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LAGRANGIANS

These exercises repeat basics in working with Lagrangians and Lagrangian densities. If you have difficulties with these, you are strongly encouraged to read up basics in classical field theory.

[H1] *Lagrangians in Classical Mechanics* **[3 pts]**
Consider a physical system, described by a Lagrangian $L(q, \dot{q}, t)$, where as usual q denotes all coordinates q_i, $i = 1, \ldots, d$ and similarly for $\dot{q}$. Show that the equations of motion derived from L are the same as those obtained from $L' \equiv L + \frac{\mathrm{d}F}{\mathrm{d}t}$, where $F = F(q, t)$ is an arbitrary function of the coordinates and the curve parameter. Do this in two different ways: On the one hand, spell out the Euler-Lagrange equations corresponding to L and L' respectively, on the other hand consider the variation of the action. Also show that the canonical momenta are *not* preserved under this change of L.

[H2] *Lagrangians in Classical Field Theory* **[3 pts]**
Repeat the reasoning of **[H1]** for the case of a local covariant field theory with Lagrangian density $\mathcal{L}(\phi, \partial_\mu\phi)$, i.e. show that the equations of motion are invariant under addition of a total divergence $\mathcal{L} \mapsto \mathcal{L}' \equiv \mathcal{L} + \partial_\mu f^\mu(\phi)$ in the two ways mentioned above.

[H3] *Toy Model I – classical considerations* **[1+1+1+1+4 pts]**
Let a non-relativistic classical field theory of a complex scalar field ψ be given by the Lagrangian density

$$\mathcal{L} = \mathrm{i}\psi^*\partial_0\psi + b(\nabla\psi^*)(\nabla\psi)\,,$$

where b is some real parameter. Of course, this is not Lorentz covariant.

(a) The Lagrangian density is also not real, but show, that the action is real and write down a manifestly real form of the kinetic term.

(b) Write $\psi = \psi_1 + \mathrm{i}\psi_2$ with real valued fields ψ_1 and ψ_2. Derive their respective equations of motion.

(c) Now, do no longer consider ψ_1 and ψ_2 as independent fields, but instead ψ and its complex conjugate ψ^*. Show that in this manner the same equations of motion as in (b) are found. (The equations for ψ^* will be the complex conjugate of those for ψ.)

(d) Make a plane wave ansatz

$$\psi(\vec{x}, t) = \exp\left(\mathrm{i}(\vec{k}\cdot\vec{x} - \omega t)\right)$$

and find the necessary dispersion relation $\omega = \omega(\vec{k})$.

(e) As already mentioned, the theory is not Lorentz invariant. It is, however, invariant under translations and rotations as well as a global change of phase

$$\psi \mapsto \mathrm{e}^{-\mathrm{i}\lambda}\psi\,, \quad \psi^* \mapsto \mathrm{e}^{\mathrm{i}\lambda}\psi^*\,,$$

with real λ. Find the conserved currents and charges that correspond to these symmetries, e.g. with the help of the Noether theorem. Fix the sign of the parameter b with the help of the requirement that energy must be bounded from below.