Elliptic multiple zeta values in string scattering amplitudes

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Nordic string meeting

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Introduction

Goal: suitable mathematical language for various scattering amplitudes

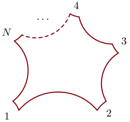
- *Feynman graphs:* very useful, but challenging: polylogarithms
  iterated integrals/nested sums, multiple zeta values
- *recently*: elliptic integrals make an appearance in field theory
- *strategy:* consider more symmetric/constrained theory
  treat realism for adequate language and omnipotent formalism
- *find:* echos of these structures in more realistic theories

Outline

Open string theory as a simple testing ground
$\Rightarrow$ only *one* topology per loop order
$\Rightarrow$ improved divergent behaviour (compared to gauge theory)

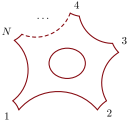
Tree-level

multiple polylogarithms
multiple zeta values ζ



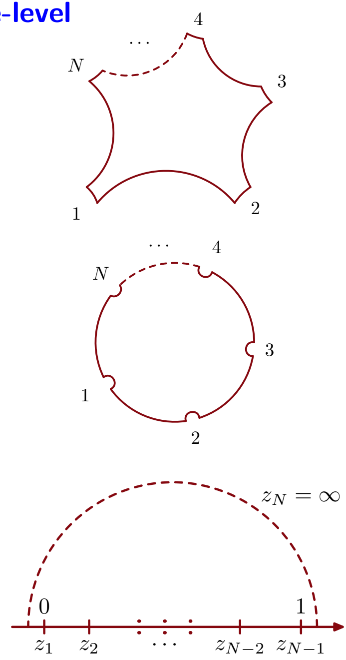
One-loop

elliptic iterated integrals
elliptic multiple zeta values $\boldsymbol{\omega}$

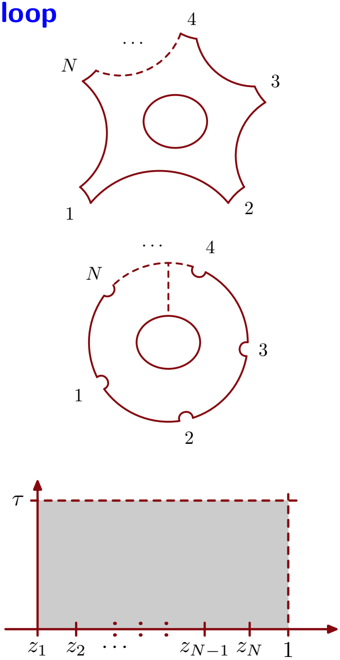


Conformal symmetry

Tree-level



One-loop

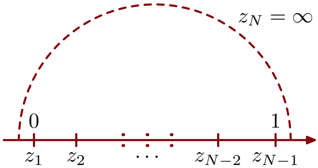


Tree-level $\Rightarrow$ multiple zeta values

N-point tree-level open-string amplitude: [Veneziano]... [Mafra, Schlotterer, Stieberger]

$$\mathbf{A}^{\text{tree}}_{\text{string}} = \boldsymbol{F}.\mathbf{A}_{\text{YM}}$$

- dependence on external states in $\mathbf{A}_{\text{YM}}$
- $\boldsymbol{F} = \boldsymbol{F}(s_{ij})$, $s_{ij} = \alpha'(k_i + k_j)^2$
- coefficients are ***multiple zeta values (MZVs)***



$$F^{(2,\dots,N-2)} = \prod_{i=2}^{N-2} \int_0^{z_{i+1}} \frac{\mathrm{d}z_i}{z_i - a_i} \prod_{i<j}^{N-1} \sum_{n_{ij}=0}^{\infty} (s_{ij})^{n_{ij}} \underbrace{\frac{(\ln|z_{ij}|)^{n_{ij}}}{n_{ij}!}}_{\text{multiple polylogs}}$$

Multiple polylogarithms

$$G(a_1, a_2, \dots, a_n; z) = \int_0^z \frac{\mathrm{d}t}{t - a_1} G(a_2, \dots, a_n; t), \quad G(;z) = 1, \quad G(\vec{a}; 0) = G(;0) = 0$$

$$G(\underbrace{0, 0, \dots, 0}_{w}; z) = \frac{1}{w!}(\ln\ z)^w \qquad G(\underbrace{1, 1 \dots, 1}_{w}; z) = \frac{1}{w!} \ln^w(1 - z)$$

Multiple zeta values

$$\zeta_{n_1,\dots,n_r} = \sum_{0<k_1<\dots<k_r} \frac{1}{k_1^{n_1} \dots k_r^{n_r}} = (-1)^r G(\underbrace{0, 0, \dots, 0, 1}_{n_r}, \dots, \underbrace{0, 0, \dots, 0, 1}_{n_1}; 1) = \zeta_{(w)}$$

