LAGRANGIANS

These exercises repeat basics in working with Lagrangians and Lagrangian densities. If you have difficulties with these, you are strongly encouraged to read up basics in classical field theory.

[H1] Lagrangians in Classical Mechanics [3 pts]
Consider a physical system, described by a Lagrangian $L(q, \dot{q}, t)$, where as usual $q$ denotes all coordinates $q_i, i = 1, \ldots, d$ and similarly for $\dot{q}$. Show that the equations of motion derived from $L$ are the same as those obtained from $L' \equiv L + \frac{dF}{dt}$, where $F = F(q, t)$ is an arbitrary function of the coordinates and the curve parameter. Do this in two different ways: On the one hand, spell out the Euler-Lagrange equations corresponding to $L$ and $L'$ respectively, on the other hand consider the variation of the action. Also show that the canonical momenta are not preserved under this change of $L$.

[H2] Lagrangians in Classical Field Theory [3 pts]
Repeat the reasoning of [H1] for the case of a local covariant field theory with Lagrangian density $\mathcal{L}(\phi, \partial_{\mu} \phi)$, i.e. show that the equations of motion are invariant under addition of a total divergence $\mathcal{L} \rightarrow \mathcal{L}' \equiv \mathcal{L} + \partial_{\mu} f^\mu(\phi)$ in the two ways mentioned above.

[H3] Toy Model I – classical considerations [1+1+1+1+4 pts]
Let a non-relativistic classical field theory of a complex scalar field $\psi$ be given by the Lagrangian density

$$\mathcal{L} = i\psi^* \partial_0 \psi + b(\nabla \psi^*)(\nabla \psi),$$

where $b$ is some real parameter. Of course, this is not Lorentz covariant.

(a) The Lagrangian density is also not real, but show, that the action is real and write down a manifestly real form of the kinetic term.

(b) Write $\psi = \psi_1 + i\psi_2$ with real valued fields $\psi_1$ and $\psi_2$. Derive their respective equations of motion.

(c) Now, do no longer consider $\psi_1$ and $\psi_2$ as independent fields, but instead $\psi$ and its complex conjugate $\psi^*$. Show that in this manner the same equations of motion as in (b) are found. (The equations for $\psi^*$ will be the complex conjugate of those for $\psi$.)

(d) Make a plane wave ansatz

$$\psi(\vec{x}, t) = \exp \left( i(\vec{k} \cdot \vec{x} - \omega t) \right)$$

and find the necessary dispersion relation $\omega = \omega(\vec{k})$.

(e) As already mentioned, the theory is not Lorentz invariant. It is, however, invariant under translations and rotations as well as a global change of phase

$$\psi \mapsto e^{-i\lambda} \psi, \quad \psi^* \mapsto e^{i\lambda} \psi^*, \quad \text{with real } \lambda.$$

Find the conserved currents and charges that correspond to these symmetries, e.g. with the help of the Noether theorem. Fix the sign of the parameter $b$ with the help of the requirement that energy must be bounded from below.