Representation theory of the Virasoro algebra

In the lecture course, we encountered the Virasoro algebra $\mathfrak{Vir}$, given by

\[ [L_n, L_m] = (n - m)L_{n+m} + C \frac{1}{12}(n^3 - n)\delta_{n+m,0}, \]
\[ [L_n, C] = 0. \]

This is the one important algebra in two-dimensional conformal field theory, since it is the algebra of the generators of arbitrary locally conformal (i.e. locally holomorphic) transformations.

In order to further analyse the nature of a two-dimensional conformally invariant quantum field theory, one has to know how physically relevant representations of the algebra $\mathfrak{Vir}$ on a space of states look like. We assume the following for a physical sensible space of states:

1. There exists precisely one state $|0\rangle$ with the property $L_n|0\rangle = 0 \forall n \geq -1$. This state is called the vacuum.
2. Each representation shall contain precisely one state $|h\rangle$ with the properties $L_n|h\rangle = 0 \forall n > 0$ and $L_0|h\rangle = h|h\rangle$. Such states are called highest weight states. The $L_0$ eigenvalue $h$ is called the highest weight. As we will see, such representations have an energy spectrum which is bounded from below admitting thus stable ground states. Finally, if $h = 0$, the representation is then the vacuum representation built upon the state $|0\rangle$, since the vacuum is unique.
3. In the case of unitary representations, we have that $L_n = L_n^\dagger$. We have seen in the lecture, that unitary representations require $C, h \geq 0$.

**[P1] Field state isomorphism**

One important thing is the so-called field-state isomorphism. Suppose we have a primary field $\Phi_h(z)$ of conformal weight $h$. This field corresponds one-to-one to a highest weight state $|h\rangle$ via

\[ |h\rangle = \lim_{z \to 0} \Phi_h(z)|0\rangle. \]

With the commutator $[L_n, \Phi_h(w)]$ from the lecture we find

\[ [L_n, \Phi_h(w)] = \frac{1}{2\pi i} \oint dze^{n+1}z^{n+1}T(z)\Phi_h(w) \]
\[ = (h(n+1)w^n + w^{n+1}\partial_w)\Phi_h(w). \]

Obviously, this vanishes for all $n > 0$, when $w \to 0$. Since

\[ L_n|h\rangle = \lim_{w \to 0} L_n\Phi_h(w)|0\rangle \]
\[ = \lim_{w \to 0} [L_n, \Phi_h(w)]|0\rangle \]
\[ = 0 \]

for all $n > 0$, the highest weight property for $|h\rangle$ follows.

**[P2] Universal enveloping algebra**

Given a highest weight state $|h\rangle$, we can construct representations with the universal enveloping algebra $U(\mathfrak{Vir})$. This is the algebra of all words $L_{n_1}L_{n_2} \cdots L_{n_k}$ for $k \in \mathbb{Z}_+$. It is easy to see that the Verma module is given by

\[ V_h = \text{span} \{ L_{-n_1}L_{-n_2} \cdots L_{-n_k}|h\rangle : n_i \geq n_{i+1} > 0 \land k \in \mathbb{Z}_+ \} \]
This is done inductively. The empty word is just the highest weight state itself, and is contained in $V_{[h]}$. Suppose now that we have shown that all words of length $N$ are of the form given above. Then, for words of length $N + 1$, one eliminates all letters $L_{n_j}$ with $n_j > 0$ by commuting them to the right and using the highest weight property. The commutators yield words of shorter length which according to our induction assumption are already ordered. Finally, one orders the negative modes all descending to the right in the same way. Unordered words are replaced via commutators by ordered words and words of shorter length, which are already accounted for by the induction assumption. The reasoning sketched here is known under the name Poincaré-Birkhoff-Witt theorem.

**[P3] Gradation**

The state $L_{-n}[h]$ has weight $h + n$. This follows by looking at the commutator $[L_0, L_{-n}][h] = n L_{-n}[h]$, since on the other hand $[L_0, L_{-n}][h] = L_0 (L_{-n}[h]) - L_{-n} L_0[h] = L_0 (L_{-n}[h]) - h(L_{-n}[h])$. The same is true for a state $L_{-n_1}L_{-n_2} \ldots L_{-n_k}[h]$ if $\sum_{i=1}^{k} n_i = n$. One sees this by applying $L_0$ to this state and commuting it element by element to the right. All intermediate terms cancel and we are left with

$$L_0 L_{-n_1}L_{-n_2} \ldots L_{-n_k}[h] = [L_0, L_{-n_1}]L_{-n_2} \ldots L_{-n_k}[h] + L_{-n_1}[L_0, L_{-n_2}] \ldots L_{-n_k}[h] + \ldots + L_{-n_1}L_{-n_2} \ldots [L_0, L_{-n_k}][h] + L_{-n_1}L_{-n_2} \ldots L_{-n_k} L_0 [h].$$

This implements a natural gradation on the Verma modules. Namely, defining

$$U_n(\mathfrak{Vir}) = \text{span} \left\{ L_{-n_1}L_{-n_2} \ldots L_{-n_k} : n_i \geq n_{i+1} \wedge k \in \mathbb{Z}_+ \wedge \sum_{i=1}^{k} n_i = n \right\},$$

