Computation of the 2-loop Coefficient with the Pure Spinor Formalism

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Motivation

- Compute the overall coefficient of the superstring 2-loop amplitude from first principles (Work in progress with H. Gomez)
- Check 2-loop unitarity in the PS formalism
- Derive general formulae and go beyond (higher points/higher loops)

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History of PS computations

Computation of superstring scattering amplitudes up to overall coefficients:

- 4-pt @ 2-loop (Berkovits,C.M.)
- 4-pt @ 1-loop (Berkovits,C.M.)
- 4-pt: tree-level, 1-loop and 2-loop are proportional (C.M.)
- Anomaly, minimal $\leftrightarrow$ non-minimal (Berkovits,C.M.)
- 5-pt @ 1-loop (C.M., C. Stahn)
- 5-pt @ tree-level and SUSY BCJ relations (C.M.)

Elegant SUSY expressions for kinematic factors in pure spinor superspace:

$$K_0 = -\langle(\lambda A^1)(\lambda\gamma^m W^2)(\lambda\gamma^n W^3)\mathcal{F}^4_{mn}\rangle$$

Present

- Computation of overall coefficients require knowing the measures of the pure spinor variables and their normalizations, e.g.

$$[d\lambda]T_{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5} = c_\lambda \epsilon_{\alpha_1\ldots\alpha_5\rho_1\ldots\rho_{11}} d\lambda^{\rho_1}\ldots d\lambda^{\rho_{11}}$$

$$c_\lambda = \left(\frac{\alpha'}{2}\right)^{-2} \frac{1}{11!} \left(\frac{A_g}{4\pi^2}\right)^{11/2}$$

- Integration over pure spinor space (H. Gomez, 2009)

$$\int [d\lambda][d\bar{\lambda}](\lambda\bar{\lambda})^n \mathrm{e}^{-(\lambda\bar{\lambda})} = \frac{(7+n)!}{7!\,60} \left(\frac{2\pi}{A_g}\right)^{11}$$

The goal

- Compute the coefficients tree-level, one- and two-loop coefficients C_0, C_1 and C_2 (omit $(2\pi)^{10}\delta^{(10)}(k)$)

$$A_0 = \kappa^4 e^{-2\mu} C_0 \left(\frac{\alpha'}{2}\right)^8 K_0 \overline{K}_0 C(s,t,u),$$

$$A_1 = C_1 \kappa^4 K_0 \overline{K}_0 \left(\frac{\alpha'}{2}\right)^8 \int \frac{d^2\tau}{\tau_2^5} \prod_{i=2}^{4} \int d^2 z_i \prod_{i<j}^{4} F_1(z_i, z_j)^{\alpha k^i \cdot k^j}$$

$$A_2 = C_2 \kappa^4 e^{2\lambda} K_0 \overline{K}_0 \left(\frac{\alpha'}{2}\right)^{10} \int_{\mathcal{M}_2} \frac{d^2\Omega_{IJ}}{(\mathrm{detIm}\Omega_{IJ})^5} \int_{\Sigma_4} |\mathcal{Y}_s|^2 \prod_{i<j} F_2(z_i, z_j)^{\alpha k^i \cdot}$$

The goal

- RNS: 2-loop coefficient found indirectly by factorization (D'Hoker, Gutperle, Phong, 2005)

$$C_1^2 = 8\pi^2 C_0 C_2,$$

- Too difficult for direct computation (functional determinants)
- Due to g_s dependence, normalization of tree-level amp matters
- Do amplitudes in PS formalism obey the factorization constraint? (unitarity)

The goal

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Non-Minimal Pure Spinor Formalism

Action (Berkovits, 2005)

$$S = \int d^2z \left(\frac{1}{2}\partial X^m \overline{\partial} X_m + p_\alpha \overline{\partial}\theta^\alpha - w_\alpha \overline{\partial}\lambda^\alpha - \overline{w}^\alpha \overline{\partial}\overline{\lambda}_\alpha + s^\alpha \overline{\partial} r_\alpha \right)$$

With bosonic pure spinors λ^α , $\overline{\lambda}_\alpha$

$$(\lambda\gamma^m\lambda) = 0$$

and a constrained fermionic r_α

$$(\overline{\lambda}\gamma^m r) = 0$$

Pure Spinor Formalism

Some important definitions for amplitude computations:

- Lorentz current
$$N^{mn} = \frac{\alpha\prime}{4}(w\gamma^{mn}\lambda)$$
- Supersymmetric momentum
$$\Pi^m = \partial X^m + \frac{1}{2}(\theta\gamma^m\partial\theta)$$
- Supersymmetric derivative
$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + \frac{1}{2}(\theta\gamma^m)_\alpha\partial_m$$