Products of differential weights (tree-level) $\Rightarrow$ partial fraction:

$$\int_0^z \mathrm{d}w\ \frac{1}{w-a_1}\frac{1}{w-a_2} = \int_0^z \mathrm{d}w \left(\frac{1}{(w-a_1)(a_1-a_2)} + \frac{1}{(w-a_2)(a_2-a_1)} \right)$$

Shuffle product:

$$\begin{aligned} \zeta(a_1,\ldots,a_r)\,\zeta(a_{r+1},\ldots,a_{r+s}) &= \zeta(a_1,\ldots,a_r \sqcup\!\sqcup\ a_{r+1},\ldots,a_{r+s}) \\ &= \sum_{\sigma\in\Sigma(r,s)} \zeta\big(a_{\sigma(1)},\ldots,a_{\sigma(r+s)}\big) \end{aligned}$$

$$a_i \in \{0,1\}$$

Stuffle relations:

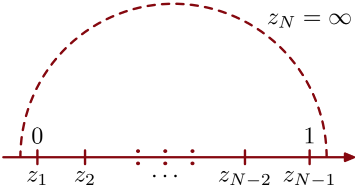
$$\zeta_n\,\zeta_m \quad = \quad \zeta_{m,n} + \zeta_{n,m} + \zeta_{m+n}$$

5-point-example:

$$\begin{aligned} F^{(23)} = 1 &- \zeta_2(s_{12}s_{23} + s_{12}s_{24} + s_{12}s_{34} + s_{13}s_{34} + s_{23}s_{34}) \\ &+ \zeta_3(s_{12}^2 s_{23} + s_{12}s_{23}^2 + s_{12}^2 s_{24} + 2s_{12}s_{23}s_{24} + s_{12}s_{24}^2 + \cdots) + \cdots \\ &+ \zeta_{3,5}(\ldots) + \cdots \end{aligned}$$

Collect tree-level results

Tree-level



$$\prod_{i=2}^{N-2} \int_0^{z_{i+1}} \mathrm{d}z_i$$

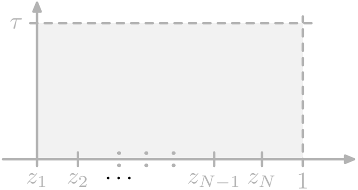
multiple polylogarithms

$$G(a_1, a_2, \ldots, a_n; z) = \int_0^z \mathrm{d}t \, \frac{1}{t - a_1} \, G(a_2, \ldots, a_n; t)$$

partial fraction

multiple zeta values ζ

One-loop



$$\int_0^1 \mathrm{d}z_N \prod_{i=1}^{N-1} \int_0^{z_{i+1}} \mathrm{d}z_i \, \delta(z_1)$$

elliptic iterated integrals

$$\Gamma\left(\begin{smallmatrix} n_1 & n_2 & \ldots & n_r \\ a_1 & a_2 & \ldots & a_r \end{smallmatrix}; z\right) = \int_0^z \mathrm{d}t \, f^{(n_1)}(t - a_1) \, \Gamma\left(\begin{smallmatrix} n_2 & \ldots & n_r \\ a_2 & \ldots & a_r \end{smallmatrix}; t\right)$$

Fay-identities

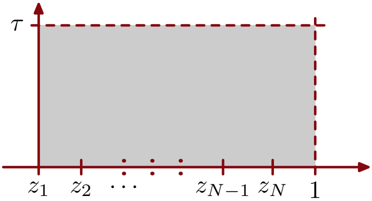
elliptic multiple zeta values ω

One-loop ⇒ elliptic multiple zeta values

***N*-point one-loop open-string amplitude:**

[Green Schwarz] ... [Broedel, Mafra Matthes, Schlotterer]

- topologies: all genus-one worldsheets with boundaries.
- here: cylinder with insertions on one boundary only: $\mathrm{Im}(z) = 0$
- one imaginary parameter: τ



$$A^{\text{1-loop}}_{\text{string}}(1,2,3,4) = s_{12}s_{23}A^{\text{tree}}_{\mathrm{YM}}(1,2,3,4)\underbrace{\int_0^\infty \mathrm{d}\tau\ I_{\text{4pt}}(1,2,3,4)(\tau)}_{F}$$

$$I_{\text{4pt}}(1,2,3,4)(\tau) = \int_0^1 \mathrm{d}z_N \prod_{i=1}^{N-1}\int_0^{z_{i+1}} \mathrm{d}z_i\, \delta(z_1) \prod_{i<j}^{N}\sum_{n_{ij}=0}^{\infty}\frac{1}{n_{ij}!}(s_{ij})^{n_{ij}}\underbrace{(\ln\chi_{ij}(\tau))^{n_{ij}}}_{???}$$

