Conformal Ward identity

We learned in the lecture course that a local conformal transformation \( z \mapsto z' = z(1 - \varepsilon(z)) \) with arbitrary meromorphic function \( \varepsilon(z) \) is generated by the Noether charge

\[ Q_\varepsilon = \frac{1}{2\pi i} \oint dz \varepsilon(z) T(z). \]

Let us now consider a generic correlation function (a.k.a. vacuum expectation value)

\[ G^{(N)}(\{w_i\}) = \langle 0| \phi_1(w_1)\phi_2(w_2)\ldots\phi_N(w_N)|0 \rangle, \]

where \( \phi_i \) are some, not necessarily primary, fields. The aim of this tutorial is to derive the conformal Ward identity and to use it in order to prove that all correlation functions involving descendant fields can entirely be computed in terms of correlation functions with solely primary fields.

**[P1] Conformal Ward identity**

To derive the conformal Ward identity, we proceed in two steps:

(a) It is rather easy to show that the variation \( \delta_\varepsilon G^{(N)}(\{w_i\}) \) is given by

\[ \delta_\varepsilon G^{(N)}(\{w_i\}) = \sum_k \langle 0| \phi_1(w_1)\ldots(\delta_\varepsilon \phi_k(w_k))\ldots\phi_N(w_N)|0 \rangle \]

for all \( \varepsilon \). It follows from a simple deformation of the integration contour. By definition, the contour for the integral defining the Noether charge has to encircle all field insertions in the correlator it acts on:

\[ \delta_\varepsilon G^{(N)}(\{w_i\}) = \langle 0| \frac{1}{2\pi i} \oint dz \varepsilon(z) T(z) \phi_1(w_1)\ldots\phi_N(w_N)|0 \rangle \]

\[ = \sum_k \langle 0| \phi_1(w_1)\ldots\phi_{k-1}(w_{k-1}) \left( \oint_{w_k} \frac{dz}{2\pi i} \varepsilon(z) T(z) \phi_k(w_k) \right) \phi_{k+1}(w_{k+1})\ldots\phi_N(w_N)|0 \rangle \]

\[ = \sum_k \langle 0| \phi_1(w_1)\ldots(\delta_\varepsilon \phi_k(w_k))\ldots\phi_N(w_N)|0 \rangle. \]

The deformation of the contour used here is depicted in the figure below:

This means that the variation of the correlation function works as a derivation, involving the variations of the individual fields. In fact, we know that \( \delta_\varepsilon \phi(z) = [Q_\varepsilon, \phi(z)] \). The Lie bracket is a derivation and hence satisfies the Leibniz rule:

\[ [Q_\varepsilon, \phi_1(w_1)\ldots\phi_N(w_N)] = \sum_k \phi_1(w_1)\ldots\phi_{k-1}(w_{k-1})[Q_\varepsilon, \phi_k(w_k)]\phi_{k+1}(w_{k+1})\ldots\phi_N(w_N). \]
Correlation functions with descendant fields

Strictly speaking, it is valid only within the context of conformal transformations generated by Noether allows us to write the above equation, the conformal Ward identity, as some kind of operator identity.

The OPE of the energy momentum tensor with a primary field can be written, including this definition of a descendant field resembles the definition of the Virasoro modes of the energy momentum tensor. The OPE of the energy momentum tensor with a primary field can be written, including all regulart terms, as

\[ \mathcal{L}_w(z) = \sum_k \left[ \frac{h_k}{(z-w_k)^2} + \frac{1}{(z-w_k)} \partial w_k \right] \phi(w_k) \]