Pure Spinor Formalism

- Supersymmetric Green-Schwarz constraint

$$d_\alpha = \frac{\alpha'}{2} p_\alpha - \frac{1}{2}(\gamma^m\theta)_\alpha \partial X_m - \frac{1}{8}(\gamma^m\theta)_\alpha(\theta\gamma_m\partial\theta)$$

- The b-ghost is a composite operator...

$$b_{\text{non}-\text{min}} = \ldots - \frac{1}{192(\lambda\overline{\lambda})^2}(\overline{\lambda}\gamma^{mnp}r)(d\gamma_{mnp}d) + \ldots$$

- Ghost current

$$J = w_\alpha\lambda^\alpha - \overline{w}^\alpha\overline{\lambda}_\alpha$$

Pure Spinor Formalism

Relevant OPE's

$$X^m(z,\overline{z})X^n(w,\overline{w}) \longrightarrow -\frac{1}{2}\eta^{mn}\ln|z-w|^2$$

$$N^{mn}(z)\lambda^\alpha(y) \longrightarrow \frac{\alpha\prime}{4}\frac{(\gamma^{mn}\lambda)^\alpha}{z-y}$$

$$d_\alpha(z)V(y,\theta) \longrightarrow \frac{D_\alpha V(y,\theta)}{z-y}$$

$$\Pi^m(z)V(y,\theta) \longrightarrow \frac{\partial^m V(y,\theta)}{z-y}$$

$$J(z)T(y) \longrightarrow \frac{3}{(z-y)^3}+\frac{J(y)}{(z-y)^2}$$

- The same ghost number anomaly as in bosonic string theory!

Issues of RNS and GS not present

Space-time SUSY

The pure spinor formalism has manifest space-time supersymmetry

- Scattering amplitudes will result in superspace expressions
- Only one computation for all multiplet states

Covariant BRST Quantization

$$Q_{\text{BRST}} = \oint \lambda^\alpha d_\alpha + \overline{w}^\alpha r_\alpha ,$$

Topological Prescription for Scattering Amplitudes

- Non-minimal pure spinor formalism is a $N = 2$ $\hat{c} = 3$ string theory (Berkovits, 2005)
- Topological string theory prescription to compute amplitudes
- Massless On-shell Vertex Operators:
  - Unintegrated

$$V = \kappa\lambda^\alpha A_\alpha(X, \theta), \quad QV = 0$$

  - Integrated

$$U = \kappa \int dz \left(\partial\theta^\alpha A_\alpha + A_m \Pi^m + d_\alpha W^\alpha + \frac{1}{2} N^{mn} \mathcal{F}_{mn} \right), \quad QU = \partial V$$

- Where $A_\alpha(x, \theta), A_m(x, \theta), W^\alpha(x, \theta)$ and $\mathcal{F}_{mn}(x, \theta)$ are the SYM superfields

$$D_\alpha A_\beta + D_\beta A_\alpha = \gamma^m_{\alpha\beta} A_m, \quad D_\alpha A_m = (\gamma_m W)_\alpha + k_m A_\alpha$$

$$D_\alpha W^\beta = \frac{1}{4} (\gamma^{mn})_\alpha{}^\beta \mathcal{F}_{mn}, \quad D_\alpha \mathcal{F}_{mn} = 2k_{[m} (\gamma_{n]} W)_\alpha$$

θ's all over the place

SYM Superfields θ-Expansion

$$A_\alpha(x,\theta) = \frac{1}{2}a_m(\gamma^m\theta)_\alpha - \frac{1}{3}(\xi\gamma_m\theta)(\gamma^m\theta)_\alpha - \frac{1}{32}F_{mn}(\gamma_p\theta)_\alpha(\theta\gamma^{mnp}\theta) + \ldots$$

$$A_m(x,\theta) = a_m - (\xi\gamma_m\theta) - \frac{1}{8}(\theta\gamma_m\gamma^{pq}\theta)F_{pq} + \frac{1}{12}(\theta\gamma_m\gamma^{pq}\theta)(\partial_p\xi\gamma_q\theta) + \ldots$$

$$W^\alpha(x,\theta) = \xi^\alpha - \frac{1}{4}(\gamma^{mn}\theta)^\alpha F_{mn} + \frac{1}{4}(\gamma^{mn}\theta)^\alpha(\partial_m\xi\gamma_n\theta)$$

$$+\frac{1}{48}(\gamma^{mn}\theta)^\alpha(\theta\gamma_n\gamma^{pq}\theta)\partial_m F_{pq} + \ldots$$

$$\mathcal{F}_{mn}(x,\theta) = F_{mn} - 2(\partial_{[m}\xi\gamma_{n]}\theta) + \frac{1}{4}(\theta\gamma_{[m}\gamma^{pq}\theta)\partial_{n]}F_{pq} + \ldots,$$

