HAUSÜBUNG XIV

Lösungsskizze

[H39] (a) Halbkreis $\Gamma : \varphi \to r(\cos\varphi, \sin\varphi, 0)$, $0 \le \varphi \le \pi$

$$\frac{d\vec{r}}{d\varphi} = r(-\sin\varphi, \cos\varphi, 0)$$

$$\vec{F} = A\begin{pmatrix}0\\0\\1\end{pmatrix} \times r\begin{pmatrix}\cos\varphi\\ \sin\varphi\\ 0\end{pmatrix} = A r\begin{pmatrix}-\sin\varphi\\ \cos\varphi\\ 0\end{pmatrix}$$

$$\frac{d\vec{r}}{d\varphi}\cdot\vec{F} = A r^2 \Rightarrow \int_0^\pi d\varphi\, \frac{d\vec{r}}{d\varphi}\cdot\vec{F} = A\pi r^2$$

(b) $F_i = -\partial_i V$ mit $V = \frac{\kappa}{4}(x^4 + 4x^2y^2 + y^4)$

$$\Rightarrow \int_{\underline{\lambda}}^{\bar{\lambda}} d\lambda\, \underbrace{\frac{d\vec{r}}{d\lambda}\cdot\vec{F}(\vec{r}(\lambda))}_{= -\frac{d}{d\lambda}V(\vec{r}(\lambda))} = -V(\vec{r}(\bar{\lambda})) + V(\vec{r}(\underline{\lambda}))$$

$$= \frac{\kappa}{4}(a^4 + 4a^2b^2 + b^4) - \frac{\kappa}{4}(a^4 + 4a^2b^2 + b^4) = 0$$

(c) Arbeit hängt nur von den Werten des Potentials am Anfangs- und Endpunkt ab, wegunabhängig!

[H40] Potentialkraft $F_i = -\partial_i V \Rightarrow \int_{\underline{\Theta}}^{\bar{\Theta}} d\lambda\, \frac{dr^i}{d\lambda} \underbrace{F_i}_{= -\partial_i V} = -\int_{\underline{\Theta}}^{\bar{\Theta}} d\lambda \frac{d}{d\lambda} V$

$$= -V(\bar{\Theta}) + V(\underline{\Theta})$$

$V_{\text{Fadenpendel}} = mgr\cos\Theta \Rightarrow$

Arbeit $= mgr\cos\underline{\Theta} - mgr\cos\bar{\Theta}$.

[H41] Parabel: $s \to (s, as^2)$, Tangente $(1, 2as)$

Länge $\sqrt{1+4a^2s^2} = \sqrt{\frac{dx}{ds}^2 + \frac{dy}{ds}^2}$

$$l = \int_0^1 ds \sqrt{1+4a^2s^2} = \frac{1}{2a}\int_0^{2a} ds' \sqrt{1+s'^2} \quad \text{mit } s' = 2as$$

Stammfunktion von $\sqrt{1+x^2} = F(x) = \frac{1}{2}\left(x\sqrt{1+x^2} + \ln|x+\sqrt{1+x^2}|\right)$

(z.B. Bronnstein / Mathematica, ...)

Probe: $$\frac{dF}{dx} = \frac{1}{2}\left(\sqrt{1+x^2} + x\,\frac{1}{2}\frac{1}{\sqrt{1+x^2}}\,2x + \frac{1}{x+\sqrt{1+x^2}}\left(1+\frac{2x}{2\sqrt{1+x^2}}\right)\right)$$

$$= \frac{1}{2}\left(\frac{1+x^2+x^2}{\sqrt{1+x^2}} + \frac{1}{\sqrt{1+x^2}}\,\frac{\sqrt{1+x^2}+x}{x+\sqrt{1+x^2}}\right)$$

$$= \frac{2}{2}\,\frac{1+x^2}{\sqrt{1+x^2}} = \sqrt{1+x^2} \quad ✓$$

$$\Rightarrow l = \frac{1}{2a}\,\frac{1}{2}\left(2a\sqrt{1+4a^2} + \ln|2a+\sqrt{1+4a^2}|\right)$$

[H42] Schnittkurve von $z(x,y) = h - ax^2 - by^2$ und $z(x,y) = 0$ ist eine Ellipse: $0 = h - ax^2 - by^2 \leadsto \frac{a}{h}x^2 + \frac{b}{h}y^2 = 1$

B = Ellipsenfläche $\Rightarrow$ Volumen $= \int_B d^2(x,y)\,(h-ax^2-by^2)$

$$= \int_{-\sqrt{h/a}}^{+\sqrt{h/a}} dx \int_{-\sqrt{\frac{h}{b}(1-\frac{a}{h}x^2)}}^{+\sqrt{\frac{h}{b}(1-\frac{a}{h}x^2)}} dy\,(h-ax^2-by^2)$$

$$= 4\int_0^{\sqrt{h/a}} dx \int_0^{\sqrt{h(1-\frac{a}{h}x^2)/b}} dy\,(h-ax^2-by^2)$$

$x' = \sqrt{\frac{a}{h}}\,x$

$y' = \sqrt{\frac{b}{h}}\,y$

$$= 4\sqrt{\frac{h}{a}}\sqrt{\frac{h}{b}}\,h\int_0^1 dx' \int_0^{\sqrt{1-x'^2}} dy'\,(1-x'^2-y'^2)$$

$$= 4\frac{h^2}{\sqrt{ab}}\int_0^1 dx' \left[(1-x'^2)y' - \frac{y'^3}{3}\right]_{y'=0}^{y'=\sqrt{1-x'^2}}$$

$$= \frac{4h^2}{\sqrt{ab}}\int_0^1 dx'\left(\sqrt{1-x'^2}\,(1-x'^2) - \frac{1}{3}\sqrt{1-x'^2}\,(1-x'^2)\right)$$

$$= \frac{8h^2}{3\sqrt{ab}}\int_0^1 dx\,\sqrt{1-x^2}\,(1-x^2)$$

= Formelsammlung oder:

$$x = \sin\varphi \qquad dx = \cos\varphi\, d\varphi \qquad 1-x^2 = \cos^2\varphi \qquad \sqrt{1-x^2} = \cos\varphi$$

$$= \frac{8h^2}{3\sqrt{ab}} \int_0^{\pi/2} d\varphi \cos\varphi \cos\varphi \cos^2\varphi = \frac{8h^2}{3\sqrt{ab}}\, d\varphi \cos^4\varphi$$

$$\cos\varphi = \tfrac{1}{2}\left(e^{i\varphi}+e^{-i\varphi}\right)$$

$$= \frac{8h^2}{16\cdot 3\sqrt{ab}} \int_0^{\pi/2} d\varphi \left(e^{4i\varphi} + 4e^{2i\varphi} + 6 + 4e^{-2i\varphi} + e^{-4i\varphi}\right)$$

$$= \frac{1}{6}\frac{h^2}{\sqrt{ab}}\Big[\underbrace{\frac{1}{4i}\left(e^{4i\varphi}-e^{-4i\varphi}\right)}_{\underbrace{\frac{1}{2}\sin 4\varphi\,\big|_0^{\pi/2}}_{=0}} + \underbrace{\frac{4}{2i}\left(e^{2i\varphi}-e^{-2i\varphi}\right)}_{\underbrace{4\sin 2\varphi\,\big|_0^{\pi/2}}_{=0}} + 6\varphi\Big]_0^{\pi/2}$$

$$= \frac{1}{6}\frac{h^2}{\sqrt{ab}}\, 3\pi$$

$$= \frac{\pi}{2}\frac{h^2}{\sqrt{ab}}$$