Compare to tree-level:

$$\prod_{i=2}^{N-2}\int_0^{z_{i+1}}\frac{\mathrm{d}z_i}{z_i - a_i}\prod_{i<j}^{N-1}\sum_{n_{ij}=0}^{\infty}\frac{1}{n_{ij}!}(s_{ij})^{n_{ij}}\underbrace{(\ln|z_{ij}|)^{n_{ij}}}_{\text{multiple polylogs}}$$

One-loop

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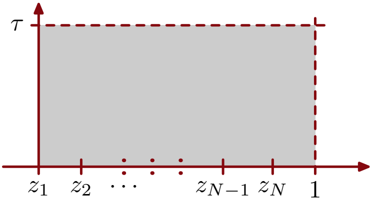

$$A^{\text{1-loop}}_{\text{string}}(1,2,3,4) = s_{12}s_{23}A^{\text{tree}}_{\text{YM}}(1,2,3,4)\underbrace{\int_0^{\infty} \mathrm{d}\tau \; I_{\text{4pt}}(1,2,3,4)(\tau)}_{F}$$

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Compare to tree-level:

$$\prod_{i=2}^{N-2}\int_0^{z_{i+1}} \frac{\mathrm{d}z_i}{z_i - a_i} \prod_{i<j}^{N-1} \sum_{n_{ij}=0}^{\infty} \frac{1}{n_{ij}!}(s_{ij})^{n_{ij}} \left(\int_{z_j+1}^{z_i} \frac{\mathrm{d}t}{t - z_j}\right)^{n_{ij}}$$

One-loop

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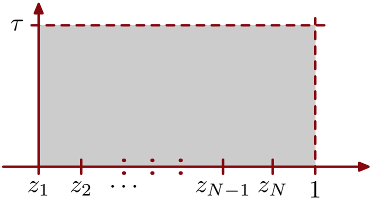

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$$I_{\text{4pt}}(1,2,3,4)(\tau)=\int_0^1 \mathrm{d}z_N\prod_{i=1}^{N-1}\int_0^{z_{i+1}}\mathrm{d}z_i\,\delta(z_1)\prod_{i<j}^{N}\sum_{n_{ij}=0}^{\infty}\frac{1}{n_{ij}!}\left(s_{ij}\right)^{n_{ij}}\underbrace{\left(\ln\chi_{ij}(\tau)\right)^{n_{ij}}}_{???}$$

$$\ln|z_{ij}|=\int_{z_j+1}^{z_i}\frac{\mathrm{d}t}{t-z_j}\qquad\Leftrightarrow\qquad \ln\chi_{ij}(\tau)=\int_{z_j}^{z_i}\mathrm{d}w\,f^{(1)}(w-z_j,\tau)$$

Natural distance on the elliptic curve:

Eisenstein–Kronecker-series: [Kronecker][Brown Levin]

$$F(z,\alpha,\tau) \equiv \frac{\theta_1'(0,\tau)\theta_1(z+\alpha,\tau)}{\theta_1(z,\tau)\theta_1(\alpha,\tau)} \;,$$

$$F(z,\alpha,\tau) \;\sim \sum_{n=0}^{\infty} f^{(n)}(z,\tau)\alpha^n$$

Natural weights for differentials on an elliptic curve: [Enriquez][Brown Levin]

$$f^{(n)}(z,\tau) = f^{(n)}(z+1,\tau) \quad \text{and} \quad f^{(n)}(z,\tau) = f^{(n)}(z+\tau,\tau) \;.$$

Explicitly: (simplification in our situation because $\mathrm{Im}(z)=0$)

$$f^{(0)}(z,\tau) \equiv 1 \qquad f^{(1)}(z,\tau) \equiv \partial \ln\theta_1(z,\tau) + 2\pi i \frac{\mathrm{Im}z}{\mathrm{Im}\tau}$$

$$f^{(2)}(z,\tau) \equiv \frac{1}{2}\Big[\Big(\partial \ln\theta_1(z,\tau) + 2\pi i \frac{\mathrm{Im}z}{\mathrm{Im}\tau}\Big)^2 + \partial^2 \ln\theta_1(z,\tau) - \frac{1}{3}\frac{\theta_1'''(0,\tau)}{\theta_1'(0,\tau)}\Big]$$

Parity: $f^{(n)}(-z,\tau) = (-1)^n f^{(n)}(z,\tau)$

Elliptic iterated integrals (suppress τ-dependence from here...)

$$\Gamma\left(\begin{smallmatrix} n_1 & n_2 & \dots & n_r \\ a_1 & a_2 & \dots & a_r \end{smallmatrix}; z\right) \equiv \int_0^z \mathrm{d}w\, f^{(n_1)}(w-a_1)\, \Gamma\left(\begin{smallmatrix} n_2 & \dots & n_r \\ a_2 & \dots & a_r \end{smallmatrix}; w\right)$$

$\Rightarrow$ can rewrite any integral $\int_{1234}\dots$ into an *elliptic iterated integral*.

