

Quantum Field Theory: Exercise Session 4

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Problem 1: Decay of a scalar particle

Consider the following Lagrangian, involving two real scalar fields Φ and φ :

$$\mathcal{L} = \frac{1}{2} \partial_\mu \Phi \partial^\mu \Phi - \frac{1}{2} M^2 \Phi^2 + \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - \frac{1}{2} m^2 \varphi^2 - \mu \Phi \varphi \varphi. \quad (1)$$

The last term is an interaction term, with coupling constant μ , which allows the particle Φ to decay into two φ 's.

- (a) Write down the momentum space Feynman rules for this theory, and hence the Feynman diagram for the decay of Φ to lowest order in μ .
- (b) Obtain the invariant matrix element \mathcal{M} , defined from the scattering matrix $S = 1 + iT$ by

$$\langle k_{\varphi_1} k_{\varphi_2} | iT | k_\Phi \rangle = (2\pi)^4 \delta^{(4)}(k_\Phi - k_{\varphi_1} - k_{\varphi_2}) i\mathcal{M}(k_\Phi \rightarrow k_{\varphi_1}, k_{\varphi_2}), \quad (2)$$

and given diagrammatically by

$$i\mathcal{M} = \{ \text{the sum of all connected amputated Feynman diagrams} \}. \quad (3)$$

- (c) Compute the decay rate

$$\Gamma = \frac{1}{2M} \prod_{f=\varphi_1, \varphi_2} \int \tilde{d}k_f |\mathcal{M}(k_\Phi \rightarrow k_{\varphi_1}, k_{\varphi_2})|^2 (2\pi)^4 \delta^{(4)}(k_\Phi - k_{\varphi_1} - k_{\varphi_2}) \quad (4)$$

and the life time $\tau = \Gamma^{-1}$ of the Φ particles in their rest frame to lowest order in μ .
Hint: A change of variables $\delta(f(x))dx = \frac{dx}{dy} \delta(y)dy$ with $y = f(x)$ shows that

$$\frac{\delta \left(\sqrt{|\vec{k}|^2 + m_1^2} + \sqrt{|\vec{k}|^2 + m_2^2} - M \right)}{\sqrt{|\vec{k}|^2 + m_1^2} \sqrt{|\vec{k}|^2 + m_2^2}} |\vec{k}|^2 d|\vec{k}| = \frac{|\vec{k}|}{M} \delta(y) dy \quad (5)$$

with $y = \sqrt{|\vec{k}|^2 + m_1^2} + \sqrt{|\vec{k}|^2 + m_2^2} - M$.

- (d) What is the lower bound on M for a decay to be possible?

Problem 2: Decay of the charged pion

The negatively charged pion can be represented by a complex scalar field φ , the muon by a Dirac field ψ and the muon neutrino by a helicity-projected left-handed Dirac field $\frac{1}{2}(1 - \gamma^5)\chi$. The pion decays almost exclusively into a muon and muon antineutrino, and this process can be described by the Lagrangian density

$$\mathcal{L} = \partial_\nu \varphi^* \partial^\nu \varphi - m_\pi^2 |\varphi|^2 + \bar{\psi}(i\not{\partial} - m_\mu)\psi + \bar{\chi}i\not{\partial}\chi + \lambda [\partial_\nu \varphi \bar{\psi} \gamma^\nu (1 - \gamma^5)\chi + \text{h.c.}], \quad (6)$$

where λ is a coupling constant. The momentum space Feynman rules for this theory read:

$$\langle \vec{p}, s | \bar{\psi} = \begin{array}{c} \text{---} \\ \diagdown \\ \text{---} \end{array} \begin{array}{c} \text{---} \xrightarrow{s} \\ \vec{p} \end{array} = \bar{u}^s(p) \quad \text{fermion} \quad (7)$$

$$\langle \vec{p}, r | \chi = \begin{array}{c} \text{---} \\ \diagdown \\ \text{---} \end{array} \begin{array}{c} \text{---} \xleftarrow{r} \\ -\vec{p} \end{array} = v^r(p) \quad \text{antifermion} \quad (8)$$

$$\langle \vec{p} | \varphi = \text{---} \begin{array}{c} \text{---} \\ \diagdown \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \vec{p} \end{array} = 1 \quad (9)$$

$$\text{---} \begin{array}{c} \diagdown \\ \diagup \end{array} = \lambda p_\nu \gamma^\nu (1 - \gamma^5). \quad (10)$$

(a) Compute the lifetime of the pions.

Hint: When computing the modulus squared of the invariant matrix element, $|\mathcal{M}|^2$, you should sum over all possible final spin states using the completeness relations

$$\sum_{s=\pm} u_a^s(p) \bar{u}_b^s(p) = (\not{p} + m_\mu)_{ab} \quad \text{where} \quad (\not{p} - m_\mu)u^s(p) = 0, \quad (11)$$

$$\sum_{r=\pm} v_a^r(p) \bar{v}_b^r(p) = \not{p}_{ab} \quad \text{where} \quad \not{p}v^r(p) = 0, \quad (12)$$

and the gamma-trace identities

$$\text{Tr}(\text{any odd number of } \gamma\text{'s}) = 0 \quad (13)$$

$$\text{Tr}(\gamma^\mu \gamma^\nu) = 4\eta^{\mu\nu} \quad (14)$$

$$\text{Tr}(\gamma^\mu \gamma^\nu \gamma^5) = 0 \quad (15)$$

$$\text{Tr}(\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma) = 4(g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho}) \quad (16)$$

$$\text{Tr}(\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma^5) = 4i\epsilon^{\mu\nu\rho\sigma}. \quad (17)$$

(b) Why is the process $\pi^- \rightarrow e^- + \bar{\nu}_e$ suppressed?