GAUGE INVARIANCE AND RENORMALIZATION

We have seen in the lecture that quantum field theories do need renormalization in the perturbative approach. We learned that renormalization is a necessity with deep physical content, and not just an annoyance. It forces upon us to introduce new physical quantities, i.e. quantities with dimensions, which enable us to connect the generic theory to the real world and experimental observation.

However, if the quantum field theory is, in addition, a gauge invariant theory, an additional problem arises which makes he naive amplitudes highly divergent. Let us state briefly what this problem is: A generic quantum field theory can schematically be written as

$$\int \mathcal{D}\varphi \exp(-\frac{1}{2}\varphi^t K\varphi - V(\varphi) + J^t\varphi) = \exp(-V(\frac{\delta}{\delta J}))\exp(\frac{1}{2}J^t K^{-1}J)\,.$$

Here, φ stand for a vector of all fields or field components, K^{-1} denotes symbolically the matrix of all propagators between the fields, J is the source term, and $V(\varphi)$ encodes the interaction between the fields by yielding the possible vertices. The question is, how to proceed if K^{-1} does not exist because K possesses zero eigen values. The simplest example of such a theory is quantum electro dynamcics (QED), the Maxwell theory. The action of QED is

$$S[A] = \int \mathrm{d}^4x \left[\frac{1}{2}A_\mu(\partial^2\eta^{\mu\nu} - \partial^\mu\partial^\nu)A_\nu + A_\mu J^\mu\right]\,,$$

where we can identify $Q^{\mu\nu} = partial^2\eta^{\mu\nu} - \partial^\mu\partial^\nu$ as our operator K. Now, gauge invariance of the theory implies that $Q^{\mu\nu}\partial_\nu\Lambda(x) = 0$ for any scalar function $\Lambda(x)$. Thus, Q has eigenvalue zero. On the other hand the Maxwell equation $\partial_\mu F^{\mu\nu} = J^\nu$ is nothing else than $Q_{\mu\nu}A^\nu = J^\nu$ with the formal solution $A^\nu = (Q^{-1})^{\nu\mu}J_\mu$. However, as it stands, Q^{-1} does not exist. The way out is to introduce an additional condition for A^μ, the so called "gauge fixing condition". Effectively, this eliminates the gauge degree of freedom and hence the zero eigen value of Q.

Trivial example.. Let us consider a toy example, namely the integral

$$\begin{aligned} &\int_{-\infty}^{+\infty} \mathrm{d}A\mathrm{e}^{-A^tKA} \quad \text{with} K = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} A = \begin{pmatrix} a \\ b \end{pmatrix} \\ = &\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \mathrm{d}a\mathrm{d}b\mathrm{e}^{-a^2}\,. \end{aligned}$$

Clearly, the integration over b does not exist. Here, b is our gauge degree of freedom, and we essentially integrate over all possible gauges. This means, that we integrate an infinite amount of physically equivalent instances of the same thing – they differ just by different gauges, i.e. different values of b. Now, we perform the gauge fixing, which is done by inserting a $\delta(f(b))$ with an arbitrary function $f(b)$ which has (at least) one zero. More specifically, we take

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \mathrm{d}a\mathrm{d}b\delta(b-\xi)\mathrm{e}^{-a^2}$$

and note that the result does not depend on our choice of ξ, i.e. on the particular choice of gauge.

Faddeev-Popov.. Let us do this now a bit more formally. Let us consider a "generic" path integral

$$I = \int \mathcal{D}A\mathrm{e}^{\mathrm{i}S[A]}\,.$$

Suppose now that there exist transformations $A \mapsto A_g$ such that $S[A] = S[A_g]$ and $\mathcal{D}A = \mathcal{D}A_g$. Thus, both the action as well as the integration measure must be invariant under gauge transformations. Obviously, these transformations yield a group structure in a natural way, $A_g \mapsto (A_g)_{g'} = A_{g'\cdot g}$. We now attempt to rewrite our I in such a way that we factorize the path integral into a gauge independent part and an integration over the group manifold of the gauge group, i.e.

