Pre-NQ-manifolds and derived brackets in generalized geometry and double field theory

Andreas Deser[1], Christian Sämann[2]
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[1]Istituto Nazionale di Fisica Nucleare, Torino
[2] Department of Mathematics, Heriot-Watt University, Edinburgh

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Motivation: What is double field theory?

Canonical momenta and winding

- Sigma model $X: \ \Sigma \to M = T^d$

$$S = \int_\Sigma h^{\alpha\beta}\partial_\alpha X^i \partial_\beta X^j G_{ij}\, d\mu_\Sigma + \int_\Sigma X^* B \ ,$$

  where $h \in \Gamma(\otimes^2 T^*\Sigma)$, $G \in \Gamma(\otimes^2 TM)$, $B \in \Gamma(\wedge^2 T^* M)$.

- Classical solutions to e.o.m. (take *closed* string $\Sigma = \mathbb{R} \times S^1$)

$$X_R^i = x_{0R}^i + \alpha_0^i(\tau - \sigma) + i\sum_{n\neq 0} \frac{1}{n}\alpha_n^i e^{-in(\tau-\sigma)} \ , \quad X_L^i = \dots \ ,$$

$$\alpha_0^i = \frac{1}{\sqrt{2}} G^{ij}\Big(p_j - (G_{jk} + B_{jk})w^k\Big) \ ,$$

  - p_k: Canonical momentum zero modes
  - w^k: *Winding* zero modes, $w^k := \frac{1}{2\pi}\int_0^{2\pi} \partial_\sigma X^k d\sigma$.

Motivation: What is double field theory?

Two sets of coordinates

- Two sets of momenta in α_0^i $\rightarrow$ differential operators:

$$p_k \simeq \frac{1}{i}\partial_k \;, \quad w^k \simeq \frac{1}{i}\tilde{\partial}^k \;.$$

- Level matching $L_0 - \bar{L}_0$, with $L_0 = \frac{1}{2}\alpha_0^i G_{ij}\alpha_0^j + N - 1$ gives

$$N - \bar{N} = \partial_i \tilde{\partial}^i$$

- Want: If two fields obey the constraint, then also their product. Thus choose a subset, which also has:

$$\partial_k \phi \, \tilde{\partial}^k \psi + \tilde{\partial}^k \phi \, \partial_k \psi = 0 \;,$$

  for all elements ϕ, ψ of the subset.

Motivation: What is double field theory?

$O(d,d)$-transformations, generalized tangent bundle, Gualtieri, Hitchin

Observation 1: The strong constraint is given by

$$\eta^{MN}\,\partial_M\phi\,\partial_N\psi = 0\,,\quad \eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

And stays the same if we apply a constant transformation that leaves η invariant:

$$A^t\,\eta\,A = \eta \quad \text{i.e.} \quad A \in O(d,d;\mathbb{R})$$

Observation 2: This is the structure group of the generalized tangent bundle, locally isomorphic to $TM \oplus T^*M$:

- Sections: $TM \oplus T^*M \ni s = s^i\,\partial_i + s_i\,dx^i$.
- Fundamental rep of $O(d,d)$: $s^M := (s^i, s_i)$.
- Bilinear pairing η: $\langle s,t\rangle = s^i t_i + s_i t^i = \eta_{MN} s^M t^N$.

Motivation: What is double field theory?

Action of DFT, C-bracket, Hohm, Hull, Zwiebach

Observation 3:

$$S_{DFT} = \int d^{2D}x\, e^{-2d}\Big(\tfrac{1}{8}\mathcal{H}_{MN}\partial^M\mathcal{H}_{KL}\partial^N\mathcal{H}^{KL} - \tfrac{1}{2}\mathcal{H}_{MN}\partial^M\mathcal{H}_{KL}\partial^L\mathcal{H}^{KN} - 2\partial^M d\,\partial^N\mathcal{H}_{MN} + 4\mathcal{H}_{MN}\partial^M d\,\partial^N d\Big)\,.$$

Properties:

- The action as a global $O(d,d;\mathbb{R})$-symmetry, and a gauge symmetry given by applying the **generalized Lie derivative**:

$$(\delta_X\mathcal{H})_{MN} := X^P\partial_P\mathcal{H}_{MN} + (\partial_M X^P - \partial^P X_M)\mathcal{H}_{PN} + (\partial_N X^P - \partial^P X_N)\mathcal{H}_{MP}\,.$$

- The commutator of two such transformations gives the **C-bracket**:

$$[V,W]_C^M := V^K\partial_K W^M - W^K\partial_K V^M - \tfrac{1}{2}\Big(V^K\partial^M W_K - W^K\partial^M V_K\Big)\,.$$

Questions

- What is the geometric meaning of double fields, such as $d(x, \tilde{x})$, $V^M(x, \tilde{x})$, $\mathcal{H}_{MN}(x, \tilde{x})$? At least locally?
- Is there an algebraic way to understand the C-bracket?
- How does this apply to the Lie- and Courant bracket?

