Quantum Field Theory: Exercise Session 6 18 January 2016

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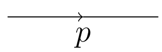
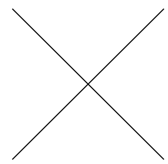
Regularization, renormalization and the one-loop structure of ϕ^4 theory

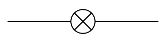
The Feynman rules in renormalized perturbation theory for the ϕ^4 theory are



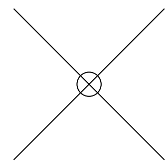
$= \dfrac{\mathrm{i}}{p^2 - m^2}$,



$= -\mathrm{i}\lambda$,



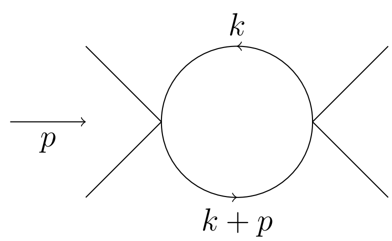
$= \mathrm{i}\left(p^2\delta_z - \delta_m\right)$,



$= -\mathrm{i}\delta_\lambda$.

(a) Write down the two-particle scattering amplitude $\mathrm{i}\mathcal{M}$ in terms of Feynman diagrams to one loop order.

(b) Use the Feynman rules to write down explicitly the integral in momentum space corresponding to the diagram



$= (\mathrm{i}\lambda)^2\, \mathrm{i}V(p^2)$.

(c) Now regularize the integral $V(p^2)$ using dimensional regularization as follows:

(i) Generalize the action for ϕ^4 theory to d spacetime dimensions, introducing an arbitrary mass parameter μ to allow the coupling λ to keep mass dimension 0. Thus, write down the corresponding momentum integral $V(p^2)$ in d dimensions.

(ii) Introduce a Feynman parameter z to combine the factors in the demominator using

$$\frac{1}{ab} = \int_0^1 \frac{\mathrm{d}z}{\left[az + b(1-z)\right]^2} \,. \tag{1}$$

(iii) Make the change of variables $k' = k + p(1-z)$ in order to perform the momentum integral by applying the relation

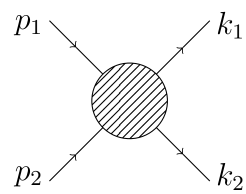
$$\int \frac{\mathrm{d}^d q}{(q^2 + 2qr - \Omega^2)^\alpha} = (-1)^{d/2}\pi^{d/2}\frac{\Gamma(\alpha - \frac{d}{2})}{\Gamma(\alpha)}\frac{1}{(-r^2 - \Omega^2)^{\alpha - \frac{d}{2}}} \,. \tag{2}$$

(iv) Take the limit $d \to 4$ using

$$\Gamma(\epsilon) = \left[\frac{1}{\epsilon} - \gamma + \mathcal{O}(\epsilon)\right] , \tag{3}$$

with γ the Euler-Mascheroni constant, and $\Gamma(n+1) = n!$ for n a natural number. Be careful with dimensionful quantities when making your expansions. Thus, express $V(p^2)$ as the sum of a divergent term $\sim 1/\epsilon$ (where $\epsilon \equiv 4 - d$) and finite terms.

(d) For a two-particle to two-particle process



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the Mandelstam variables are given by $s = (p_1 + p_2)^2 = (p_3 + p_4)^2$, $t = (p_3 - p_1)^2 = (p_4 - p_2)^2$, $u = (p_4 - p_1)^2 = (p_3 - p_2)^2$. Write down the entire amplitude $\mathrm{i}\mathcal{M}$ in terms of the physical mass m and coupling λ, the Mandelstam variables as $V(s), V(t), V(u)$, and the counterterms, $\delta_\lambda, \delta_m, \delta_Z$.

(e) By applying the renormalization conditions on the two-particle scattering amplitude, compute the shift δ_λ from the bare coupling constant to the physical coupling constant in the limit $d \to 4$.

(f) Combine your results to write down a finite expression for the two-particle scattering amplitude in terms of physically observable quantities.

(g) Compute the propagator to determine the remaining counterterms, δ_Z and δ_m, working to one-loop order.