Integrability and Non-planarity

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Work in progress

The Niels Bohr Institute
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Hannover, Feb. 22., 2010

Outline

- Integrability of the spectral problem of planar $\mathcal{N} = 4$ SYM
- Beyond the planar limit
- Non-planar ABJM theory and integrability
- Non-planar ABJ theory, integrability and parity
- $\mathcal{N} = 4$ SYM with gaugegroup $SO(N)$
- Summary and outlook

The spectral problem of planar $\mathcal{N}=4$ SYM

$\mathcal{N}=4$ SYM, gauge group SU(N) $\longleftrightarrow$ IIB strings on $AdS_5 \times S^5$

$$\underbrace{\lambda = g_{\rm YM}^2 N,}_{\text{loop expansion}} \quad \underbrace{\frac{1}{N}}_{\text{topological exp.}} \qquad \underbrace{\frac{R^2}{\alpha'} = \sqrt{\lambda},}_{\text{spectrum}} \quad \underbrace{g_s = \frac{\lambda}{N}}_{\text{interactions}}$$

Local gauge invariant operators $\longleftrightarrow$ string states
Conformal dimensions, $\Delta \longleftrightarrow$ energies of string states

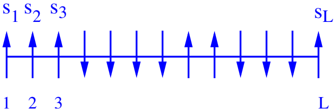
The planar spectral problem of $\mathcal{N}=4$ SYM: **INTEGRABLE**

Determine $\Delta = \Delta(\lambda)$ for $N \to \infty$
Diagonalize dilatation operator D

Theme of the talk: What happens when we go beyond the planar limit (i.e. N finite)

Ex: SU(2) sector, one loop order, $\mathcal{O} = \mathrm{Tr}(ZZZXXXXZZXXXZ)$

[Minahan &Zarembo '02]



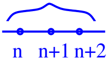
$S_{L+m} = S_m$

$$\hat{D} = \frac{\lambda}{2} \sum_{n=1}^{L} (1 - \bar{\sigma}_n \cdot \bar{\sigma}_{n+1}) = \lambda \sum_{n=1}^{L} \left(1 - P_{n,n+1}\right) \equiv \lambda \sum_{n=1}^{L} \hat{H}_{n,n+1}$$

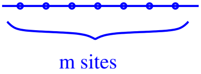
Conserved charges: $\exists \; \hat{Q}_i, \qquad i = 1, \ldots L: \qquad \left[\hat{Q}_i . \hat{Q}_j\right] = 0$

$$\hat{Q}_1 = \sum_n e^{i\hat{P}_n}, \qquad \hat{Q}_2 = \hat{D}$$

$$\hat{Q}_3 = \sum_n [\hat{H}_{n,n+1}, \hat{H}_{n+1,n+2}] \; =$$



$\hat{Q}_m$:



Beyond one-loop order

Higher orders in λ:

Spin chain with long range interactions

Order λ^n: interactions between $n+1$ nearest neighbours

Still integrable:

$\exists$ conserved charges Q_i, $i = 1, \ldots, L$:

at n-loop order: $Q_i = Q_i^0 + \lambda Q_i^1 + \ldots + \lambda^n Q_i^n$,

$$\left[Q_i, Q_j\right] = \mathcal{O}(\lambda^{n+1}), \quad Q_i^n \text{ of range } (i+n)$$

(Almost) proved to be true at any loop order

Discovery: Observation of otherwise unexplained degeneracies in the spectrum [Beisert, C.K. & Staudacher '03]

Parity

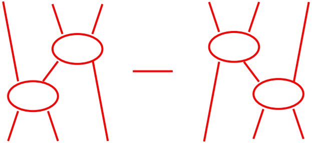
$$\hat{P}\mathrm{Tr}(Z^3X^2ZX) = \mathrm{Tr}(XZX^2Z^3) = \mathrm{Tr}(Z^3XZX^2), \qquad \hat{P}^2 = 1$$

$[\hat{P}, \hat{H}] = 0$, i.e. eigenstates of $\hat{H}$ of definite parity, $P = \pm 1$

Observation: Pairs of operators with opposite parity but the same energy. Survive loop corrections.

