The importance of symmetries in modern theoretical physics cannot be overestimated. They determine to a large extent the form of physical laws, for example the form of the Lagrangian. Symmetries and conservation laws are in a one-to-one correspondence, which is mediated by the Noether theorem.

**H1** Noether Theorem

The Noether theorem is often formulated as follows: We assume that the action be invariant under infinitesimal transformations \( \delta \varphi_a(x) = \theta^A V^A_a \), where the \( \theta^A \) are parameters and the \( V^A_a \) functions of the fields \( \varphi_b(x) \) and possibly of their derivatives \( \partial_{\mu} \varphi_b(x) \). The index label \( A \) can, but need not, run over a basis of generators of a Lie group. Please note that the statement, that the action does not change, must hold without the use of the equations of motion! Ultimately, the Euler-Lagrange equations are precisely derived from the fact, that all variations \( \delta S \) with respect to \( \delta \varphi \) vanish (under suitable boundary conditions). In the lecture, we discussed the example of a scalar field theory with internal \( O(n) \) symmetry: Here, \( \delta S = 0 \) holds precisely because the action has been constructed out of scalar products of \( O(n) \)-vectors.

i. We now look at a situation where the parameters \( \theta^A \) shall depend on the position in space-time, \( \theta^A = \theta^A(x) \), and \( \delta \varphi_a(x) = \theta^A(x)V^A_a[\varphi_b(x), \partial_{\mu} \varphi_b(x)] \). Of course, we do not necessarily have \( \delta S = 0 \) now, but we know that the variation of the action for constant \( \theta^A \) vanishes. Verify that \( \delta S \) can thus only have the form

\[
\delta S = \int d^4x J^\mu(x) \partial_\mu \theta^A(x) .
\]

With this, you can easily determine the current \( J^\mu(x) \), it is simply given by the coefficient of \( \partial_\mu \theta^A(x) \).

ii. Compute the current \( J^\mu(x) \) explicitly for the Lagrange density

\[
\mathcal{L} = \frac{1}{2} \left[ (\partial \varphi)^2 - m^2 \varphi^2 \right] - \frac{\lambda}{4} (\varphi^2)^2 ,
\]

which we encountered in the lecture.

**H2** Propagator

In the lecture, we introduced the propagator \( D_{ab}(x) \) for the scalar \( O(n) \) field theory. Show, that this propagator must be proportional to \( \delta_{ab} \).

**H3** Representations

Consider a theory with an internal \( SO(3) \)-symmetry. Construct the Lagrangian of such a theory with a Lorentz-scalar field \( \varphi \), which transforms in a five-dimensional representation of \( SO(3) \), up to terms of fourth order. Hint: It is helpful, to write \( \varphi \) as a symmetric, traceless \( 3 \times 3 \) matrix (can you say, why?).