THE CANONICAL FORMALISM

In the lecture, we introduce and develop QFT mainly via the path integral formalism. Historically, QFT was first developed within the formalism of canonical quantization. This alternative approach is still important, since there are results in QFT, which can be seen or derived much easier in the canonical formalism than in the path integral formalism. Moreover, the canonical formalism, although it is not free from some deficiencies, has a better developed mathematical foundation. The following exercises shall help you to gain some practice with the canonical formalism.

To recapitulate: In the lecture, we introduced the canonical commutation relations

\[ [\pi(\vec{x}, t), \varphi(\vec{x}', t)] = [\partial_0 \varphi(\vec{x}, t), \varphi(\vec{x}', t)] = -i \delta^{(D)}(\vec{x} - \vec{x}') \].

These led to a Fourier expansion of the real massive scalar field in the form

\[ \varphi(\vec{x}, t) = \int \frac{d^Dk}{(2\pi)^D2\omega_k} \left[ a(\vec{k})e^{-i(\omega_k t - \vec{k} \cdot \vec{x})} + a^\dagger(\vec{k})e^{i(\omega_k t - \vec{k} \cdot \vec{x})} \right], \]

where \( \omega_k = +\sqrt{\vec{k}^2 + m^2} \). The Fourier modes are on their parts operators, which satisfy the commutation relations

\[ [a(\vec{k}), a^\dagger(\vec{k}')] = \delta^{(D)}(\vec{k} - \vec{k}'), \]

which are often referred to as Heisenberg algebras.

[H1] Lorentz covariance

The definition of creation and annihilation operators is done slightly different by some authors. Namely, they define the algebra via

\[ \varphi(\vec{x}, t) = \int \frac{d^Dk}{(2\pi)^D2\omega_k} \left[ a(\vec{k})e^{-i(\omega_k t - \vec{k} \cdot \vec{x})} + a^\dagger(\vec{k})e^{i(\omega_k t - \vec{k} \cdot \vec{x})} \right], \]

i.e. without the square root in the denominator. Again, we have put \( \omega_k = +\sqrt{\vec{k}^2 + m^2} \).

i. Show that for an arbitrary function \( f \) one has:

\[ \int d^4k \delta(k^2 - m^2) \theta(k^0) f(k^0, \vec{k}) = \int \frac{d^3k}{2\omega_k} f(\omega_k, \vec{k}). \]

ii. Argue that the integration measure \( \frac{d^3k}{2\omega_k} \) is a Lorentz invariant measure. Hint: Lorentz transformations can never change the sign of \( k^0 \).

iii. This gives rise to an implicit definition of creation and annihilation operators, which are Lorentz covariant. Derive their commutation relations.

[H2] Expectation values

Let \( H \) be a Hamilton operator \( \int d^Dx H = \int d^Dx (\pi(\vec{x}, t)\partial_0 \varphi(\vec{x}, t) - \mathcal{L}) \). Compute its expectation value \( \langle \vec{k}' | H | \vec{k} \rangle \), where we have \( | \vec{k} \rangle = a^\dagger(\vec{k}) | 0 \rangle \).

[H3] Complex scalar field

Up to now, we considered the real (or Hermitean) scalar field. Let us now look at a complex scalar field with Lagrangian density \( \mathcal{L} = \partial_\varphi \partial^\varphi - m^2 \varphi^2 \).

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i. Write down the Euler-Lagrange equations of motion, where $\phi^\dagger$ and $\phi$ are considered as independent variables.

ii. Since $\phi$ is not Hermitean, we have to replace our definition of the field by

$$\phi(\vec{x}, t) = \int \frac{dDk}{\sqrt{(2\pi)^D2\omega_k}} \left[ a(\vec{k})e^{-i(\omega_k t - \vec{k} \cdot \vec{x})} + b^\dagger(\vec{k})e^{i(\omega_k t - \vec{k} \cdot \vec{x})} \right].$$

Show that the canonical commutation relations imply that the pairs $(a, a^\dagger)$ and $(b, b^\dagger)$ form two independent sets of creation and annihilation operators.

iii. Compute $\langle 0 | T(\phi(x)\phi^\dagger(0)) | 0 \rangle$. **Hint:** Keep in mind that $\partial_0 \phi$ is conjugate to $\phi^\dagger$, not to $\phi$.

[H4] **Time ordered product**

Let us consider an arbitrary field $A$ and let us assume $q^0 > 0$. Show that

$$\text{Im} \left( i \int d^4xe^{iqx} \langle 0 | T(A(x)A(0)) | 0 \rangle \right) = \frac{1}{2} \int d^4xe^{iqx} \langle 0 | [A(x), A(0)] | 0 \rangle,$$

by inserting $1 = \sum_n |n\rangle\langle n|$ for a complete set $|n\rangle$ of states between $A(x)$ and $A(0)$ on the left hand side. **Hint:** Use the integral representations

$$\theta(t) = -i \int \frac{d\omega}{2\pi} \frac{e^{i\omega t}}{\omega - i\epsilon}, \quad \theta(-t) = +i \int \frac{d\omega}{2\pi} \frac{e^{i\omega t}}{\omega + i\epsilon}.$$