SCALAR FIELD THEORY

In the lecture, we discussed the Feynman rules for the scalar field. The following exercises shall help you to gain a bit of praxis with these rules.

[H1] Loops [5 pts]
Derive the amplitude of the diagram

from first principles within the Schwinger formalism, i.e. directly from

\[ Z[J, \lambda] = Z[0, 0] \exp \left( -\frac{i}{4!} \lambda \int d^4w \frac{\delta^4}{\delta J(w)^4} \right) \exp \left( -\frac{i}{2} \int \int d^4xd^4y J(x)D(x-y)J(y) \right). \]

Show in particular that there is a symmetry factor of 1/2. Your result should read

\[ \frac{1}{2} \left( -i\lambda \right)^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon} \frac{i}{(k_1 + k_2 - k)^2 - m^2 + i\epsilon}. \] (1)

[H2] Diagrams [5 pts]
Draw all diagrams up to order \( \lambda^2 \), in which two particles \( \varphi \) interact and finally four particles \( \varphi \) survive. Use the Feynman rules to write down the corresponding amplitudes in momentum space.

[H3] Real Particles [5 pts]
Lorentz invariance implies that we may choose \( k_1 + k_2 = (E, \vec{0}) \) in (1) by going into the center of mass system of the two incoming particles. The integral can then be studies as a function of the energy \( E \). Show that the two internal intermediate particles can only become real, if \( E > 2m \). Interpret this in physical terms.

[H4] Wick Contractions [3 pts]
Extract out of (1) the terms of order \( \lambda^1 \) and four external lines. Hint: This is done be replacing

\[ \exp \left( -\frac{i}{4!} \lambda \int d^4w \frac{\delta^4}{\delta J(w)^4} \right) \mapsto \frac{i}{4!} \lambda \int d^4w \frac{\delta^4}{\delta J(w)^4}, \]

\[ \exp \left( -\frac{i}{2} \int \int d^4xd^4y J(x)D(x-y)J(y) \right) \mapsto \frac{i^4}{4!2^4} \left( \int \int d^4xd^4y J(x)D(x-y)J(y) \right)^4. \]

Introduce the abbreviations \( J_a = J(x_a), \int_a = \int d^4x_a \), and \( D_{ab} = D(x_a - x_b) \). Your result shoud then be proportional to

\[ i\lambda \int_w \frac{\delta^4}{\delta J_w^4} \int \int \int \int \int \int D_{ax}D_{bf}D_{cg}D_{dh}J_aJ_bJ_cJ_dJ_fJ_gJ_h. \]

The four functional derivatives \( (\delta/\delta J_w) \) hit the eight \( J \)'s in all possible combinations producing many terms which you should write out. In the end, there are three different graphs, one connected one, and two disconnected ones.