we may write

$$V_{[h]} = \bigoplus_{n=0}^{\infty} U_n(\mathfrak{Vir})[h].$$

For any given level $n$, the vector space $U_n(\mathfrak{Vir})$ is finite dimensional. Indeed, for $n = 0$, the statement is trivial. For positive $n$, we see that $U_n(\mathfrak{Vir})$ is spanned by words $L_{-n_1}L_{-n_2} \ldots L_{-n_k}$ where the positive numbers $n_i$ sum up to $n$. Obviously, there are only finitely many different ways to distribute $n$ items into piles. Start with just one pile of $b$ items. Take away one to form a pile of a single item. Taking away a second item leaves you with two possibilities: One large pile plus two piles of one item each, or one large pile and one pile with two items. Continue recursively at each step with all piles with more than one item until one reaches the extremal distribution of $n$ piles of one item each. As there are in any step only finitely many piles with finitely many items, the process must stop after a finite number of steps. Thus, the dimensions of the level $n$ vector spaces must all be finite. What we actually compute here is the number of partitions of a number $n$, written $p(n)$. This means that on each level of excitation above the energy of the ground states, only the finite number $p(n)$ of different excitation states exist. Explicitly the dimensions for $n = 1, 2, \ldots, 5$ are $p(1) = 1, p(2) = 2, p(3) = 3, p(4) = 5$ and $p(5) = 7$. One further defines $p(0) = 1$. For example the partitions of $n = 4$ are $\{4, 3 + 1, 2 + 2, 2 + 1 + 1, 1 + 1 + 1 + 1\}$. And for $n = 5$, they are $\{5, 4 + 1, 3 + 2, 3 + 1 + 1, 2 + 2 + 1, 2 + 1 + 1 + 1, 1 + 1 + 1 + 1 + 1\}$.

**[P4] Kac determinant**

Let us now concentrate specifically on $U_1(\mathfrak{Vir}) = \text{span}\{L_{-1}\}$ and $U_2(\mathfrak{Vir}) = \text{span}\{L_{-1}L_{-1}, L_{-2}\}$. We now construct for these vector spaces the (Hermitean) Cauchy matrices

$$K_n = \left( \langle h | (L_{-n})_a \rangle (L_{-n})_b | h \rangle \right),$$

where $L_{-\{n\}}_a$ denotes an arbitrary enumeration of the words in $U_n(\mathfrak{Vir})$. These matrices contain all possible pairings of words of the fixed level $n$. So, we specifically find

$$K_1 = (\langle h | L_1 L_{-1} | h \rangle)$$
and
\[
K_2 = \begin{pmatrix}
\langle h|L_2 L_{-2}|h\rangle & \langle h|L_2 L_{-1} L_{-1}|h\rangle \\
\langle h|L_1 L_1 L_{-2}|h\rangle & \langle h|L_1 L_1 L_{-1} L_{-1}|h\rangle
\end{pmatrix}.
\]

Here, we used that $L_{-n}^\dagger = L_n$ to simplify matters\(^1\). Clearly, $\det(K_1) = \langle h|[L_1, L_{-1}]|h\rangle = 2\langle h|L_0|h\rangle = 2h\langle h|h\rangle$, where we could take the commutator due to the highest weight property of the state $|h\rangle$. Requiring unitarity implies $h \geq 0$.

Now, let us compute $\det(K_n)$ and find its zeroes. Taking commutators and using the highest weight property extensively, we find
\[
K_2 = \begin{pmatrix}
4h + \frac{1}{2}c & 6h \\
6h & 4h + 8h^2
\end{pmatrix} \langle h|h\rangle.
\]

If $\det(K_2) \leq 0$, than there exists an eigenvector $v = \begin{pmatrix} r \\ s \end{pmatrix}$ such that $v^\dagger K_2 v \leq 0$, i.e. that $\| \alpha L_{-2}|h\rangle + \beta L_{-1} L_{-1}|h\rangle \| \leq 0$. This is, why the determinants of the matrices $K_n$ are important. Their zeroes mark the points in the $(h, c)$ plane, where the norm of one eigenvector of $K_n$ changes sign. For $K_2$, we find
\[
\det(K_2) = 2h \left(16h^2 - 10h + 2hc + c\right) \langle h|h\rangle^2
= 32(h - h_{1,1})(h - h_{1,2})(h - h_{2,1})\langle h|h\rangle^2
= 16 \det(K_1)(h - h_{1,2})(h - h_{2,1})\langle h|h\rangle.
\]

Here, we used the customary parametrization of the zeroes in the form
\[
h_{r,s}(c) = \frac{(m + 1)r - ms)^2 - 1}{4m(m + 1)},
\]
where
\[
m = \frac{1}{2} + \frac{1}{2} \sqrt{\frac{25 - c}{1 - c}}.
\]
The choice of sign simply interchanges $r \leftrightarrow s$ and $m \leftrightarrow m - 1$. More directly, we find that $\det(K_2)$ has three zeroes, one of them being the zero of $\det(K_1)$, the other two are
\[
h_{\pm}(c) = \frac{1}{16} \left(5 - c \pm \sqrt{(c - 1)(c - 25)}\right).
\]

Actually, in this case, it is simpler to give the inverse solution
\[
c = c(h) = \frac{2h(5 + 8h)}{2h + 1}.
\]

One can show that the general Kac determinant at level $n$ is of the form
\[
\det(K_n) \propto \det(K_{n-1}) \prod_{r,s=n} (h - h_{r,s}).
\]

The zeroes of the Kac determinant tell us where we have zero norm states. More precisely, the zeroes hint towards states which are orthogonal to all other states. Hence, the Verma module can be decomposed into an orthogonal sum of a module built on the eigenvector with eigenvalue zero, and its orthogonal complement. In the end not all states in the Verma module are linearly independent. We can throw away some states together with the Verma modules built on them. The irreducible modules are then the Verma modules with the maximal proper ideal divided out.

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\(^1\)Instead of using the unitarity condition, one can work directly with a formal pairing of words of negative modes with words of positive modes. This is called the Shapovalov form, which does not in itself induce a well defined scalar product. However, we do not need a scalar product and the norm induced by it, if we do not require unitarity. All we need is that we can decide whether all states in the Verma module are linearly independent, or whether there are relations among them. This will be discussed in detail in the lecture.