack them into a four vector in the standard way as $j^m = (\rho, \vec{j})$.

a) Warm-up and preliminaries: Show that the relativistic equation

$$\eta^{mn}\partial_m F_{np} = j_p \tag{3}$$

is equivalent to Maxwell's equations when expressed in terms of $\vec{E}$ and $\vec{B}$ in the presence of sources ρ and $\vec{j}$. Moreover, considering the electric field produced by a static point charge Q localized at the origin, $\vec{E} = Q\vec{r}/4\pi r^3$, notice that the energy of this field configuration, $\mathcal{E} = \frac{1}{2}\int d^3x\, (\vec{E} \cdot \vec{E})$, is infinite.

b) Show that equation (3) follows from extremizing the action

$$S = -\int d^4x \left[\frac{1}{4}F_{mn}F^{mn} + A_m j^m\right] \tag{4}$$

with respect to variation of the gauge field A_m.

c) For a general action of the form

$$S = -\int d^4x \left[f(\vec{E}, \vec{B}) + A^m j_m \right], \tag{5}$$

where f is an arbitrary function of the electric and magnetic fields, show that extremizing the action with respect to A_m implies the equations of motion

$$-\frac{d}{dt}\vec{D} + \vec{\nabla} \times \vec{H} = \vec{j}, \qquad \vec{\nabla} \cdot \vec{D} = \rho, \tag{6}$$

where $\vec{D} := -\frac{\partial f}{\partial \vec{E}}$ and $\vec{H} := \frac{\partial f}{\partial \vec{B}}$.

d) In the action proposed by Born and Infeld,

$$f(\vec{E}, \vec{B}) = b^2 \sqrt{1 + \frac{1}{2b^2} F_{mn} F^{mn} - \left(\frac{1}{8b^2} \epsilon^{mnpq} F_{mn} F_{pq} \right)^2} - b^2 \tag{7}$$

where b is a constant with mass dimension 2. Show that Born-Infeld theory is equivalent to Maxwell theory in the limit that $b \to \infty$.

e) Show that the Born-Infeld action can be concisely expressed as

$$f(\vec{E}, \vec{B}) = b^2 \sqrt{-\det \left(\eta_{mn} + \frac{1}{b} F_{mn} \right)} - b^2. \tag{8}$$

In your argument, you can use the fact that at any point, one can always choose a Lorentz frame such that the only nonzero components of F_{mn} are $-F_{10} = F_{01} = E_1$ and $-F_{23} = F_{32} = B_1$, i.e. $\vec{E}$ and $\vec{B}$ are both pointing in the x direction.

f) Write the equations of motion in terms of $\vec{E}$ and $\vec{B}$ that come from extremizing the Born-Infeld action with respect to A_m. The formulas in the introduction may be useful.

g) Find the electric field in Born-Infeld theory produced by a static point charge Q localized at the origin. Hint: First find the expressions for $\vec{D}$ and $\vec{H}$. You can also use that $\vec{B} = 0$ for the static point charge in order to simplify the calculations here and in the next items.

h) Show that the electric field has a maximum value of $|\vec{E}| = b$. Sketch the magnitude $|\vec{E}|$ as a function of the distance r for $b = 1$ and compare it with the usual Maxwell case corresponding to $b \to \infty$.

i) Show that the total energy $\mathcal{E}$ of the electric field in Born-Infeld theory is finite where the total energy of the electric field is defined by

$$\mathcal{E} = \int d^3x \left(\vec{D} \cdot \vec{E} + f(\vec{E}, \vec{B}) \right). \tag{9}$$