(Pre)-NQ-manifolds and derived brackets

Motivation: An easy calculation...

Given a manifold M, consider $T[1]M$ with local coordinates (x^μ, ξ^μ). Its cotangent bundle locally has $(x^\mu, \xi^\mu, p_\mu, \xi^*_\mu)$ and is Poisson:

$$\{p_\mu, x^\nu\} = \delta^\nu_\mu \qquad \{\xi^*_\mu, \xi^\nu\} = \delta^\nu_\mu \, .$$

Let's take the operator $Q = \xi^\mu p_\mu$, and vector fields $X = X^\mu \xi^*_\mu$, $Y = Y^\nu \xi^*_\nu$, then we can do the following exercise:

$$\begin{aligned}
\Big\{\{Q, X\}, Y\Big\} &= \Big\{\{\xi^\mu p_\mu, X^\nu \xi^*_\nu\}, Y^\rho \xi^*_\rho\Big\} \\
&= \Big\{\xi^\mu \partial_\mu X^\nu \xi^*_\nu + X^\nu p_\nu, Y^\rho \xi^*_\rho\Big\} \\
&= - Y^\rho \partial_\rho X^\nu \xi^*_\nu + X^\rho \partial_\rho Y^\nu \xi^*_\nu \\
&= [X, Y]^\nu_{\mathrm{Lie}} \xi^*_\nu \, .
\end{aligned}$$

We say, that the Lie bracket is a **derived bracket** (due to Kosmann-Schwarzbach, Roytenberg, Voronov).

Definition 1.

A **symplectic pre-NQ-manifold of $\mathbb{N}$-degree** *n is an $\mathbb{N}$-graded manifold $\mathcal{M}$, together with symplectic form ω of degree n and a vector field Q of degree 1, satisfying $L_Q\omega = 0$.*

Examples

An important class where in addition $Q^2 = 0$, are the **Vinogradov Lie** n**-algebroids**:

$$\mathcal{V}_n(M) := T^*[n]T[1]M\ .$$

They have the following properties:

- Local coordinates $(x^\mu, \xi^\mu, \zeta_\mu, p_\mu)$ of degrees $0, 1, n-1, n$.
- Symplectic form $\omega = dx^\mu \wedge dp_\mu + d\xi^\mu \wedge d\zeta_\mu$
- Nilpotent vector field Q with Hamiltonian $\mathcal{Q} = \xi^\mu p_\mu$, i.e. $\{\mathcal{Q}, \mathcal{Q}\} = 0$.

Constructing the brackets

Getzler, Fiorenza, Manetti

Let $\mathcal{M}$ be a symplectic pre-NQ-manifold. Functions on the body M are degree-0 objects, i.e. $f \in C_0^\infty(\mathcal{M})$. We choose as analogue of vector fields degree $(n-1)$-objects $X \in \mathcal{C}_{n-1}^\infty(\mathcal{M})$, call them **extended vector fields**. Then define n-ary brackets by

$$\mu_1(V) = \begin{cases} \{\mathcal{Q}, V\}\,, & \text{if } V \text{ has degree } 0 \\ 0\,, & \text{otherwise} \end{cases}$$

$$\mu_2(V, W) = \tfrac{1}{2}(\{\delta V, W\} - \{\delta W, V\})$$

$$\mu_3(V, W, U) = -\tfrac{1}{12}(\{\{\delta V, W\}, U\} \pm \dots)$$

$$\dots$$

$$\text{where } \delta V := \begin{cases} \{\mathcal{Q}, V\}\,, & \text{if } V \text{ has degree } n-1 \\ 0\,, & \text{otherwise} \end{cases}$$

Conditions for L_∞-structure

If $Q^2 = 0$, the above brackets form an L_∞ structure. In our case we want to investigate conditions that this is also true, especially for $n = 2$, where we found the following

Theorem 1.