This follows, because in each contour integral \( \oint_{w_k} \frac{dz}{2\pi i} \varepsilon(z) T(z) \phi_k(w_k) \) we can choose the contour to be a very small circle around \( w_k \). Hence \( |z-w_k| \) is small and the OPE

\[ T(z) \phi_k(w_k) \sim \left( \frac{h_k}{(z-w_k)^2} + \frac{1}{(z-w_k)} \partial w_k \right) \phi_k(w_k) \]

can be inserted (for simplicity, we omit the symbol for radial ordering). Furthermore, each contour integral will indeed pick up only the singular parts of the OPE, independent of how \( \varepsilon(z) \) is chosen. This allows us to write the above equation, the conformal Ward identity, as some kind of operator identity. Strictly speaking, it is valid only within the context of conformal transformations generated by Noether charges. It is useful to introduce the following differential operator yielding the action of the energy momentum tensor on a correlation function of primary fields,

\[ \mathcal{L}_{w}(z) = \sum_k \left[ \frac{h_k}{(z-w_k)^2} + \frac{1}{(z-w_k)} \partial w_k \right] \phi(w_k) \]

[P2] Correlation functions with descendant fields

Let us now consider a correlation function with \( N - 1 \) primary fields \( \phi_i(w_i) \), and one descendant field

\[ \phi_N^{(-k)}(w_N) = \frac{1}{2\pi i} \oint dz \frac{1}{(z-w_N)^{k-1}} T(z) \phi_N(w_N) \]

This definition of a descendant field resembles the definition of the Virasoro modes of the energy momentum tensor. The OPE of the energy momentum tensor with a primary field can be written, including all regulart terms, as

\[ \mathcal{R}T(z) \phi(w) = \sum_{k \geq 0} (z-w)^{k-2} \phi^{(-k)}(w) \]

from which we can read off \( \phi^{(0)}(w) = h \phi(w) \) and \( \phi^{(-1)}(w) = \partial \phi(w) \). Furthermore, the energy momentum tensor itself can be regarded as a descendant field of the identity: \( T(w) = \mathbb{1}^{(-2)}(w) \).

We want to express a correlation function \( \langle 0 | \phi_1(w_1) \ldots \phi_{N-1}(w_{N-1}) \phi_N^{(-k)}(w_N) | 0 \rangle \) involving one descendant field in terms of the function \( \mathcal{G}^{(N)}(\{w_i\}) \). To do so, we proceed as follows:

(a) We first use the conformal Ward identity to rewrite this in the following way:

\[
\langle 0 | \phi_1(w_1) \ldots \phi_{N-1}(w_{N-1}) \phi_N^{(-k)}(w_N) | 0 \rangle
= \langle 0 | \phi_1(w_1) \ldots \phi_{N-1}(w_{N-1}) \int \frac{dz}{2\pi i (z-w_N)^{k-1}} T(z) \phi_N(w_N) | 0 \rangle
= \int \frac{dz}{2\pi i (z-w_N)^{k-1}} \left[ \langle 0 | T(z) \phi_1(w_1) \ldots \phi_{N}(w_N) | 0 \rangle \right.
- \sum_{j=1}^{N-1} \left[ \frac{h_j}{(z-w_j)^2} + \frac{1}{(z-w_j)} \partial w_j \right] \langle 0 | \phi_1(w_1) \ldots \phi_{N}(w_N) | 0 \rangle \right].
\]

In the first line, the contour encircles only the point \( w_N \), while in the second line, the contour encircles all the \( w_j \), \( j = 1, \ldots, N \). Finally, in the third line, we have individual small contours encircling all the \( w_j \) except \( w_N \).
So, we used contour deformation to replace the small contour encircling \( w_N \) by a large contour encircling all the \( w_i, \ i = 1, \ldots, N \) and \( N - 1 \) small contours encircling the individual \( w_j, \ j = 1, \ldots, N - 1 \) in the opposite direction (note the minus sign!). We also plugged in the OPE of \( T(z) \) with the primary fields \( \phi_j \) in the latter \( N - 1 \) contour integrals.