$$I = \left(\int \mathcal{D}g\right) J \quad \text{with } J \text{ independent of } g\,.$$

In fact, $\mathcal{D}g$ should be a so called invariant measure. For the case that the gauge group is a compact Lie group, it can be taken to be the Haar measure. As a trivial example, take polar coordinates and an action which does depend only on the distance:

$$I = \int \mathrm{d}x\mathrm{d}y\mathrm{e}^{\mathrm{i}S(s,y)}\,, \quad S(x,y) \text{ a function of } x^2+y^2\,.$$

Clearly, in polar coordinates, this reads

$$I = \left(\int \mathrm{d}\theta\right) J \quad \text{with } J = \int \mathrm{d}r\, r\mathrm{e}^{\mathrm{i}S(r^2)}\,.$$

Note that $\int \mathrm{d}\theta = 2\pi = \mathrm{vol}(O(2)) = \mathrm{vol}(U(1))$ is the volume of the gauge group.

This trivial example helps us to understand the generic procedure as devised by Faddeev and Popov. One start by the following implementation of the gauge fixing,

$$1 = \Delta(A)\int \mathcal{D}g\delta(f(A_g))\,,$$

which implicitly defines $\Delta(A)$ which is called the Faddeev-Popov determinant. It is easy to see that the Faddeev-Popov determinant is gauge invariant,

$$(\Delta(A_{g'}))^{-1} = \int \mathcal{D}g\delta(f(A_{g'g})) = \int \mathcal{D}g''\delta(f(A_{g''})) = (\Delta(A))^{-1}\,,$$

where $g'' = g'g$ and where we used the invariance of the integration measure, $\mathcal{D}g = \mathcal{D}g''$. As we defined (formally) a complicated one, we can insert this into our path integral I. Hence,

$$\begin{aligned} I &= \int \mathcal{D}A\mathrm{e}^{\mathrm{i}S[A]} \\ &= \int \mathcal{D}A\mathrm{e}^{\mathrm{i}S[A]}\Delta(A)\int \mathcal{D}g\delta(f(A_g)) \\ &= \int \mathcal{D}g\int \mathcal{D}A\mathrm{e}^{\mathrm{i}S[A]}\Delta(A)\delta(f(A_g)) \\ &= \left(\int \mathcal{D}g\right)\int \mathcal{D}A\mathrm{e}^{\mathrm{i}S[A]}\Delta(A)\delta(f(A_g))\,, \end{aligned}$$

where we substituted $A_{g^{-1}}$ for A to get to the last line, since $\mathcal{A}$, $S[A]$ and $\Delta(A)$ are all invariant under gauge transformations by assumption. Thus, we arrive at the desired situation where we can factorize out the integration over the volume of the gauge group.

Gauge fixing for the electromagnetic field.. Let us put this into action for our quantum electrodynamics (QED). The gauge transformation is $(A_\mu)_g = A_\mu - \partial_\mu\Lambda$. We choose for the gauge fixing $f(A) = \partial A - \sigma$, where $\sigma = \sigma(x)$ is a function. Note that the integral I does not depend on the choice of f although it superficially does not look like it. The point is that I is independent of σ. Therefore, we can integrate I with an arbitrary functional of σ, for example $\exp(-\frac{\mathrm{i}}{2\xi}\int \mathrm{d}^4x\sigma(x)^2)$. Furthermore, we find for the Faddeev-Popov determinant

$$(\Delta(A))^{-1} = \int (D)g\delta(f(A_g)) = \int \mathcal{D}\Lambda\delta(\partial A - \partial^2\Lambda - \sigma)\,.$$