Products of differential weights (tree-level) $\Rightarrow$ *partial fraction:*

$$\int_0^z \mathrm{d}w\, \frac{1}{w-a_1}\frac{1}{w-a_2} = \int_0^z \mathrm{d}w \left(\frac{1}{(w-a_1)(a_1-a_2)} + \frac{1}{(w-a_2)(a_2-a_1)} \right)$$

Products of differential weights (one-loop) $\Rightarrow$ *Fay identities*

$$\int_0^z \mathrm{d}w\, f^1(w-a_1) f^1(w) = \int_0^z \mathrm{d}w \Big(f^{(1)}(w-a_1) f^{(1)}(a_1) - f^{(1)}(w) f^{(1)}(a_1) \\ + f^{(2)}(w) + f^{(2)}(a_1) + f^{(2)}(w-a_1) \Big)$$

Fay identity: trisecant equation for Eisenstein–Kronecker series:

$$F(z_1,\alpha_1)F(z_2,\alpha_2) = F(z_1,\alpha_1+\alpha_2)F(z_2-z_1,\alpha_2) \\ + F(z_2,\alpha_1+\alpha_2)F(z_1-z_2,\alpha_1)$$

Elliptic multiple zeta values (eMZV's)

$$\omega(n_1,n_2,\ldots,n_r,\tau) \equiv \int\limits_{0\le z_i\le z_{i+1}\le 1} f^{(n_1)}(z_1,\tau)\mathrm{d}z_1\, f^{(n_2)}(z_2,\tau)\mathrm{d}z_2\,\ldots f^{(n_r)}(z_r,\tau)\mathrm{d}z_r$$

$$= \Gamma(n_r,\ldots,n_2,n_1;1) = \Gamma\left(\begin{smallmatrix} n_r & n_{r-1} & \ldots & n_1 \\ 0 & 0 & \ldots & 0\end{smallmatrix};1\right)$$

Shuffle relation:

$$\omega(n_1,n_2,\ldots,n_r)\,\omega(k_1,k_2,\ldots,k_s) = \omega\big((n_1,n_2,\ldots,n_r)\sqcup\!\sqcup\,(k_1,k_2,\ldots,k_s)\big)$$

Reflection identity:

$$\omega(n_1,n_2,\ldots,n_{r-1},n_r) = (-1)^{n_1+n_2+\ldots+n_r}\omega(n_r,n_{r-1},\ldots,n_2,n_1)$$

Numerous other relations, e.g.

$$0 = \omega(2,3) - \omega(0,5),$$
$$0 = \omega(2,5) - \omega(3,4) - 2\,\omega(0,7)$$
$$0 = \omega(0,0,5) + \omega(0,1,4) + \omega(2,0,3)$$
$$0 = 10\,\omega(0,0,0,5) + 4\,\omega(0,0,3,2) + 2\,\omega(0,2,0,3) - \omega(2)\omega(0,3) - \omega(0,5)$$

Four-point result

$$\begin{aligned} I_{\text{4pt}}(1,2,3,4)(\tau) = {} & \omega(0,0,0) \;-\; 2\,\omega(0,1,0,0)\,(s_{12}+s_{23}) \\ & +\; 2\,\omega(0,1,1,0,0)\,(s_{12}^2+s_{23}^2) \;-\; 2\,\omega(0,1,0,1,0)\,s_{12}s_{23} \\ & +\; \beta_5\,(s_{12}^3+2s_{12}^2 s_{23}+2s_{12}s_{23}^2+s_{23}^3) \\ & +\; \beta_{2,3}\,s_{12}s_{23}(s_{12}+s_{23}) \;+\; \mathcal{O}(\alpha'^4) \end{aligned}$$

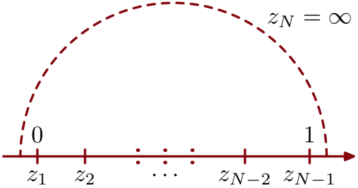
with

$$\beta_5 = \frac{4}{3}\left[\,\omega(0,0,1,0,0,2)+\omega(0,1,1,0,1,0)-\omega(2,0,1,0,0,0)-\zeta_2\,\omega(0,1,0,0)\right]$$

$$\begin{aligned} \beta_{2,3} = {} & \frac{1}{3}\,\omega(0,0,1,0,2,0)-\frac{3}{2}\,\omega(0,1,0,0,0,2)-\frac{1}{2}\,\omega(0,1,1,1,0,0) \\ & -2\,\omega(2,0,1,0,0,0)-\frac{4}{3}\,\omega(0,0,1,0,0,2)-\frac{10}{3}\,\zeta_2\,\omega(0,1,0,0) \end{aligned}$$

