Exercise sheet 12: Further cosmology

Please prepare your solutions, ready to present in the class on **20.07.2022** at **16:00**.

1. The Friedmann–Robertson–Walker metric can be written as[1]

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a(t)^2 \left((1 - kr^2)^{-1} \, \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta \, \mathrm{d}\phi^2) \right), \tag{1}$$

and Einstein's equations (for a comoving perfect fluid with pressure p and energy density ρ) reduce to the Friedmann equations

$$\tfrac{\dot{a}^2}{a^2} = \tfrac{8\pi}{3} G\rho - \tfrac{k}{a^2} \qquad \text{and} \qquad \tfrac{\ddot{a}}{a} + \tfrac{4\pi}{3} G(\rho + 3p) = 0. \tag{2}$$

   (a) Assume the equation of state $p = w\rho$ to show that the equations (2) imply the covariant conservation of energy equation $\frac{\mathrm{d}}{\mathrm{d}t}\left(\log\rho + 3(1+w)\log a\right) = 0$.

   (b) Which balance of dust (w=0) and cosmological constant (w=-1) allows for a static solution to Einstein's equations?

   (c) Given $\dot{a} > 0$ today, deduce the ultimate fate of the Universe for each of the three values $k = -1, 0, 1$, in the three cases $w = 0, \frac{1}{3}, -1$ (dust, radiation, dark energy).

2. Given a perfect fluid with $p = w\rho$, show that we can interpret the Friedmann equations (2) for the scale of the Universe as the Newtonian equations of motion for the position $a(t)$ of a particle under the influence of a potential V, by rewriting them in the form

$$\tfrac{\dot{a}^2}{2} + V = \mathcal{E} \qquad \text{and} \qquad \ddot{a} = -\tfrac{\partial V}{\partial a},$$

for some conserved "energy" $\mathcal{E}$. Plot $V(a)$ for the cases considered in question 1c. Use your plots to comment on the qualitative evolution of the scale factor in each case.

3. One defines the Hubble and density parameters by $H = \frac{\dot{a}}{a}$ and $\Omega = \frac{8\pi}{3H^2} G\rho$, respectively.

   (a) By rearranging the first Friedmann equation for k, discuss the evolution of $\Omega^{-1} - 1$.

   (b) Suppose the Universe began at time t=0, and t_0 is the age of the Universe today.

      i. Show that t_0 may be determined in terms of the *redshift* $z = \frac{1}{a} - 1$ as

$$t_0 = H_0^{-1} \int_0^\infty \frac{\mathrm{d}z'}{(1 + z')E(z')},$$

      where $H_0 = 70.4 \pm 1.4$ km/s/Mpc is Hubble today, and $E(z) = H(z)/H_0$.

      ii. Calculate the age of the Universe for k=0 (thus Ω=1) and an arbitrary mixture of dust and dark energy. Insert the numbers to three digits for a combination of 32% dust and 68% dark energy, and also for the single-ingredient cases (pure dust, pure radiation, pure vacuum energy).

[1] Here we take the coordinate r to be dimensionless, and $a(t)$ has dimension length.