In I, the combination $(\Delta(A))\delta(f(A))$ appears. Effectively, this means that we can set $f(A) = \partial A - \sigma$ to zero. This implies that the determinant is something like

$$\Delta(A) \text{ ``=''} \left(\int \mathcal{D}\Lambda\delta(\partial^2\Lambda)\right)^{-1}$$

which does not depend on A at all. In cases where the Faddeev-Popov determinant does not depend on the fields, one can actually simply drop it. Then, we find formally

$$I \text{ ``=''} \int \mathcal{D}A\mathrm{e}^{\mathrm{i}S[A]}\delta(\partial A - \sigma)$$

up to gauge invariant factors which we all dropped. We now get rid of the function σ with the help of the functional chosen above:

$$\begin{aligned} Z &= \int \mathcal{D}\sigma\mathrm{e}^{-\frac{\mathrm{i}}{2\xi}\int \mathrm{d}^4x\sigma(x)^2}\int \mathcal{D}A\mathrm{e}^{\mathrm{i}S[A]}\delta(\partial A - \sigma) \\ &= \int \mathcal{D}A\mathrm{e}^{\mathrm{i}S[A] - \frac{\mathrm{i}}{2\xi}\int \mathrm{d}^4x(\partial A)^2}\,. \end{aligned}$$

The physical meaning of this is that our action is mapped to an effective action

$$\begin{aligned} S[A] \mapsto S_{\text{eff}}[A] &= S[A] - \frac{1}{2\xi}\int \mathrm{d}^4x(\partial A)^2 \\ &= \int \mathrm{d}^4x\left[\frac{1}{2}A_\mu\left(\partial^2\eta^{\mu\nu} - (1-\frac{1}{\xi})\partial^\mu\partial^\nu\right)A_\nu + A_\mu J^\mu\right] . \end{aligned}$$

Turning back to our initial operator Q, this implies the map

$$\begin{aligned} Q^{\mu\nu} \mapsto Q^{\mu\nu}_{\text{eff}} &= \partial^2\eta^{\mu\nu}(1-\frac{1}{\xi})\partial^\mu\partial^\nu , \\ \tilde{Q}^{\mu\nu}_{\text{eff}} &= -k^2\eta^{\mu\nu} + (1-\frac{1}{\xi})k^\mu k^\nu , \\ \Longrightarrow \tilde{Q}^{\mu\nu}_{\text{eff}}\left(-\eta_{\nu\lambda} + (1-\xi)\frac{k_\nu k_\lambda}{k^2}\right)\frac{1}{k^2} &= \delta_\lambda^{\ \mu} . \end{aligned}$$

Finally, we see that $\tilde{Q}_{\text{eff}}$ has an inverse, while Q did not have one. And we see that the photon propagator is what we already found by other means:

$$\mathrm{i}\tilde{D}_{\mu\nu}(k) = \frac{-\mathrm{i}}{k^2}\left(\eta_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right) .$$

You now recognize, that we effectively integrated out the gauge degree of freedom with the help of an auxiliary field σ leaving us only with the variable ξ which we can choose to our liking. Note that ξ is assumed to be fixed throughout all computations, but can be chosen arbitrarily beforehand. Its appearance assures that the propagator, the inverse of the differential operator in the action in the path integral, exists.

The so called Faddeev-Popov trick discussed above thus gets rid of the zero eigen values of the operator K in the action of gauge invariant theories. This is achieved by factoring out an integration over the volume of the gauge group. Typically, as every single gauge-inequivalent field configuration comes with the whole volume of all gauge-equivalent field configurations, this amounts to effectively extract an infinite factor which then can be thrown away. More elegantly, this infinite factor is dealt with by integrating out the gauge degree of freedom with the help of an auxiliary field. As this trick is extremely useful, the procedure has been applied to other problems of renormalization as well. In the method of Faddeev-Popov, one can renormalize a quantum field theory with the help of auxiliary "ghost" fields very efficiently. These are fields with the wrong statistics, i.e. Fermi statistics for Bosons and vice versa. As you can guess, what these auxiliary fields achieve is that they get the determinant of the operator K in the Gauss-type integrals in the numerator for Bosons and in the denominator for Fermions, since they just do it the wrong way. So, these determinants can then cancel the determinants of the physical fields effectively getting rid of these often infinite factors.