Solution VI for the lecture course CONFORMAL FIELD THEORY

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Free field realization

FREE FIELD REALIZATION AND VERTEX OPERATORS

We learned in the lecture that in conformal field theory, fields are divided into two classes, the primary fields $\Phi_h(z,\bar{z})$, and their descendant fields. If we know how to compute correlation functions of primary fields, we can compute any correlation function. What we miss so far is an explicit realization of a primary field. We know one exception so far: Considering a theory of a free massless scalar Boson $\phi(z,\bar{z})$, we saw that the current $J(z) = \mathrm{i}\partial_z\phi(z,\bar{z})$ is a chiral primary field of weight $h = 1$. But how does one construct primary fields of other scaling dimensions?

[P1] *Vertex operators*

Consider the normal ordered expression

$$V_k(z,\bar{z}) = {:}\exp(ik\phi(z,\bar{z})){:}\,,$$

and compute its OPE with the energy momentum tensor. We remember that $T(z) = -\frac{1}{2}{:}\partial\phi(z)\partial\phi(z){:}$, and that $\langle\phi(z,\bar{z})\phi(w,\bar{w})\rangle = -\log|z-w|^2$. We make use of Wick's theorem to contract products of normal ordered quantities into fully normal ordered quantities times expectation values in the usual way, and find

$$\begin{aligned} T(z)V_k(w,\bar{w}) &= -\tfrac{1}{2}{:}\partial\phi(z)\partial\phi(z){:}\sum_{n=0}^{\infty}\frac{1}{n!}(\mathrm{i}k)^n{:}\phi^n{:} \\ &= -\tfrac{1}{2}\sum_{n=0}^{\infty}\frac{1}{n!}(\mathrm{i}k)^n\frac{n(n-1)}{(z-w)^2}{:}\phi^{n-2}{:} + \sum_{n=0}^{\infty}\frac{1}{n!}(\mathrm{i}k)^n\frac{n}{(z-w)}{:}\partial\phi(z)\phi^{n-1}{:} \\ &= \frac{k^2/2}{(z-w)^2}V_k(w,\bar{w}) + \frac{1}{(z-w)}\partial_w V_k(w,\bar{w}) + \textit{reg. terms}\,. \end{aligned}$$

The same calculation goes through for $\bar{T}(\bar{z})$. We now immediately read off from the form of these OPEs that the vertex operators $V_k(z,\bar{z})$ are primary fields ith conformal weights$h = \bar{h} = \frac{k^2}{2}$. Note that V_k has the same conformal weight as V_{-k}.

[P2] *Operator algebra*

Compute the leading term of the OPE of two vertex operators with Wick's theorem or by inserting the mode expansion of the free field and using the Baker-Hausdorff formula. This yields

$$\begin{aligned} V_{k'}(z,\bar{z})V_k(w,\bar{w}) &= {:}\exp(\mathrm{i}k'\phi(z,\bar{z})){:}{:}\exp(\mathrm{i}k\phi(w,\bar{w})){:} \\ &= (z-w)^{\frac{1}{2}(k+k')^2-\frac{1}{2}k^2-\frac{1}{2}k'^2}(\bar{z}-\bar{w})^{\frac{1}{2}(k+k')^2-\frac{1}{2}k^2-\frac{1}{2}k'^2}{:}\exp(\mathrm{i}(k+k')\phi(w,\bar{w})){:} + \ldots \\ &= |z-w|^{2kk'}V_{k+k'}(w,\bar{w}) + \ldots \end{aligned}$$

as its leading term. Thus, the vertex operators form an operator algebra. The number k is called the charge of the vertex operator. To verify that this is indeed the $U(1)$ charge with respect to the current $J(z) = \mathrm{i}\partial\phi(z,\bar{z})$, we either use the OPE $J(z)V_k(w,\bar{w})$, or we note that $T(z) = \frac{1}{2}{:}J(z)J(z){:}$ which implies that the charge q with respect to the current $J(z)$ is $q(k) = k$. The charge of a product of operators is therefore additive. It is a rather typical feature of CFTs that the scaling dimensions are quadratic expressions in charges due to the fact that the energy momentum tensor is often a bilinear expression in currents.