Scattering Amplitude Prescriptions

- Tree-level

$$\mathcal{A} = \kappa^4 e^{-2\mu} \int d^2 z_4 \langle |\mathcal{N}\, V^1(0) V^2(1) V^3(\infty) U^4(z_4)|^2 \rangle$$

- One-loop

$$\mathcal{A} = \frac{1}{2}\kappa^4 \int d^2\tau_1 \langle |\mathcal{N}(b,\mu_1) V^1(0) \prod_{i=2}^{4} \int d^2 z_i U^i(z_i)|^2 \rangle$$

- Two-loops

$$\mathcal{A}_2 = \frac{1}{2} e^{2\mu} \kappa^4 \int \prod_{I=1}^{3} d^2\tau_I \prod_{i=1}^{4} \int d^2 z_i \langle |\mathcal{N}(b,\mu_I) U^i(z_i)|^2 \rangle$$

Scattering Amplitude Prescriptions

- b-ghost insertion the same as in bosonic string theory

$$(b, \mu_j) = \frac{1}{2\pi} \int d^2 y_j b_{zz} \mu_{j\,\bar{z}}^{z}$$

- $0 \cdot \infty$ is regulated by

$$\mathcal{N} = \mathrm{e}^{-(\lambda\bar{\lambda})-(w^I\overline{w}^I)-(r\theta)+(s^I d^I)}$$

- $\langle\,\rangle$ denote integration over

$$\prod_{I=1}^{g} \int [d\theta][dd^I][dr][ds^I][d\overline{w}^I][dw^I][d\lambda][d\bar{\lambda}]$$

Measures for zero-modes

$$[d\lambda]T_{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5} = c_\lambda \epsilon_{\alpha_1\ldots\alpha_5\rho_1\ldots\rho_{11}} d\lambda^{\rho_1}\ldots d\lambda^{\rho_{11}}$$

$$[d\overline{\lambda}]\overline{T}^{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5} = c_{\overline{\lambda}}\epsilon^{\alpha_1\ldots\alpha_5\rho_1\ldots\rho_{11}} d\overline{\lambda}_{\rho_1}\ldots d\overline{\lambda}_{\rho_{11}}$$

$$[d\omega] = c_\omega T_{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5}\epsilon^{\alpha_1\ldots\alpha_5\rho_1\ldots\rho_{11}} d\omega_{\rho_1}\ldots d\omega_{\rho_{11}}$$

$$[d\overline{w}]T_{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5} = c_{\overline{w}}\epsilon_{\alpha_1\ldots\alpha_5\rho_1\ldots\rho_{11}} d\overline{w}^{\rho_1}\ldots d\overline{w}^{\rho_{11}}$$

$$[dr] = c_r \overline{T}^{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5}\epsilon_{\alpha_1\ldots\alpha_5\delta_1\ldots\delta_{11}}\partial_r^{\delta_1}\ldots\partial_r^{\delta_{11}}$$

$$[ds^I] = c_s T_{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5}\epsilon^{\alpha_1\ldots\alpha_5\rho_1\ldots\rho_{11}}\partial^{s^I}_{\rho_1}\ldots\partial^{s^I}_{\rho_{11}}$$

$$[d\theta] = c_\theta d^{16}\theta, \quad [dd^I] = c_d d^{16} d^I$$

$$c_\lambda = \left(\frac{\alpha'}{2}\right)^{-2} \frac{1}{11!} \left(\frac{A_g}{4\pi^2}\right)^{11/2} \quad c_\omega = \left(\frac{\alpha'}{2}\right)^{2} \frac{(4\pi^2)^{-11/2}}{11!5!\, Z_g^{11/g}}$$

$$c_{\overline{\lambda}} = \left(\frac{\alpha'}{2}\right)^{2} \frac{2^6}{11!} \left(\frac{A_g}{4\pi^2}\right)^{11/2} \quad c_{\overline{w}} = \left(\frac{\alpha'}{2}\right)^{-2} \frac{(4\pi^2)^{-11/2}(\lambda\overline{\lambda})^3}{11!\, Z_g^{11/g}}$$