Consider the subset of $\mathcal{C}^\infty(\mathcal{M})$ consisting of functions and extended vector fiels, i.e. $\mathcal{C}_0^\infty(\mathcal{M}) \oplus \mathcal{C}_1^\infty(\mathcal{M})$. If the Poisson brackets and the maps μ_i close on this subset, the latter is an L_∞-algebra if and only if

$$
\begin{aligned}
\{Q^2 f, g\} + \{Q^2 g, f\} &= 0\,, \\
\{Q^2 X, f\} + \{Q^2 f, X\} &= 0\,, \\
\{\{Q^2 X, Y\}, Z\}_{[X,Y,Z]} &= 0\,,
\end{aligned}
$$

for all functions f, g and extended vector fields X, Y, Z. The notation $Q^2 f$ means $\{\mathcal{Q}, \{\mathcal{Q}, f\}\}$ and the subscript $[X, Y, Z]$ means the alternating sum over X, Y, Z.

$n = 1$: Standard Lie bracket

Obtaining our toy example

Now, lets apply the construction to different cases.

Let M be a Riemannian manifold. As graded manifold, we take $\mathcal{M} = \mathcal{V}_1(M)$, which has coordinates (x^μ, ζ_μ) of degree 0 and (ξ^μ, p_μ) of degree 1. Then we have:

- As $\mathcal{Q} = \xi^\mu p_\mu$ which squares to zero in $\mathcal{V}_1(M)$, there are no further restrictions.
- Vector fields are linear functions in ζ_μ.
- μ_1 gives the de Rham differential.
- $\mu_2(X, Y) = \frac{1}{2}\left(\left\{\{\mathcal{Q}, X\}, Y\right\} - \left\{\{\mathcal{Q}, Y\}, X\right\}\right) = [X, Y]^\mu_{\mathrm{Lie}} \zeta_\mu$.
- μ_3 vanishes due to the Jacobi identity for the Lie bracket.

So, we recover the standard Lie bracket as a derived bracket.

$n = 2$: Courant bracket

Roytenberg, Weinstein

For a manifold M, take now $\mathcal{V}_2(M)$. Locally, coordinates are $(x^\mu, \xi^\mu, \zeta_\mu, p_\mu)$ of degrees $0, 1, 1, 2$. We get

- $\mathcal{Q} = \xi^\mu p_\mu$ squares to zero.
- Extended vectors, i.e. degree 1 objects, are now the "generalized vectors", i.e. $V = X^\mu \zeta_\mu + \alpha_\mu \xi^\mu$, $W = Y^\mu \zeta_\mu + \beta_\mu \xi^\mu$.
- μ_1 is the de Rham differential.
- $\mu_2(V, W) = [X, Y]^\mu \zeta_\mu + (L_X \beta - L_Y \alpha - \frac{1}{2} d(\iota_X \beta - \iota_Y \alpha))_\mu \xi^\mu$, i.e. we get the Courant bracket.
- μ_3 gives the the defect to the Jacobi identity for Courant algebroids.

So we recover generalized geometry on a Courant algebroid.

$n = 2$: C-bracket

New result: Interpretation of the C-bracket

We take the same setting as before, but instead of M as base, we take T^*M, i.e. we take $\mathcal{V}_2(T^*M)$. Local coordinates are now $(x^M, \xi^M, \zeta_M, p_M)$ of degree $(0, 1, 1, 2)$.

Problem: We now have too many "vectors". We solve this by defining

$$\theta^M := \frac{1}{\sqrt{2}}(\xi^M + \eta^{MN}\zeta_N) \quad \text{and} \quad \beta^M := \frac{1}{\sqrt{2}}(\xi^M - \eta^{MN}\zeta_N) \,,$$

and taking only θ^M as degree-1 coordinates. Taking

$$\omega = dx^M \wedge dp_M + \frac{1}{2}\eta_{MN} d\theta^M \wedge d\theta^N, \quad \mathcal{Q} = \theta^M p_M \,,$$

we get a pre-NQ-manifold (as $\mathcal{Q}$ doesn't square to zero, but we have $L_Q\omega = 0$ for the corresponding vector field).

$n = 2$: C-bracket

New result: Interpretation of the C-bracket

With this we get the following results:

- $\mu_1(f) = \theta^M \partial_M f$, i.e. the de Rham differential on the doubled space.
- For vectors $X = X_M \theta^M$, $Y = Y_M \theta^M$ we get, using $\eta^{MN} X_M \partial_N = X^N \partial_N$ etc.

  $$\mu_2(X, Y) = (X^M \partial_M Y_K - Y^M \partial_M X_K - \tfrac{1}{2}(Y^M \partial_K X_M - X^M \partial_K Y_M))\theta^K ,$$

  i.e. the C-bracket of double field theory.
- μ_3 gives the defect to the Jacobi identity of the C-bracket.