[P3] *Correlators*

The two-point function of two such vertex operators can be found in many ways, e.g. by exploiting $SL(2,\mathbb{C})$ invariance or the OPE computed above, and turns out as

$$\langle V_k(z,\bar{z})V_{k'}(w,\bar{w})\rangle = (z-w)^{-2\frac{k^2}{2}}(\bar{z}-\bar{w})^{-2\frac{k^2}{2}}\delta_{k+k',0} = |z-w|^{-k^2}\delta_{k+k',0}\,.$$

Alternatively, one can use the general identity

$$\langle \exp(\mathrm{i}k\phi(z,\bar{z}))\exp(-\mathrm{i}k\phi(w,\bar{w}))\rangle = \exp(k^2\langle\phi(z,\bar{z})\phi(w,\bar{w})\rangle)$$

of free field theory. In fact, the two fields must contract to the identity field, since the only non-vanishing one-point function is $\langle \mathbb{1}\rangle = 1$. The OPE above tells us that the leading term would be the identity operator if $k + k' = 0$, as $V_0(z,\bar{z}) = \mathbb{1}$. In deed, the identity operator not only has scaling dimension $h = \bar{h} = 0$, but also must have vanishing charge with respect to $J(z)$, as $h = k^2/2$.

The n-point functions of arbitrary vertex operators of the free bosonic CFT can all be computed yielding the quite simple result

$$\langle\prod_i V_{k_i}(z_i,\bar{z}_i)\rangle = \prod_{i>j}|z_j - z_i|^{k_ik_j}\delta_{\sum_i k_i,0}\,,$$

provided $|z_i| > |z_j|$ for $i < j$. This follows from Wick's theorem that demands to consider all possible contractions. Only complete contractions down to a product of two-point functions (i.e. propagators) will contribute, as this is the only term in Wick's expansion with no non-trivial normal ordered product left (which, by definition, would yield a vanishing expectation value). The only complete contraction is just the one yielding the product above. Thus, these n-point functions are trivially zero unless the charge balance $\sum_i k_i = 0$ is kept, i.e. total momentum is conserved.

Alternatively, we can use the OPE of two vertex operators., Contracting all fields via successive OPEs will finally result in the vertex operator $V_K(0,0)$, $K = \sum_i k_i$, which must be of conformal weight zero. The OPE can only be applied for short distances. However, global conformal invariance always admitts to achieve this situation by a global translation of all points $z_i \mapsto z_i + Z$ with $|Z| \gg 1$ and a following inversion.

[P4] *Conservation of momentum*
The condition $\sum_i k_i = 0$ comes from the existence of a conserved charge. Actually, the operator $J(z) = \mathrm{i}\partial\phi(z)$ is a conserved current with zero mode $a_0 = p$, as can be infered from its mode expansion $J(z) = pz^{-1} + \sum_{n\neq 0} a_n z^{-n-1} \equiv \sum_n a_n z^{-n-1}$. Since the vacuum was defined in such a way that $\langle 0|p = p|0\rangle = 0$, it follows that

$$\begin{aligned}
0 &= \langle 0|\overset{(\leftarrow)}{p}\prod_i V_{k_i}(z_i,\bar{z}_i)|0\rangle \\
&= \langle 0|\overset{(\rightarrow)}{p}\left(\prod_i V_{k_i}(z_i,\bar{z}_i)\right)|0\rangle \\
&= \sum_j \langle 0|\prod_{i>j} V_{k_i}(z_i,\bar{z}_i)(pV_{k_j}(z_i,\bar{z}_i))\prod_{i<j}V_{k_i}(z_i,\bar{z}_i)|0\rangle \\
&= \sum_j (k_j)\langle 0|\prod_i V_{k_i}(z_i,\bar{z}_i)|0\rangle\,,
\end{aligned}$$

where we have indicated the direction in which p is applied. That is in fact consistent with the operator produkt expansion of the CFT. Due to global conformal invariance, the only non-vanishing one-point function must be of a field of zero conformal weight (which in general is the identity).

We learn from this that the two-point function is non-zero only for $k' = -k$, meaning that the correct definition of the in- and out-states is

$$|k\rangle = V_k(0,0)|0\rangle\,,\quad \langle k| = \lim_{z\to\infty}\langle 0|(V_k(z,\bar{z}))^\dagger z^{k^2}\bar{z}^{k^2} = \lim_{z\to\infty}\langle 0|V_{-k}(z,\bar{z})z^{k^2}\bar{z}^{k^2}\,,$$

such that $\langle k'|k\rangle = \delta_{k,k'}$. One says that the field $V_{-k}(z,\bar{z})$ is the *conjugate* field of $V_k(z,\bar{z})$. Note that $h(k) = \frac{1}{2}k^2 = h(-k)$ such that conjugate fields have the same conformal weights, as it must be.