Collect one-loop results

Tree-level



$$\prod_{i=2}^{N-2} \int_0^{z_{i+1}} \mathrm{d}z_i$$

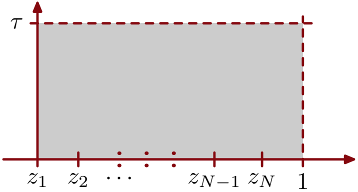
multiple polylogarithms

$$G(a_1, a_2, \dots, a_n; z) = \int_0^z \mathrm{d}t \, \frac{1}{t - a_1} G(a_2, \dots, a_n; t)$$

partial fraction

multiple zeta values ζ

One-loop



$$\int_0^1 \mathrm{d}z_N \prod_{i=1}^{N-1} \int_0^{z_{i+1}} \mathrm{d}z_i \, \delta(z_1)$$

elliptic iterated integrals

$$\Gamma\left(\begin{smallmatrix} n_1 & n_2 & \dots & n_r \\ a_1 & a_2 & \dots & a_r \end{smallmatrix}; z\right) = \int_0^z \mathrm{d}t \, f^{(n_1)}(t - a_1) \, \Gamma\left(\begin{smallmatrix} n_2 & \dots & n_r \\ a_2 & \dots & a_r \end{smallmatrix}; t\right)$$

Fay-identities

elliptic multiple zeta values ω

Picture not yet complete...

- Known how to count multiple zeta values, basis identified, algebraic structure clear.
- How to find a basis for elliptic multiple zeta values?

A basis for multiple zeta values

$$\zeta_{n_1,n_2,\ldots,n_r} = \int_{0\le z_i\le z_{i+1}\le 1} \omega_1 \underbrace{\omega_0\ldots\omega_0}_{n_1-1}\omega_1\underbrace{\omega_0\ldots\omega_0}_{n_2-1}\ldots\omega_1\underbrace{\omega_0\ldots\omega_0}_{n_r-1}$$

$$= \zeta(1\underbrace{0\ldots0}_{n_1-1}1\underbrace{0\ldots0}_{n_2-1}\ldots1\underbrace{0\ldots0}_{n_r-1}) \qquad \omega_0=\frac{1}{z},\ \omega_1=\frac{1}{z-1}$$

Rewrite as *noncommutative words* (employ coproduct): [Goncharov][Brown]

$$\begin{array}{rcl} \phi:\text{algebra of }\zeta^{\mathfrak{m}} & \longrightarrow & \text{Hopf algebra comodule}\\ \zeta^{\mathfrak{m}} & \longmapsto & \phi(\zeta^{\mathfrak{m}}) \end{array}$$

Result: words $f_{2i_1+1}\ldots f_{2i_r+1}\ f_2^k$, with $r,k\ge 0$ and $i_1,\ldots,i_r\ge 1$

Examples: $\phi(\zeta^{\mathfrak{m}}_n)=f_n$, $\phi(\zeta^{\mathfrak{m}}_{3,9})=-6f_5f_7-15f_7f_5-27f_9f_3$

shuffle relations: obvious, no other relations (e.g. stuffle)

Basis: [Zagier][Broadhurst Kreimer]

basis MZVs ⇔ shuffle-independent words in f's

An analogous construction for eMZVs?

Different representation of eMZVs: Iterated Eisenstein integrals

- q-derivative: [Enriquez][Hain][Brown]

$$q = e^{2\pi i\tau}: \qquad q\frac{\mathrm{d}}{\mathrm{d}q}\underbrace{\omega(n_1,\ldots,n_r;q)}_{w_\omega,\ \ell_\omega} \rightarrow \sum \mathrm{G}_{\tilde{n}}\,\underbrace{\omega(\tilde{n}_1,\ldots,\tilde{n}_{r-1};q)}_{\text{shorter}}$$

- instead of iteration in z, use iteration in q (or τ): $\mathrm{G}_0 \equiv -1$

$$\gamma(k_1,k_2,\ldots,k_n;q) \equiv \frac{1}{4\pi^2}\int\limits_{0\leq q'\leq q} \mathrm{d}\log q'\ \gamma(k_1,\ldots,k_{n-1};q')\,\mathrm{G}_{k_n}(q') \qquad k_1 \neq 0$$

- *shuffle relations*:

$$\gamma(n_1,\ldots,n_r;q)\,\gamma(k_1,\ldots,k_s;q) = \gamma\left((n_1,\ldots,n_r)\sqcup\!\sqcup(k_1,\ldots,k_s);q\right),$$

no other relations: tested to very high power in q

Example: $\omega(0,3,9) = -729\,\gamma(10,4) - 315\,\gamma(8,6) - 5616\,\gamma(14,0) +$ non-max. terms

