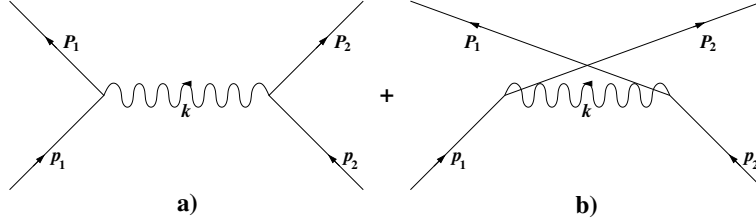


ELECTRON-ELECTRON SCATTERING VIA PHOTON EXCHANGE

As term is coming to an end soon, we have to hurry up a bit. Thus, it may be that I have to skip some material in the lecture which you might need to solve the exercise sheet. Therefore, I collect here briefly all the stuff, which I should (have) present(ed) in the lecture course. I start with the situation I considered on Friday (16. January 2009):



We found that the amplitude for the graphs (a) and (b) together could be written as

$$\mathcal{M} = A(P_1, P_2) - A(P_2, P_1),$$

such that the cross section becomes proportional to

$$|\mathcal{M}|^2 = |A(P_1, P_2)|^2 + |A(P_2, P_1)|^2 - 2\text{Re}(A(P_2, P_1)^* A(P_1, P_2)).$$

Furthermore, we found that all the squares of absolute values factorized,

$$|A(P_1, P_2)|^2 = \frac{e^4}{(P_1 - p_1)^4} [\bar{u}(P_1)\gamma^\mu u(p_1)\bar{u}(p_1)\gamma^\nu u(P_1)\bar{u}(P_2)\gamma_\mu u(p_2)\bar{u}(p_2)\gamma_\nu u(P_2)],$$

while the interference term does not factorize.

The most simple experiment.. In reality, a scattering experiment is usually conducted in such a way that the incoming electrons are unpolarized, and the polarization of the outgoing electrons is not measured. Hence, we have to average over the polarization of the incoming electrons, and must sum over the polarizations of the outgoing electrons. For that we can use the formula

$$\sum_s u(p, s)\bar{u}(p, s) = \not{p} + m.$$

With this it follows that

$$\begin{aligned} |A(P_1, P_2)|^2 &= \frac{e^4}{(P_1 - p_1)^4} \tau^{\mu\nu}(P_1, p_1)\tau_{\mu\nu}(P_2, p_2), \\ \tau^{\mu\nu}(P_1, p_1) &= \sum_s \sum_S \bar{u}(P_1, S)\gamma^\mu u(p_1, s)\bar{u}(p_1, s)\gamma^\nu u(P_1, S) \\ &= \frac{1}{(2m)^2} \text{tr} \left((\not{P}_1 + m)\gamma^\mu (\not{p}_1 + m)\gamma^\nu \right). \end{aligned}$$

For the interference term, one finds in an analogous manner

$$\begin{aligned} A(P_2, P_1)^* A(P_1, P_2) &\propto \kappa \\ \kappa &= \sum \sum \sum \sum \bar{u}(P_1)\gamma^\mu u(p_1)\bar{u}(P_2)\gamma_\mu u(p_2)\bar{u}(p_1)\gamma^\nu u(P_2)\bar{u}(p_2)\gamma_\nu u(P_1), \end{aligned}$$

where we have suppressed the spin labels.

Traces of gamma matrices.. It turns out that in order to evaluate such expressions, one very often has to compute traces over products of gamma matrices. If one keeps in mind that $(\gamma^\mu)^2 = \pm 1$ and $\{\gamma^\mu, \gamma^\nu\} = 0$ for $\nu \neq \mu$, it follows that the trace over an odd number of gamma matrices must vanish, $\text{tr}(\gamma^{\mu_1} \dots \gamma^{\mu_{2k+1}}) = 0$. Since there are only four different, linear independent, gamma matrices, it suffices to know the traces of products of two and four gamma matrices, as all higher products can be reduced to these with the help of the Clifford algebra. For example, we find:

$$\begin{aligned} \tau^{\mu\nu}(P_1, p_1) &= \frac{1}{(2m)^2} \left[\text{tr}(\not{P}_1 \gamma^\mu \not{p}_1 \gamma^\nu) + m^2 \text{tr}(\gamma^\mu \gamma^\nu) \right] \\ &= \frac{4}{(2m)^2} (P_1^\mu p_1^\nu - \eta^{\mu\nu} (P_1 \cdot p_1) + P_1^\nu p_1^\mu + m^2 \eta^{\mu\nu}), \end{aligned}$$

where we rewrite the term $\text{tr}(\not{P}_1 \gamma^\mu \not{p}_1 \gamma^\nu) = P_{1,\rho} p_{1,\lambda} \text{tr}(\gamma^\rho \gamma^\mu \gamma^\lambda \gamma^\nu)$, so we can more easily compute it. In a similar fashion, we can compute κ , which, however, leads to a rather long expression.