$$c_r = \left(\frac{\alpha'}{2}\right)^{-2} \frac{R}{11!5!} \left(\frac{2\pi}{A_g}\right)^{11/2} \quad c_s = \left(\frac{\alpha'}{2}\right)^{2} \frac{(2\pi)^{11/2}}{2^6 11!5!(\lambda\overline{\lambda})^3} Z_g^{11/g} R^{-1}$$

$$c_\theta = \left(\frac{\alpha'}{2}\right)^{4} \left(\frac{2\pi}{A_g}\right)^{16/2} \quad c_d = \left(\frac{\alpha'}{2}\right)^{-4} (2\pi)^{16/2} Z_g^{16/g}$$

A_g is the area of the Riemann surface and

$$Z_g = \frac{1}{\sqrt{\det(2\mathrm{Im}(\Omega_{IJ}))}}$$

- They are measures in the phase space like the standard $\frac{dx}{\sqrt{2\pi}}\frac{dp}{\sqrt{2\pi}}$ in quantum mechanics (H. Gomez, 2009)
- 11! are due to number of d.o.f, 5! are due to contractions of $T_{\alpha_1\ldots\alpha_5}$
- Z_g appear to make basis of holomorphic 1-forms orthonormal
- Integration over non zero modes $(\det\partial\overline{\partial})^{-11-11+16+11} = (\det\partial\overline{\partial})^5$ cancels $(\det\partial\overline{\partial})^{-5}$ from exponential factors in the vertices
- Use the following formula for their combined result

$$\langle\prod_{i=1}^{4} : e^{ik\cdot x} :\rangle_g = (2\pi)^{10}\delta^{(10)}(k)\frac{A_g^5}{(2\pi^2\alpha')^5}\prod_{i<j} F_g(z_i,\overline{z}_j)^{\alpha k^i\cdot k^j}$$

- No need for difficult determinant computations!

- Integration over pure spinor space (H. Gomez, 2009)

$$\int [d\lambda][d\bar\lambda](\lambda\bar\lambda)^n \mathrm{e}^{-(\lambda\bar\lambda)} = \frac{(7+n)!}{7!\,60}\left(\frac{2\pi}{A_g}\right)^{11}$$

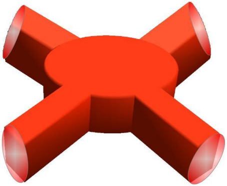
therefore

$$N_n^{(g)} = \int [d\theta][dr][d\lambda][d\bar\lambda]\frac{e^{-(\lambda\bar\lambda)-(r\theta)}}{(\lambda\bar\lambda)^{3-n}}(\lambda^3\theta^5)$$

$$= 2^7 R\left(\frac{2\pi}{A_g}\right)^{5/2}\left(\frac{\alpha'}{2}\right)^2\frac{(7+n)!}{7!}, \quad n\geq 0,$$

- A_g cancels out in $|N_n^{(g)}|^2\langle\prod_{i=1}^4 : e^{ik\cdot x} :\rangle_g$
- Closed string amplitudes don't depend on the area

Four gravitons at tree-level



Tree-level 4-graviton computation

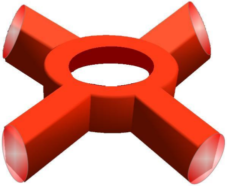
- Computed up to overall coeff in 2008 (with several ids in pure spinor superspace) (C.M.)
- Overall coefficient easy to fix

$$A_0 = C_0 \kappa^4 e^{-2\mu} (\frac{\alpha'}{2})^8 K^{\rm RNS} \overline{K}^{\rm RNS} C(s,t,u),$$

$$C_0 = \left(\frac{\sqrt{2}}{2^{12}\pi^6 \alpha'^5} \right)$$

- Trivial agreement with RNS

Massless 4-point one-loop amplitude



Massless 4-point one-loop amplitude

- Computed with the minimal pure spinor formalism (Berkovits 2004)

$$K_1 = \int d^{16}\theta(\epsilon T^{-1})^{((\alpha\beta\gamma))}_{[\rho_1\ldots\rho_{11}]}\theta^{\rho_1}\ldots\theta^{\rho_{11}}(\gamma_{mnpqr})_{\beta\gamma}\times$$

$$\Big[A_{1\alpha}(\theta)(W_2(\theta)\gamma^{mnp}W^3(\theta))\mathcal{F}_4^{qr}(\theta)\Big]$$

  and shown to agree with the RNS and GS results (C.M. 2005)

$$K_1 = \langle(\lambda A)(\lambda\gamma^m W)(\lambda\gamma^n W)\mathcal{F}_{mn}\rangle = t_8 F^4 + \ldots$$