Translate iterated Eisenstein integrals into noncommutative words:

$$\psi[\gamma(k_1,k_2,\ldots,k_n)] \equiv \frac{g_{k_n}g_{k_{n-1}}\cdots g_{k_2}g_{k_1}}{\prod_{j=1}^{n}(k_j-1)}\ .$$

Comparison between ϕ and ψ

MZV

$$\phi(\zeta^{\mathfrak{m}}) = \sum f_{2i_1+1} \dots f_{2i_r+1} \; f_2^k$$

$$\phi(\zeta^{\mathfrak{m}}_{n_1,\dots,n_r} \, \zeta^{\mathfrak{m}}_{k_1,\dots,k_s}) = \phi(\zeta^{\mathfrak{m}}_{n_1,\dots,n_r}) \sqcup\!\sqcup \, \phi(\zeta^{\mathfrak{m}}_{k_1,\dots,k_s}) \, .$$

$$\phi(\zeta^{\mathfrak{m}}) = \sum_{3 \leq 2k+1 \leq w} f_{2k+1} \xi_{2k+1}$$

ξ_{2k+1}: component of weight $(2k+1) \otimes (w-2k-1)$ in the coaction for motivic MZVs

eMZV

$$\psi(\omega) = \sum g_{2i_1} \dots g_{2i_r}$$

$$\psi(\gamma(n_1,\dots,n_r)\gamma(k_1,\dots,k_s)) = \psi(\gamma(n_1,\dots,n_r)) \sqcup\!\sqcup \, \psi(\gamma(k_1,\dots,k_s))$$

$$\frac{\mathrm{d}}{\mathrm{d}\log q} \, \omega(n_1, n_2, \dots, n_r) = \frac{1}{4\pi^2} \sum_{k=0}^{\infty} \xi_{2k} \, \mathrm{G}_{2k}$$

ξ_{2k}: component of weight $(2k) \otimes (w-2k)$ in the q-derivative

Does the counting work for eMZVs?

indecomposable eMZVs $\overset{??}{\Leftrightarrow}$ shuffle-independent words in g's

Assume label 2 to appear in the divergent $\gamma(2) = \omega(0,1)$ *exclusively*.

Comparison:

$\omega(0,n)$	$\leftrightarrow$	$\gamma(n+1)$	g_{n+1}	✓
$\omega(0,0,n)$	$\leftrightarrow$	$\gamma(n+2,0)$	$g_0 g_{n+2}$	✓
$\omega(0,0,8)$	$\leftrightarrow$	$\gamma(10,0)$	$g_0 g_{10}$	✓
$\omega(0,3,5)$	$\leftrightarrow$	$\gamma(6,4)$	$g_4 g_6$	
$\omega(0,0,10)$	$\leftrightarrow$	$\gamma(12,0)$	$g_0 g_{12}$	✓
$\omega(0,3,7)$	$\leftrightarrow$	$\gamma(8,4)$	$g_4 g_8$	
$\omega(0,0,12)$	$\leftrightarrow$	$\gamma(14,0)$	$g_0 g_{14}$	
$\omega(0,3,9)$	$\leftrightarrow$	$\gamma(10,4)$	$g_4 g_{10}$	✗
$\omega(0,5,7)$	$\leftrightarrow$	$\gamma(8,6)$	$g_6 g_8$	

Very unfortunate. What's the problem here?

... need some calculation, group theory, KZB associators and Eisenstein series to identify ...

a special derivation algebra $\mathfrak{u}$.

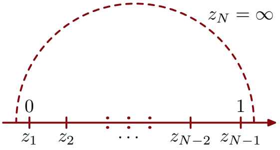
Not a free Lie algebra!
Relations in $\mathfrak{u}$ can be mapped to linear combinations of iterated Eisenstein integrals.

Taking all available relations in $\mathfrak{u}$ into account, the counting matches!

Conjecture: Indecomposable eMZVs can be enumerated by shuffle-independent iterated Eisenstein integrals, taking all relations in $\mathfrak{u}$ into account in addition.

Comparison

Tree-level



$$\prod_{i=2}^{N-2} \int_0^{z_{i+1}} \mathrm{d}z_i$$

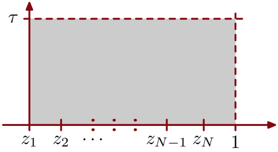
multiple polylogarithms

$$G(a_1,a_2,\ldots,a_n;z)$$

$$= \int_0^z \mathrm{d}t \, \frac{1}{t-a_1} \, G(a_2,\ldots,a_n;t)$$

partial fraction
multiple zeta values ζ
Drinfeld method - no integrals
basis elements:
shuffle-independent words in letters f