Explanation: The existence of $\hat{Q}_3$, i.e. integrability

$$Q_3 = \sum_n [H_{n,n+1}, H_{n+1,n+2}] =$$



$\{\hat{Q}_3, P\} = 0, \qquad [\hat{Q}_3, \hat{H}] = 0$

The operators in a degenerate pair are connected via $\hat{Q}_3$.

Beyond the planar limit

$\mathcal{O} = \text{Tr}(X \ldots XZ \ldots)\text{Tr}(X \ldots XZ \ldots) \subset SU(2)$ sector.

[Constable et al '02], [Beisert, C.K., Plefka, Semenoff & Staudacher '02]

$$\hat{D} = -g_{\text{YM}}^2 : \text{Tr}[Z, X][\check{Z}, \check{X}] :, \qquad (\check{Z})_{\alpha\beta} = \frac{\delta}{\delta Z_{\beta\alpha}}$$

$$= \lambda(D_0 + \underbrace{\frac{1}{N} D_+}_{\text{adds a trace}} + \underbrace{\frac{1}{N} D_-}_{\text{removes a trace}})$$

Origin: Quartic interaction between scalars

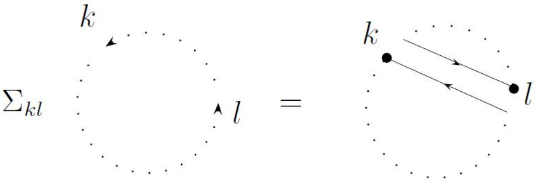
Example:

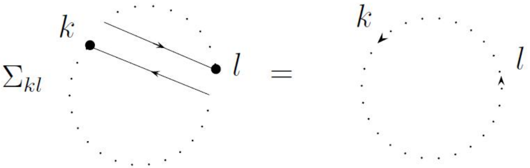
$$\text{Tr}(ZX\check{Z}\check{X}) \cdot \text{Tr}(XZXXZ)\,\text{Tr}(XZ) = \text{Tr}(ZX\check{Z}ZXXZ)\,\text{Tr}(XZ)$$

$$= N\text{Tr}(ZXXXZ)\,\text{Tr}(XZ) + \text{Tr}(ZX)\,\text{Tr}(ZXX)\,\text{Tr}(XZ) + \text{Tr}(ZXZZZXXZ)$$

The non-planar part of $\hat{D}$

$$D_{+} + D_{-} = \sum_{k} \sum_{l \neq k+1} \left(1 - P_{k,l}\right) \Sigma_{k+1,l} \equiv \sum_{k} H_k^{(1)}$$





$\frac{1}{N}$-corrections to short operators

Easy to evaluate

- $D_+\mathcal{O}$, $D_-\mathcal{O}$ involves a finite (small) number of operations
- Only diagonalization of finite-dim. matrix

Strategy:

- Consider closed set of operators. Ex: Length 8 with 3 excitations
- Find the planar eigenvalues and eigenstates (can be checked by Bethe eqns.).
- Write down $\hat{D}$ in the basis of planar eigenstates and do perturbation theory in $\frac{1}{N}$.

$\frac{1}{N}$ corrections to short operators—Lessons learned

Lessons learned

- Δ does not always have a well-defined expansion in λ and $\frac{1}{N}$ but D has. (Higher loop effect.)
  [Ryzhov '01], [Arutyunov et al. '02] [Bianchi, Kovacs Rossi,Stanev '02] [Beisert, C.K. Staudacher '03]
- Degeneracies between single and double trace states (of equal parity) lead to $\frac{1}{N}$ as opposed to $\frac{1}{N^2}$ corrections.
- Including $\frac{1}{N}$ corrections, degeneracies between parity pairs are lifted, but still $[H, P] = 0$
  $\Longrightarrow$ absence of Q_3 (and integrability), at least in its previous form [Beisert, C.K. Staudacher '03]

Conserved charges beyond the planar limit ?

$$D = D^0 + \tfrac{1}{N} D^1, \qquad Q = Q^0 + \tfrac{1}{N} Q^1$$

Determine Q^1 such that

$$0 = \left[D^0, Q^1\right] + \left[D^1, Q^0\right]$$

Q^1 must involve splitting and joining.

