Exercise sheet 11: Cosmology and conformal transformations

Please prepare your solutions, ready to present in the class on **13.07.2022** at **16:00**.

1. Consider the spatially-flat, dust-filled FRW universe
$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a(t)^2\left(\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\right).$$
Suppose you are at $r = 0$ and you want to send a message to a distant galaxy by engraving the message on a bullet and firing it from a (very powerful) gun.
   (a) The bullet will (of course) follow a timelike geodesic, which we can assume is radial by isotropy. Show that such a geodesic satisfies
$$a^2\dot{r} = k, \quad \text{and} \quad \dot{t}^2 = 1 + \tfrac{k^2}{a^2},$$
where k is a constant, and $\dot{}$ denotes derivatives with respect to the proper time τ.
   (b) Relate the constant k to the rest mass m and initial energy E_0 of the bullet (as observed by an observer at rest), and the scale factor $a_0 = a(t_0)$, where t_0 is the coordinate time at which the bullet is fired.
   (c) Using $a(t) \propto t^{2/3}$ and assuming $E_0 < \infty$, show that at late times the bullet asymptotically approaches a finite value $r = r_\infty$. I.e. even if you wait an infinite amount of time, the bullet will never reach points with r larger than r_∞.

2. The *Kasner universe* is a cosmological model described by the metric
$$\mathrm{d}s^2 = -\mathrm{d}t^2 + t^{2p_1}\mathrm{d}x_1^2 + t^{2p_2}\mathrm{d}x_2^2 + t^{2p_3}\mathrm{d}x_3^2, \tag{1}$$
where p_i are distinct constants.
   (a) Is the metric (1) spatially homogeneous? Is it isotropic?
   (b) Derive constraints for the p_i so that (1) solves the vacuum Einstein's equations.

3. Consider de Sitter space in coordinates where the metric takes the form
$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{e}^{Ht}\mathrm{d}\vec{x}\cdot\mathrm{d}\vec{x}.$$
Solve the (affinely parameterised) geodesic equations for comoving observers (x_i = constant) to write the affine parameter λ as a function of t. Show that the geodesics can be extended to $t = -\infty$ in finite λ. What can we therefore conclude about these coordinates?

4. Suppose $\widetilde{g}_{\mu\nu} = e^{\alpha(x)}g_{\mu\nu}$, and let K^μ be a Killing vector for the metric $g_{\mu\nu}$.
   (a) Show that $\widetilde{g}_{\mu\nu}K^\nu p^\mu$ is conserved for a light-ray with 4-momentum p^μ.
   (b) What is the constraint on the conformal factor α so that K^μ is also a Killing vector for the metric $\widetilde{g}_{\mu\nu}$?