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Fundamental Interactions

Problem 10: Proton wave function In this problem we consider the strong interactions between up and down quarks. Since $m_u \approx m_d$, the strong interaction possesses an approximate SU(2) flavour symmetry. Choosing a basis

$$u = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad d = \begin{pmatrix} 0 \\ 1 \end{pmatrix} , \tag{1}$$

the invariance of the strong interaction under exchange $u \leftrightarrow d$ is expressed as invariance under rotations in the *isospin space* via

$$\begin{pmatrix} u' \\ d' \end{pmatrix} = U \begin{pmatrix} u \\ d \end{pmatrix} = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \begin{pmatrix} u \\ d \end{pmatrix} . \tag{2}$$

Here, U is a 2x2 special unitary matrix. We already know that any SU(2) matrix can be generated by a set of three linearly independent Hermitian generators, which we choose to be the Pauli spin matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{3}$$

Thus, the suggested flavour symmetry of the strong interaction has the same transformation properties as the spin. One defines the *isospin generators* via $T_i = \frac{1}{2}\sigma_i$, which satisfy $[T_i, T_j] = i\epsilon_{ijk}T_k$. As known from quantum mechanics, one can find simultaneous eigenvectors of the total isospin as well as the third component, i.e.

$$T^2|I, I_3\rangle = I(I+1)|I, I_3\rangle , \qquad T_3|I, I_3\rangle = I_3|I, I_3\rangle . \tag{4}$$

In the isospin eigenbasis the two light quarks are represented by

$$u = |\tfrac{1}{2}, +\tfrac{1}{2}\rangle , \qquad d = |\tfrac{1}{2}, -\tfrac{1}{2}\rangle . \tag{5}$$

Define the isospin ladder operators $T_\pm = T_1 \pm iT_2$, then the action of the ladder operatos on any isospin-multiplet is given by

$$T_\pm|I, I_3\rangle = \sqrt{I(I+1) - I_3(I_3 \pm 1)}|I, I_3 \pm 1\rangle . \tag{6}$$

From which one observes

$$T_+u = 0 , \qquad T_+d = u , \qquad T_-u = d , \qquad T_-d = 0 . \tag{7}$$

Now, we derive a baryonic wave function. Recall that the wave function has to by anti-symmetric with respect to the exchange of any two of its constituents. We proceed in two steps: firstly, we study the combined system of two quarks and, secondly, we add the third quark.

a) Take the isospin doublet (u, d), denoted by $\underline{2}$, and decompose the product representation $\underline{2} \otimes \underline{2}$ into irreducible isospin representations. (This should be very familiar to you, because it is the same mathematics as the coupling of two spin-$\frac{1}{2}$-particles.) Determine all appearing states and express them in terms of product states of u and d. What is the eigenvalue of the total isospin in the appearing multiplets?

b) Now, we build the tensor product of another doublet (u, d), i.e. the third quark, with the direct sum of the triplet $\underline{3}$ and singlet $\underline{1}$ found above. Thus, decompose $\underline{2} \otimes \underline{3}$ and $\underline{2} \otimes \underline{1}$ into irreducible isospin multiplets and express the states in triple products of u and d.

c) Investigate the symmetry properties of the states in the isospin quadruplet $\underline{4}$ and the two isospin doublets under the exchange $q_i \leftrightarrow q_j$ for $q_1q_2q_3$ where the $q_k \in \{u, d\}$.

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The (total) baryon wave function can be expressed as

$$\Psi = \Phi_{\text{flavour}}\, \chi_{\text{spin}}\, \xi_{\text{colour}}\, \eta_{\text{space}} \,, \tag{8}$$

which is the product of the wave functions for all degrees of freedom. It is known that the colour wave function for all bound qqq states is antisymmetric under exchange of any two quarks. Moreover, restricting to vanishing angular momentum, the spatial wave function is symmetric under exchange of any two quarks.

d) What is the symmetry property of the product $\Phi_{\text{flavour}}\, \chi_{\text{spin}}$ under exchange of any two quarks?

e) We have already determined the colour wave function Φ_{flavour}. What is the spin wave function for a baryon made up of u and/or d quarks? (Recall: isospin and spin have the same mathematical foundations.)

f) Knowing the symmetry properties of $\Phi_{\text{flavour}}\, \chi_{\text{spin}}$ from part (f) and the symmetry behaviour of the spin and isospin multiplet from (e), what are the possibilities to obtain a valid total wave function?

g) For the proton wave function we combined the mixed-symmetry spin and the mixed-symmetry isospin states. Let the two doublets be denoted by $\underline{2}_S$ and $\underline{2}_A$. Firstly, check the symmetry properties of $\phi(\underline{2}_S)\chi(\underline{2}_S)$ and of $\phi(\underline{2}_A)\chi(\underline{2}_A)$ under the exchange $1 \leftrightarrow 2$. Secondly, check the behaviour under $1 \leftrightarrow 3$. How can you obtain a totally symmetric combination, i.e. symmetric under $1 \leftrightarrow 2$, $1 \leftrightarrow 3$, $2 \leftrightarrow 3$? To identify the proton wave function you need consider the $I_3 = \frac{1}{2}$ state.

As a comment, if we would have chosen to combine the isospin quadruplet with the spin quadruplet then the resulting baryons are the Δ resonances.

Problem 11: Light mesons To include anti-quarks, we define

$$\bar{u} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \,, \qquad \bar{d} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \,, \tag{9}$$

which furnishes the anti-fundamental doublet $\bar{\underline{2}}$ of the $SU(2)$ isospin. (For $SU(2)$, $\bar{\underline{2}}$ is equivalent to the fundamental representation $\underline{2}$.)

a) Compute the isospin eigenvalues of $\bar{u}$, $\bar{d}$ and the action of the ladder operators on the anti-quarks. Compare the result to the action on the quarks from Problem 10.

b) Decompose the tensor product $\underline{2} \otimes \bar{\underline{2}}$ and express the normalised states in terms of $q\bar{q}'$ states. You should find an isospin triplet and a singlet.

Examples for these light mesons are the $\pi^\pm, \pi^0$ and η pseudoscalar mesons as well as the $\rho^\pm, \rho^0$ and ω vector mesons. Their full treatment, however, requires the inclusion of the strange quark s and leads to an approximate $SU(3)$ flavour symmetry of the strong interaction.