$n = 2$: C-bracket

New result: Interpretation of the C-bracket

So the formulas give us double field theory, but what about the constraints for Q^2?

$n = 2$: C-bracket

The strong constraint

To have a proper L_∞-structure, we still have to implement the constraints of our theorem. What are they? Let f be a function and $X = X_M \theta^M$, Y and Z be extended vectors.

- $\{Q^2 f, g\} + \{Q^2 g, f\} = 2\partial_M f\, \eta^{MN} \partial_N g = 0$
  This is the strong constraint.
- $\{Q^2 X, f\} + \{Q^2 f, X\} = 2(\partial_M X_K \theta^K)\eta^{MN}\partial_N f = 0$
  This is the strong constraint for vectors and functions.
- $\{\{Q^2 X, Y\}, Z\}_{[X,Y,Z]} = 2\theta^K((\partial^M X_K)(\partial_M Y^N) Z_N)_{[X,Y,Z]} = 0$
  Additional constraint for vectors? ...Shows up in properties of the Riemann tensor of double field theory

So the strong constraint together with the third constraint ensure the L_∞-structure for vectors and functions.

Outlook

What we did...

We found a unifying language to describe the Lie bracket, Courant bracket and C-bracket. The strong constraint plays a role to ensure an L_∞-structure on functions and vectors. What I didn't describe in the talk:

- In all three cases, arbitrary tensors can be defined, extended Lie derivatives and the action of infinitesimal extended diffeomorphisms.
- We get a whole "derived geometry", including torsion, Gualtieri-torsion and Riemann tensors (so far for $n = 1, 2$).
- Writing down integration densities and Einstein-Hilbert actions for extended metrics for the cases $n = 1$ and $n = 2$ was possible. In general?...long calculations!
- NS-NS fluxes can be added by twisting the corresponding Vinogradov algebroids. This is interesting for flux compactification and T-duality.

Outlook

Open questions

- Everything was local. Global analysis? Gerbes, groupoids... What is the global description of double field theory? T-duality?
- For higher Vinogradov algebroids $\mathcal{V}_n(M)$, degree $n-1$-objects are

$$X = X^{\mu}\zeta_{\mu} + X_{\mu_1 \dots \mu_{n-1}} \xi^{\mu_1} \cdots \xi^{\mu_{n-1}} ,$$

  i.e. sections of $TM \oplus \wedge^{n-1} T^* M$. For $n = 3$, we get the easiest case of *exceptional generalized geometry*, where the U-duality group is $SL(5, \mathbb{R})$. How about the other exceptional tangent bundles?
- What are torsion and Riemann tensors in exceptional generalized geometries? Do they have a meaning in Poisson geometry on certain Vinogradov algebroids?
- Quantization: If we can write the brackets in terms of Poisson brackets, we can do deformation quantization!

Definition 2.

An **extended covariant derivative** ∇ *on a pre-NQ-manifold* $\mathcal{M}$ *is a linear map from the set* $\mathcal{X}(\mathcal{M})$ *of extended vectors to* $\mathcal{C}^\infty(\mathcal{M})$*, such that the image* ∇_X *for* $X \in \mathcal{X}(\mathcal{M})$ *gives a map* $\{\nabla_X, \cdot\} : \mathcal{X}(\mathcal{M}) \to \mathcal{X}(\mathcal{M})$*, which satisfies*

$$\{\nabla_{fX}, Y\} = f\{\nabla_X, Y\} \quad \mathit{and} \quad \{\nabla_X, fY\} = \{\{\mathcal{Q}, X\}, f\}Y + f\{\nabla_X, Y\} \,,$$

for all functions f *and extended vectors* Y*. For arbitrary extended tensors extend this by the graded Leibniz rule of the Poisson bracket*

$$\{V, W \otimes U\} := \{V, W\} \otimes U + (-1)^{(n-|W|)|U|} W \otimes \{V, U\} \,,$$

where $V, W, U \in \mathcal{C}^\infty(\mathcal{M})$ *and* $|W|$ *denotes the degree.*

