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Fundamental Interactions

Problem 14: angular momentum of a magnetic monopole The free Maxwell equations (in suitable units)

$$\nabla \cdot \vec{E} = 0 \,, \qquad \nabla \times \vec{E} = -\frac{\partial}{\partial t}\vec{B} \,, \tag{1a}$$

$$\nabla \cdot \vec{B} = 0 \,, \qquad \nabla \times \vec{B} = \frac{\partial}{\partial t}\vec{E} \,, \tag{1b}$$

enjoy the so-called *electromagnetic duality*, i.e. the invariance under the transformation

$$\begin{pmatrix} \vec{E}' \\ \vec{B}' \end{pmatrix} = \begin{pmatrix} \cos\zeta & -\sin\zeta \\ \sin\zeta & \cos\zeta \end{pmatrix} \begin{pmatrix} \vec{E} \\ \vec{B} \end{pmatrix} \qquad \text{for} \quad \zeta \in [0, 2\pi) \,. \tag{2}$$

Inspired by symmetry of the equations, it is legit to consider magnetic point charges just like we are used to do for electric point charges. The generalised Maxwell equations read

$$\nabla \cdot \vec{E} = \rho_e \,, \tag{3a}$$

$$\nabla \cdot \vec{B} = 0 \qquad \mapsto \qquad \nabla \cdot \vec{B} = \rho_m \,, \tag{3b}$$

$$\nabla \times \vec{E} = -\frac{\partial}{\partial t}\vec{B} \qquad \mapsto \qquad \nabla \times \vec{E} = -\frac{\partial}{\partial t}\vec{B} - \vec{j}_m \,, \tag{3c}$$

$$\nabla \times \vec{B} = \frac{\partial}{\partial t}\vec{E} + \vec{j}_e \,, \tag{3d}$$

and the duality transformations generalise. As a consequence, the Lorentz force now reads

$$\vec{F}_L = q\left(\vec{E} + \vec{v} \times \vec{B}\right) + g\left(\vec{B} - \vec{v} \times \vec{E}\right) \,. \tag{4}$$

The magnetic analogue of a electric point charge has been investigated by Dirac, and the magnetic field takes the form

$$\vec{B} = \frac{g}{4\pi}\frac{\vec{r}}{r^3} \qquad \text{such that} \quad \nabla \cdot \vec{B} = g\,\delta^{(3)}(\vec{r}) \,. \tag{5}$$

This configuration is known as *Dirac monopole* (of charge g) located at the origin.

(i) A Dirac monopole of magnetic charge g and a point particle with electric charge q remain motionless at a distance of $2a$. What forces do they exert at one another?

(ii) Compute the angular momentum

$$\vec{L} = \int d^3r\; \vec{r} \times \left(\vec{E} \times \vec{B}\right) \tag{6}$$

of the electromagnetic field given by

$$\vec{E} = \frac{q}{4\pi}\frac{\vec{r}+\vec{a}}{|\vec{r}+\vec{a}|^3} \quad \text{and} \quad \vec{B} = \frac{g}{4\pi}\frac{\vec{r}-\vec{a}}{|\vec{r}-\vec{a}|^3} \,. \tag{7}$$

How does $\vec{L}$ depend on the distance $2a$?
Hint: You may choose a convenient coordinate system in which $\vec{a} = a\vec{e}_z$, such that you evaluate the integral in cylindrical coordinates ($x = \rho\sin\phi$, $y = \rho\cos\phi$, z). Moreover, you might want to use

$$\int_{-\infty}^{+\infty} dt \int_0^{\infty} ds\; s^3 \left[\left(s^2+t^2+1\right)^2 - 4t^2\right]^{-\frac{3}{2}} = 1 \,. \tag{8}$$

(iii) What condition on (q, p) does the quantisation of the angular momentum, $L_z = \frac{n}{2}\hbar$ for $n \in \mathbb{Z}$, imply?

Problem 15: charge lattice for dyons A dyon is a hypothetical particle with both electric and magnetic charge. Thus, a magnetic monopole is a dyon with zero electric charge.

(i) Generalise the quantisation condition from Problem 14(iii) to the scenario of two dyons (q_1, g_1) and (q_2, g_2). The result is named after Dirac (1931), Zwanziger (1968), and Schwinger (1969), in short: DZS-condition.

(ii) Given an electron of charge $(q, g) = (e, 0)$, determine the general solution (q, g) of the DZS-condition.

(iii) A CP transformation acts on the charges via $(q, g) \mapsto (-q, g)$. In which cases is the charge spectrum CP invariant?

Problem 16: charged particle in a magnetic monopole field The Newton's equation of motion for an electrically charged point particle (mass m, electric charge q, position $\vec{r}$, velocity $\vec{v} = \dot{\vec{r}}$) in the field of a magnetic monopole (charge q) located at the origin is as follows:

$$m\ddot{\vec{r}} = q\left(\vec{v} \times \vec{B}\right) = \kappa\, \vec{v} \times \frac{\vec{r}}{r^3} \qquad \text{with} \quad \kappa = \frac{qg}{4\pi}\,. \tag{9}$$

(i) Prove that the kinetic energy $T = \frac{1}{2}m\vec{v}^2$ is conserved.

(ii) Compute the change over time of the angular momentum $\vec{L} = m\vec{r} \times \vec{v}$. Define a quantity $\vec{J}$ which is conserved. Reminding yourself of the result from Problem 14, interpret your result.
Hint: You may want to recall (9) and the identity $\dot{\vec{e}}_r = \left(r\,\vec{v} - (\vec{e}_r \cdot \vec{v})\,\vec{r}\right)/r^2$.

(iii) Show that $\vec{J} \cdot \vec{e}_r = -\kappa =$ constant. What is the geometric implication on the particle's trajectory?

(iv) Compute $\vec{J}^2$ and prove that $\vec{L}^2$ is conserved, too. What is the change over time of $\vec{L}$?

(v) Prove that $\frac{d^2}{dt^2}r^2 = \frac{d^2}{dt^2}(\vec{r} \cdot \vec{r}) = 2v^2 =$ constant. Solve for $r(t)$.