A natural guess: $\quad Q^1 = \sum_{n=1}^{L} \left[D_n^0, D_{n+1}^1\right] + \left[D_n^1, D_{n+1}^0\right]$
where

$$D_n^0 = 1 - P_{n,n+1}, \qquad D_n^1 = \underbrace{\sum_{l \neq n+1} (1 - P_{n,l}) \Sigma_{n,l}}_{\text{extremely non-local}},$$

Very complicated — seems not to work

Idea of the asymptotic S-matrix does also not work

ABJM theory — Summary

ABJM theory: 3D $\mathcal{N} = 6$ $U(N)_k \times \overline{U(N)}_{-k}$ superconformal CSM

[Aharony, Bergman, Jafferis & Maldacena '08]

'tHooft expansion: $\underbrace{\lambda = \frac{N}{k}}_{\text{loop expansion}}$, $\underbrace{\frac{1}{N}}_{\text{topological exp.}}$

In $SU(2) \times SU(2)$ sector:

$$D_{planar} = \lambda^2 \sum_{l=1}^{2L} (1 - P_{l,l+2}) \Longrightarrow \text{Planar parity pairs}$$

$$D_{full} = D_{planar} + \lambda^2 (\frac{1}{N}(D_+ + D_-) + \underbrace{\frac{1}{N^2}(D_{++} + D_{--} + D_{00})}_{\text{New type of terms}})$$

Degeneracies lifted at the non-planar level but parity conserved
$\Longrightarrow$ absence of Q_3 (in its previous form) [C.K., Orselli & Zoubos '08]

ABJ theory — Summary

ABJ theory: $3D$, $\mathcal{N} = 6$ $U(N)_k \times \overline{U(M)}_{-k}$ superconformal CSM

[Aharony, Bergman & Jafferis '08]

't Hooft expansion: $\lambda = \frac{N}{k}$, $\bar{\lambda} = \frac{M}{k}$, $\frac{1}{N}$, $\frac{1}{M}$.

In $SU(2) \times SU(2)$ sector:

$$D_{planar} = \lambda\bar{\lambda}\sum_{l=1}^{2L}(1 - P_{l,l+2}) \Longrightarrow \text{No signs of parity breaking}$$

$$D_{full} = D_{planar} + \lambda\bar{\lambda}(\frac{1}{\mathcal{M}}(D_+ + D_-) + \frac{1}{\mathcal{M}^2}(D_{++} + D_{--} + D_{00}))$$

where $\frac{1}{\mathcal{M}} = \frac{1}{N}$ or $\frac{1}{M}$ and $\frac{1}{\mathcal{M}^2} = \frac{1}{M^2}$ or $\frac{1}{N^2}$ or $\frac{1}{MN}$.

Parity is broken at the non-planar level
(and degeneracies lifted). [Caputa, C.K., & Zoubos '09]

Other gauge groups

$\mathcal{N}=4$ SYM, gauge group SO(N) $\longleftrightarrow$ IIB strings on $AdS_5 \times RP^5$ [Witten '98]

$RP^5 = S^5/Z_2, \quad (z^i \equiv -z^i)$, orientifold

Planar spectral problem $\subset$ planar spectral problem for $SU(N)$

Parity is gauged:

$$X^T = -X \implies \hat{P}\mathrm{Tr}(X_{i_1}\ldots X_{i_L}) = (-1)^L \mathrm{Tr}(X_{i_1}\ldots X_{i_L})$$