- Computed also in the non-minimal pure spinor formalism (Berkovits 2005, Berkovits & C.M. 2006)

How to get it (Non-minimal)

- One can compute it quickly by using symmetry alone
- Recall the regulator

$$N = \exp\left(-(\lambda\bar{\lambda}) - (r\theta) - (w\overline{w}) + (sd)\right)$$

- Zero modes:
  - s^α has 11 zero-modes
  - d_α has 16 zero-modes
- N contributes 11 s and 11 d, the b-ghost 2 d and the external vertices 3 d's.
- Therefore one gets (λA) and $(dW)^3$ from the external vertices and $(\bar{\lambda}\gamma^{mnp}r)(d\gamma_{mnp}d)$ from the b-ghost

Unique contraction

There is only one Lorentz invariant contraction for these fields

$$\langle(\bar{\lambda}\gamma_{mnp}D)(\lambda A)(\lambda\gamma^m W)(\lambda\gamma^n W)(\lambda\gamma^p W)\rangle$$

Overall coefficient

- Computed in 2009 and agreement with RNS found (H. Gomez, 2009)
- However, $1/4$ mistake! (work in progress)

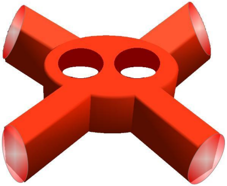
$$A_1 = C_1 \kappa^4 K_0 \overline{K}_0 \left(\frac{\alpha'}{2}\right)^8 \int \frac{d^2\tau}{\tau_2^5} \prod_{i=2}^{4} \int d^2 z_i \prod_{i<j}^{4} F_1(z_i, z_j)^{\alpha k^i \cdot k^j}.$$

$$C_1 = \frac{1}{2^9 \pi^2 \alpha'^5}$$

- Disagreement with RNS of D'Hoker, Gutperle and Phong!

$$A_1^{\mathrm{PS}} = \frac{1}{4} A_1^{\mathrm{RNS}}$$

Massless 4-point two-loop amplitude



Massless 4-point two-loop amplitude

- Can be computed quickly using zero-mode saturation (Berkovits, 2005)
- Kinematic factor

$$K_2 = \langle (\lambda\gamma^{mnpqr}\lambda)\mathcal{F}^1_{mn}\mathcal{F}^2_{pq}\mathcal{F}^3_{rs}(\lambda\gamma^s W^4)\rangle$$

- Shown to agree with RNS results of D'Hoker and Phong up to overall coefficient (Berkovits, C.M, 2005)
- Pure spinor superspace identity (C.M., 2008)

$$K_2 = -16(k^1 \cdot k^2)\langle (\lambda A^2)(\lambda\gamma^r W^1)(\lambda\gamma^s W^4)\mathcal{F}^3_{rs}\rangle = -16(k^1 \cdot k^2)K_1$$

Massless 4-point two-loop amplitude

- Overall coeff: after lots of pure spinor covariant manipulations

$$A_2 = C_2 \kappa^4 e^{2\lambda} K_0 \overline{K}_0 \left(\frac{\alpha'}{2}\right)^{10} \int_{\mathcal{M}_2} \frac{d^2\Omega_{IJ}}{(\mathrm{det}\mathrm{Im}\Omega_{IJ})^5} \int_{\Sigma_4} |\mathcal{Y}_s|^2 \prod_{i<j} F_2(z_i, z_j)^{\alpha k^i \cdot}$$

$$C_2 = \frac{\sqrt{2}}{2^{10}\alpha'^5}$$

- Disagreement with RNS of D'Hoker, Gutperle and Phong again!

$$A_2^{\mathrm{PS}} = \frac{1}{16} A_2^{\mathrm{RNS}}$$

Was ist los?

- Work in progress possibilities:
  1. We made an embarrassing mistake
  2. The PS formalism is not unitary ($4 = 2^2$ at 1-loop and $16 = 2^4$ at 2-loops: 2^{2g} spin structures...)
  3. D'Hoker et al. made a mistake...
- However, the PS amplitudes satisfy the same factorization condition!
  - This is reassuring, as the factorization condition can be derived from S-duality (D'Hoker, Gutperle, Phong, 2005)
  - RNS: 2-loop coefficient found by

$$C_2 = \frac{C_1^2}{8\pi^2 C_0}$$

  - $1/4$ mistake in C_1 leads to $1/16$ mistake in C_2

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