Some remarks

- Pointwise, $\mathcal{X}$ is a vector space, so we can consider also its dual. For torsion and curvature we use $\hat{\mathcal{X}} := \mathcal{X} \oplus \mathcal{X}^*$.
- We also denote by $\pi : \hat{\mathcal{X}} \to \mathcal{X}$ the projection to the first summand.
- In the following we will deal with $\mathcal{V}_1$ and the restricted $\mathcal{V}_2$ suitable for double field theory. In these cases, one can show that the following functions have the right properties:

$$\nabla_X = X^\mu p_\mu - X^\mu \Gamma^\rho{}_{\mu\nu} \zeta_\rho \xi^\nu \ ,$$
$$\nabla_X = X^M p_M - \tfrac{1}{2} X^M \Gamma_{MNK} \theta^N \theta^K \ .$$

Definition 3.
Let $\mathcal{M}$ be a pre-NQ-manifold. Given an extended connection ∇, we define the **extended torsion tensor** $\mathcal{T} : \otimes^3 \hat{\mathcal{X}}(\mathcal{M}) \to \mathcal{C}^\infty(M)$ *for* $X, Y, Z \in \hat{\mathcal{X}}(\mathcal{M})$ *by*

$$\mathcal{T}(X,Y,Z) := 3\Big((-1)^{n|X|}\left\{X, \{\nabla_{\pi(Y)}, Z\}\right\}\Big)_{[X,Y,Z]} + \frac{(-1)^{n(|Y|+1)}}{2}\left(\{X,\{QZ,Y\}\} - \{Z,\{QX,Y\}\}\right) ,$$

where $|X|, |Y|$ denote the respective degrees, π is the above defined projection and $n = 1, 2$ is the degree of the underlying Vinogradov algebroid.

With this definition, we are able to show the following results relating extended torsion to standard ones:

Appendix: Covariant derivatives, torsion and curvature

Extended torsion

Theorem 2.
For $\mathcal{M} = \mathcal{V}_1(M)$, let $X \in \mathcal{X}^*(\mathcal{M})$ and $Y, Z \in \mathcal{X}(\mathcal{M})$, then the extended torsion reduces to the torsion operator $T(X, Y, Z) = \langle X, \nabla_Y Z - \nabla_Z Y - [Y, Z] \rangle$, where the bracket is the Lie bracket of vector fields. More generally, this is true whenever we take one element of $\mathcal{X}^*(\mathcal{M})$ and the other two in $\mathcal{X}(\mathcal{M})$. In all other cases the extended torsion vanishes. In case of double field theory, for extended vector fields X, Y, Z, the extended torsion tensor equals the Gualtieri torsion of generalized geometry.

Definition 4.

Let $\mathcal{M}$ be a pre-NQ-manifold. Given an extended connection ∇, the **extended curvature operator** $\mathcal{R} : \otimes^4 \hat{\mathcal{X}}(\mathcal{M}) \to \mathcal{C}^\infty(M)$ *for* $X, Y, Z, W \in \hat{\mathcal{X}}(\mathcal{M})$ *is defined by*

$$\mathcal{R}(X,Y,Z,W) := \frac{1}{2}\Big(\Big\{\big\{\{\nabla_X, \nabla_Y\} - \nabla_{\mu_2(X,Y)}, Z\big\}, W\Big\} - (-1)^n (Z \leftrightarrow W) + \Big\{\big\{\{\nabla_Z, \nabla_W\} - \nabla_{\{\nabla_Z, W\} - \{\nabla_W, Z\}}, X\big\}, Y\Big\} - (-1)^n (X \leftrightarrow Y)\Big) .$$

Reminder: μ_2 is the C-bracket in the derived-bracket form.

Theorem 3.
For $\mathcal{M} = \mathcal{V}_1(M)$, let $X, Y, Z \in \mathcal{X}(\mathcal{M})$ and $W \in \mathcal{X}^*(\mathcal{M})$. Then the extended curvature reduces to the standard curvature:

$$\mathcal{R}(X, Y, Z, W) = \langle W, \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \rangle \, .$$

Furthermore, if $X, Y, Z, W \in \mathcal{X}(\mathcal{M})$ or if two, three or all of X, Y, Z, W are in $\mathcal{X}^*(\mathcal{M})$, we have $\mathcal{R}(X, Y, Z, W) = 0$. Moreover, in case of double field theory, extended curvature is the Hohm-Zwiebach curvature. Tensoriality holds by the constraints given in theorem 1.

It is the last sentence, where the algebraic setting becomes important for geometry.