(b) The first term with the large contour can be deformed on the Riemann sphere into a small contour encircling the point infinity, as depicted in the following figure:

Imagine that you pull the contour all around the whole sphere. However, this term does not contribute for any \( k \geq -1 \) due to the highest weight property of the state \( \langle 0 \rangle \). Using \( T(z) = \sum_m L_m z^{-m-2} \), this can be seen as follows by performing the contour deformation via the conformal map \( z \mapsto u = z^{-1} \), the inversion:

\[
\oint \frac{dz}{2\pi i} \frac{1}{(z - w_N)^{k-1}} L_m z^{-m-2} = \oint \frac{du}{2\pi i} \frac{u^{k-1}}{u^2 (1 - uw_N)^{k-1}} L_m u^{m+2} = - \oint \frac{du}{2\pi i} \frac{1}{(1 - uw_N)^{k-1}} L_m u^{m+k-1} \]

since \( \langle 0 | L_m = 0 \) for all \( m \leq 1 \) as we have no singularities within the contour. The \( u \)-contour is a small circle around \( u = 0 \), and hence has a pole only for \( (m + k - 1) < 0 \). Now, as \( k > 0 \), we must have \( m < 0 \), but then the highest weight property annihilates this contribution (remember that for \( z = \infty \), the modes \( L_m \) automatically are applied to the out-state \( \langle 0 \rangle \), which sits, in radial ordering, at infinity, the infinite future.

(c) Putting everything together we find the result

\[
\langle 0 | \phi_1(w_1) \ldots \phi_{N-1}(w_{N-1}) \phi_N^{(-k)}(w_N) | 0 \rangle = \mathcal{L}_{-k}(w_N) \langle 0 | \phi_1(w_1) \ldots \phi_N(w_N) | 0 \rangle
\]

with the differential operator

\[
\mathcal{L}_{-k}(w_N) = \sum_{j=1}^{N-1} \left[ -\frac{(1-k)h_j}{(w_j-w_N)^k} + \frac{1}{(w_j-w_N)^{k-1}} \partial_w^j \right].
\]

In deed, as discussed in part (b), the first term does not contribute, while the second term consists of the operator

\[
\mathcal{L}_{-k}(w_N) = - \oint \frac{dz}{2\pi i} \frac{1}{(z-w_N)^{k-1}} \sum_{j=1}^{N-1} \left[ \frac{h_j}{(z-w_j)^2} + \frac{1}{(z-w_j)^{k-1}} \partial_w^j \right] = \sum_{j=1}^{N-1} \left[ -\frac{(1-k)h_j}{(w_j-w_N)^k} + \frac{1}{(w_j-w_N)^{k-1}} \partial_w^j \right]
\]

applied to the correlation function \( G^{(N)}(\{z_i\}) \). The second line follows simply from Cauchy’s integral formula

\[
\frac{1}{2\pi i} \oint \frac{dz}{(z-w)^n} = \frac{1}{(n-1)!} \partial^{n-1} f(w).
\]
This can be generalized to many multiple descendant fields \( \phi_1^{(-k_1,-k_2,...,-k_l)}(w_{1}) \). Firstly, it is clear that we can repeat the construction above with a descendant of the descendant of the field at \( w_{N} \). This can be done arbitrarily often in successive order. Working from the inside out, we can always apply the conformal Ward identity for the case of primary fields (i.e. using the specific OPE of the energy momentum tensor with primary fields). The result is then simply

\[
\langle 0 | \phi_1(w_1) \cdots \phi_{N-1}(w_{N-1}) \phi_N^{(-k_1,-k_2,...,-k_m)}(w_N) | 0 \rangle = L_{-k_1}(w_N) \cdots L_{-k_m}(w_N) \langle 0 | \phi_1(w_1) \cdots \phi_N(w_N) | 0 \rangle .
\]

Secondly, if other fields are also descendant, we can work in a similar way. The difference is now only, that we cannot use the conformal Ward identity of primary fields. Although it is possible to compute such arbitrary correlators of descendant fields, the resulting expressions are much more complicated. In fact, no closed forms as for the case of just one arbitrary descendant field do exist.

Remark: Conformal symmetry implies that the two-point correlation function of descendants of primary fields must vanish.

Thus, all correlation functions can be expressed in terms of correlation functions of solely primary fields. This means that conformal symmetry alone allows us to compute arbitrary correlation functions as long as we know how to compute the correlation functions of solely primary fields.