One-loop



$$\int_0^1 \mathrm{d}z_N \prod_{i=1}^{N-1} \int_0^{z_{i+1}} \mathrm{d}z_i \, \delta(z_1)$$

elliptic iterated integrals

$$\Gamma\left(\begin{smallmatrix} n_1 & n_2 & \ldots & n_r \\ a_1 & a_2 & \ldots & a_r \end{smallmatrix};z\right)$$

$$= \int_0^z \mathrm{d}t \, f^{(n_1)}(t-a_1) \, \Gamma\left(\begin{smallmatrix} n_2 & \ldots & n_r \\ a_2 & \ldots & a_r \end{smallmatrix};t\right)$$

Fay-identities
elliptic multiple zeta values ω
elliptic associator? [Knizhnik, Bernard Zamolodchikov]
indecomposable elements:
shuffle-independent words in letters g and relations in derivation algebra $\mathfrak{u}$

Summary

- eMZVs are the natural generalization of MZVs to the elliptic domain
- one-loop amplitudes in open string theory: perfect laboratory
- regularization: subtle, but under control
- no divergent eMZVs: open-string result finite
- full amplitude only after τ-integration and consideration of other topologies
- eMZVs might not be the only ingredient to one-loop amplitudes: Euler sums?

Give it a try ...

- numerous relations for eMZVs: https://tools.aei.mpg.de/emzv

Future...

- closed/recursive form of the integrand for the one-loop open-string amplitude in terms of iterated Eisenstein integrals (analogous to Drinfeld-method)?
- relation to the elliptic functions from [Adams, Bogner Weinzierl] ?
- higher genera?

Thanks!

Consider generating function for eMZVs:

Elliptic associator: [Enriquez]

$$A(\tau) \equiv \sum_{r\geq 0}(-1)^r \sum_{n_1,n_2,\ldots,n_r\geq 0} \omega(n_1,n_2,\ldots,n_r)\,\mathrm{ad}_x^{n_r}(y)\ldots\mathrm{ad}_x^{n_2}(y)\,\mathrm{ad}_x^{n_1}(y)$$

$$\mathrm{ad}_x(y) \equiv [x,y]\ , \qquad \mathrm{ad}_x^n(y) = \underbrace{[x,\ldots[x,[x,}_{n\ \text{times}}y]]\ldots]$$

Differential equation: [Enriquez]

$$\frac{\mathrm{d}}{\mathrm{d}\log q}A(\tau) = \frac{1}{4\pi^2}\left(\sum_{n=0}^{\infty}(2n-1)\mathrm{G}_{2n}\,\epsilon_{2n}\right)A(\tau)\,.$$

Algebra of derivations $\mathfrak{u}$: [Hain Matsumoto][Pollack]

$$\epsilon_{2m}(x) = (\mathrm{ad}_x)^{2m}(y)\ , \qquad m\geq 0$$

$$\epsilon_{2m}(y) = [y,(\mathrm{ad}_x)^{2m-1}(y)] + \sum_{1\leq j<m}(-1)^j\Big[(\mathrm{ad}_x)^j(y),(\mathrm{ad}_x)^{2m-1-j}(y)\Big] \qquad m>0$$

$$\epsilon_0(y) = 0$$

Does it solve the conundrum?

First discrepancy occurs at weight 12, length 3:

$$\omega(0,0,12) = -\frac{\zeta_{12}}{3} - 156\,\gamma(14,0)$$

$$\omega(0,3,9) = -729\,\gamma(10,4) - 315\,\gamma(8,6) - 5616\,\gamma(14,0) - 210\,\gamma(6)\,\gamma(8)$$
$$- 1350\,\zeta_{10}\,\gamma(4,0) - 630\,\zeta_8\,\gamma(6,0) + 630\,\zeta_6\,\gamma(8,0) + 1458\,\zeta_4\,\gamma(10,0)$$

$$\omega(0,5,7) = -1134\,\gamma(10,4) - 490\,\gamma(8,6) - 5642\,\gamma(14,0)$$
$$- 1260\,\zeta_{10}\,\gamma(4,0) - 700\,\zeta_8\,\gamma(6,0) + 980\,\zeta_6\,\gamma(8,0) + 2268\,\zeta_4\,\gamma(10,0)$$

ratio of $\gamma(8,6)$ and $\gamma(10,4)$ is *equal* in *all* eMZVs of weight 12 and length 3:

$$81\,\gamma(10,4) + 35\,\gamma(8,6)\,.$$

There are further non-obvious relations X in the algebra of derivations $\mathfrak{u}$. [Pollack]
In particular:

$$0 = [\epsilon_{10}, \epsilon_4] - 3\,[\epsilon_8, \epsilon_6]\,.$$

… solution

Define a suitable derivative (already appeared in the context of the ϕ-map for MZVs):

$$\partial_j g_{k_1} \dots g_{k_n} = \delta_{j,k_1} g_{k_2} \dots g_{k_n} \ .$$

Identify derivations ϵ_{2m} with ∂_{2m}. Thus

$$\Big([\partial_{10}, \partial_4] - 3[\partial_8, \partial_6]\Big) \ \psi\Big[81\,\gamma(10,4) + 35\,\gamma(8,6)\Big] =$$

$$\Big([\partial_{10}, \partial_4] - 3[\partial_8, \partial_6]\Big) \ \Big[3\, g_4\, g_{10} + g_6\, g_8\Big] = 0\,.$$