High Energy Physics.. Many things become much simpler when going to the relativistic limit, where the rest mass of the electron can be neglected in comparison to the momenta, $m \ll p$. One then finds

$$\begin{aligned} \kappa &\sim \frac{1}{(2m)^4} \text{tr}(\not{P}_1 \gamma^\mu \not{p}_1 \gamma^\nu \not{P}_2 \gamma_\mu \not{p}_2 \gamma_\nu) \\ &= \frac{-2}{(2m)^4} \text{tr}(\not{P}_1 \gamma^\mu \not{p}_1 \not{p}_2 \gamma_\mu \not{P}_2) \\ &= \frac{-32}{(2m)^4} (p_1 \cdot p_2)(P_1 \cdot P_2). \end{aligned}$$

In the relativistic limit, also the expression $\tau^{\mu\nu}$ simplifies, namely to

$$\tau^{\mu\nu} \sim \frac{4}{(2m)^2} (P_1^\mu p_1^\nu + P_1^\nu p_1^\mu - \eta^{\mu\nu} (P_1 \cdot p_1)).$$

Putting things together, we obtain in the relativistic limit

$$\begin{aligned} \tau^{\mu\nu}(P_1, p_1) \tau_{\mu\nu}(P_2, p_2) &= \frac{16}{(2m)^4} (P_1^\mu p_1^\nu + P_1^\nu p_1^\mu - \eta^{\mu\nu} (P_1 \cdot p_1)) (2P_2^\mu p_2^\nu - \eta^{\mu\nu} (P_2 \cdot p_2)) \\ &= \frac{32}{(2m)^4} ((p_1 \cdot p_2)(P_1 \cdot P_2) + (p_1 \cdot P_2)(p_2 \cdot P_1)). \end{aligned}$$

Explicit computation.. In order to really and explicitly compute this, we now move into the center-of-mass system and choose without loss of generality the coordinate system in such a way that $p_1 = E(1, 0, 0, 1)$, $p_2 = E(1, 0, 0, -1)$ for the incoming momenta, and correspondingly $P_1 = E(1, \sin \theta, 0, \cos \theta)$, $P_2 = E(1, -\sin \theta, 0, \cos \theta)$ for the outgoing momenta. With this choice we find for the scalar products $(p_1 \cdot p_2) = (P_1 \cdot P_2) = 2E^2$, $(p_1 \cdot P_1) = (p_2 \cdot P_2) = 2E^2 \sin^2 \theta/2$, $(p_1 \cdot P_2) = (p_2 \cdot P_1) = 2E^2 \cos^2 \theta/2$ and $(P_1 - p_1)^4 = (-2p_1 \cdot P_1)^2 = 16E^4 \sin^4 \theta/2$.

Result in the relativistic limit.. Now, we can put all this together and finally obtain in the relativistic limit the result

$$\begin{aligned} \frac{1}{2} \sum_s \sum_S |\mathcal{M}|^2 &= \frac{e^4}{4m^4} f(\theta), \\ f(\theta) &= \frac{1 + \cos^4 \theta/2}{\sin^4 \theta/2} + \frac{2}{\sin^2 \theta/2 \cos^2 \theta/2} + \frac{1 + \sin^4 \theta/2}{\cos^4 \theta/2}. \end{aligned}$$

The first term is the contribution from forward scattering, the third term then corresponds to backward scattering which is fixed by the symmetry $\theta \leftrightarrow \pi - \theta$, as the two electrons are indistinguishable. The second term, however, is the quantum interference term. The positive sign results from the fact that the two electrons are Fermions.

In the exercise sheet, you will show that one can obtain the differential cross section from this result, which then reads

$$\frac{d\sigma}{d\Omega} = \left(\frac{e^2}{4\pi} \right)^2 \frac{1}{8E^2} f(\theta).$$