New $\frac{1}{N}$-effects not involving splitting and joining

Feynman diags w/ cross-caps $\longleftrightarrow$ non-orientable world sheets

$\frac{1}{N}$ effects for gauge group $SO(N)$

Restrict to $SU(2)$ sector: $\mathcal{O} = \mathrm{Tr}(X \ldots XZ \ldots)\mathrm{Tr}(X \ldots XZ \ldots)$

$$\hat{D} = -g_{YM}^2 \mathrm{Tr}[Z, X][\check{Z}, \check{X}], \qquad (\check{Z})_{\alpha\,\beta} Z_{\gamma\epsilon} = \frac{1}{\sqrt{2}}(\delta_{\alpha\epsilon}\delta_{\beta\gamma} - \delta_{\alpha\gamma}\delta_{\beta\epsilon})$$

$$= \lambda\Big(D_0 + \frac{1}{N} D_+ + \frac{1}{N}\tilde{D}_- + \underbrace{\frac{1}{N} D_{flip}}_{\text{Acts inside a trace}}\Big)$$

$$D_{flip} \cdot \mathrm{Tr}(XWZY) = \mathrm{Tr}(XZW^TY) + \mathrm{Tr}(XZYW^T) - \mathrm{Tr}(XW^TYZ) - \mathrm{Tr}(XYW^TZ)$$

Energy corrections generically of order $\frac{1}{N}$: $\quad E_1 = \langle \mathcal{O} | D_{flip} | \mathcal{O} \rangle$

Search for integrability with gauge group $SO(N)$

- No degenerate parity pairs (parity is gauged).
- Degeneracy between single and multiple trace states lifted by $\frac{1}{N}$-corrections.
- Considering only the perturbation D_{flip}
  (restrict to single trace states, not degenerate with multi-trace states)

  - Try to construct conserved charges $Q = Q^0 + \frac{1}{N} Q^1$

    $$0 = [D_0, Q^1] + [D_{flip}, Q^0], \qquad \text{does not work}$$

  - Try to look for perturbed Bethe equations

Considering only D_{flip}

Two excitation states: $O_p^J = \mathrm{Tr}(XZ^pXZ^{J-p})$, J even

Planar eigenstates: $D_0|n^J\rangle = E_n^0|n^J\rangle$

$$|n^J\rangle = \frac{1}{J+1}\sum_{p=0}^{J}\cos\left(\frac{\pi n(2p+1)}{J+1}\right)O_p^J, \qquad 0 \le n \le \frac{J}{2}$$

$$E_n^0 = 8\sin^2\left(\frac{\pi n}{J+1}\right)$$

Non-planar correction: $E_n = E_n^0 + \frac{1}{N}E_n^{flip}$ (prediction for strings)

$$\begin{aligned} E_n^{flip} &= \langle n^J|D_{flip}|n^J\rangle \\ &= \underbrace{2\sin^2\left(\frac{\pi n}{J+1}\right)}_{\text{correction of disp. rel.?}} \\ &\quad \underbrace{-\frac{1}{J+1}\left\{4\tan^2\left(\frac{\pi n}{J+1}\right) - \tan^2\left(\frac{2\pi n}{J+1}\right) - \cos\left(\frac{2\pi n}{J+1}\right)\right\}}_{\text{correction of momenta?}} \end{aligned}$$

E_n^{flip} from a perturbed Bethe ansatz?

Bethe eqn. for length L and M excitations

$$e^{ip_k L} = \prod_{m\neq k}^{M} \frac{u_k - u_m + \frac{i}{2}}{u_k - u_m - \frac{i}{2}}, \qquad \text{where} \qquad e^{ip} = \frac{x(u+\frac{i}{2})}{x(u-\frac{i}{2})}$$

Dispersion relation: $E = 16\sin^2\left(\frac{p}{2}\right) + \delta E(p)$

Parametrizing $x(u) = u(1 - \frac{1}{N}f(u))$ we find from explicit solution

$$f(u+\frac{i}{2}) - f(u-\frac{i}{2}) = -i\frac{1}{16u^3(4u^2-1)}$$

From symmetry arguments

$$f(u+\frac{i}{2}) + f(u-\frac{i}{2}) = 2iu\left(f(u+\frac{i}{2}) - f(u-\frac{i}{2})\right)$$

No solution — equations incompatible

Summary and outlook

- No sign of integrability beyond the planar limit (yet?)
- Need to rethink the concept of integrability when going beyond the planar limit