Conjecture: For any relation

$$X = \sum_{\{n_1,n_2,\dots,n_r\}} \alpha_{n_1,n_2,\dots,n_r} [[\dots[[\epsilon_{n_1}, \epsilon_{n_2}], \epsilon_{n_3}], \dots], \epsilon_{n_r}] = 0$$

in the algebra of derivations $\mathfrak{u}$ one finds:

$$X\Big|_{\epsilon\to\partial} \psi[A(\tau)] = 0\,.$$

Further evidence

Central element

ϵ_2 is central in derivation algebra $\mathfrak{u}$:

$$[\epsilon_{2m}, \epsilon_2] = 0 \quad \leftrightarrow \quad \text{only divergent eMZV: } \gamma(2) = \omega(0,1)\,.$$

Fits with observed absence of label 2 in iterated Eisensteins integrals except for $\gamma(2)$.

Additional derivations

Generators of the free Lie algebra $\mathbb{L}(z_3, z_5, z_7, z_9, ...)$ induce derivations $\widetilde{z}$ in $\mathfrak{u}$:

$$0 = [\widetilde{z}_{2k+1}, \epsilon_0] = [\widetilde{z}_{2k+1}, \epsilon_2]\ , \qquad k = 1, 2, 3, \ldots\ ,$$

whose commutators with $\epsilon_{2m}, m > 1$ can be constructed. [Pollack]

Further irreducible relations

Numerous further *irreducible* relations are known, e.g.:

$$0 = 2\,[\epsilon_{14},\epsilon_4] - 7\,[\epsilon_{12},\epsilon_6] + 11\,[\epsilon_{10},\epsilon_8]$$

$$\begin{aligned} 0 = {} & 80\,[\epsilon_{12},[\epsilon_4,\epsilon_0]] + 16\,[\epsilon_4,[\epsilon_{12},\epsilon_0]] - 250\,[\epsilon_{10},[\epsilon_6,\epsilon_0]] \\ & - 125\,[\epsilon_6,[\epsilon_{10},\epsilon_0]] + 280\,[\epsilon_8,[\epsilon_8,\epsilon_0]] - 462\,[\epsilon_4,[\epsilon_4,\epsilon_8]] - 1725\,[\epsilon_6,[\epsilon_6,\epsilon_4]]\,. \end{aligned}$$

$$\vdots \qquad\qquad \vdots$$

Reducible relations:

$$X = 0 \quad \Rightarrow \quad \mathrm{ad}_{n_1,n_2,\ldots,n_k}(X) \equiv [\epsilon_{n_1},[\epsilon_{n_2},[\ldots,[\epsilon_{n_k},X]\ldots]]] = 0\ .$$

Simple example:

$$\big[\epsilon_n, 2\,[\epsilon_{14},\epsilon_4] - 7\,[\epsilon_{12},\epsilon_6] + 11\,[\epsilon_{10},\epsilon_8]\big] = 0$$

but as well

$$\big[\widetilde{z}_3, [\epsilon_{10},\epsilon_4] - 3[\epsilon_8,\epsilon_6]\big] = 0$$

at $w_\gamma = 20$ and $\ell_\gamma = 5$.

Vanishing nested commutators

The nested commutator

$$[[[[\partial_4, \partial_0], \partial_0], \partial_0], \partial_{2m}]$$

annihilates all eMZVs starting from $w_\gamma = 8,\ \ell_\gamma = 5$.

- Consider $\gamma(4,0,0,0)$: corresponds to ω of weight 0 and length 5.
- only known eMZV is $\omega(0,0,0,0,0) = 1/120 \neq \gamma(4,0,0,0)$ (no q-expansion)

Indeed

$$[[[\epsilon_4, \epsilon_0], \epsilon_0], \epsilon_0](x) = 0\ , \qquad [[[\epsilon_4, \epsilon_0], \epsilon_0], \epsilon_0](y) = 0\,.$$

Adding further 0's leads to ω's of *negative* weight: further reducible relations.

Taking all available relations in $\mathfrak{u}$ into account, the counting matches!

Conjecture: Indecomposable eMZVs can be enumerated by shuffle-independent iterated Eisenstein integrals, taking all relations in $\mathfrak{u}$ into account in addition.