Exercise sheet 10: Charged black holes

Please prepare your solutions, ready to present in the class on **06.07.2022** at **16:00**. Throughout this question sheet, we use units where $c = G = 1$.

1. Recall that Maxwell's equations on a four-dimensional spacetime (M, g) are

$$\nabla^\mu F_{\mu\nu} = 0, \quad \nabla_{[\mu} F_{\nu\rho]} = 0.$$

   (a) Write down Einstein's equations in the presence of an electromagnetic field, and show that any solution must have vanishing scalar curvature ($R = 0$). (*Hint: use the result of one of the exercises on sheet 5*). How does this affect Einstein's equations?

   (b) Now consider a metric of the form

$$\mathrm{d}s^2 = -H(\vec{x})^{-2}\,\mathrm{d}t^2 + H(\vec{x})^2\,\mathrm{d}\vec{x}\cdot\mathrm{d}\vec{x}, \qquad (1)$$

   where H is some non-negative time-independent scalar field. You are given that the non-zero Christoffel symbols are

$$\Gamma^t{}_{ti} = \Gamma^t{}_{it} = -\Gamma^i{}_{ii} = H^4\Gamma^i{}_{tt} = -\partial_i \log H, \quad i = 1, 2, 3.$$

   Show that for the electrostatic vector potential A_μ given by $A_t = H^{-1}$ and $A_i = 0$, the metric (1) simultaneously solves Maxwell's and Einstein's equations when H is a harmonic function, i.e. $\nabla^2 H = 0$, where $\nabla^2 = \partial_1^2 + \partial_2^2 + \partial_3^2$. (*Hint: show that*

$$H^{-1}\partial_i\partial_j H = \partial_i\partial_j \log H + (\partial_i \log H)(\partial_j \log H)$$

   holds for all i, j. You may also wish to use symbolic computation software.)

   (c) Discuss the general solutions H which are well-defined as $|\vec{x}| \to \infty$.

2. Consider a charge Q, mass M, Reissner–Nordström black hole in coordinates (t, r, θ, ϕ).

   (a) For $Q < M$, the metric is singular at $r = 0$ and $r = r_\pm$ ($r_+ > r_-$). Define a new coordinate u (analogous to the Eddington–Finkelstein coordinates) so that the metric in (u, r, θ, ϕ) coordinates is regular at $r = r_+$.

   (b) What magnetic field is seen by an observer in a free circular orbit at radius R?

   (c) In the presence of an electromagnetic field, the 4-velocity $\dot{x}^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}$ of a particle of charge e and mass m obeys the generalised Lorentz force law

$$\dot{x}^\lambda \nabla_\lambda \dot{x}^\mu = \tfrac{e}{m} F^\mu{}_\nu \dot{x}^\nu.$$

   For such a particle in the field of a Reissner–Nordström black hole, show that

$$E = m\left(1 - \tfrac{2M}{r} + \tfrac{Q^2}{r^2}\right)\dot{t} + \tfrac{eQ}{r}$$

   is the conserved energy.