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FIELD THEORY IN CURVED SPACETIME

Although we are mainly interested in QFT set in a flat background with Minkowski metric $\eta_{\mu\nu}$, it is often very helpful to work with the general formalism for arbitrary metric $g_{\mu\nu}$. On reason is that the general formulæ allow for the definition of the energy momentum tensor, which is a very powerful tool, and which can be used in the flat case as well. The other reason is that coupling of QFT to a classical gravitational field is possible, and the case of small deviations from the flat metric, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, can be dealt with in a perturbative manner.

[H1] *Energy* **[4 pts]**

Use the definition of the energy momentum tensor,

$$T^{\mu\nu}(x) = -\frac{2}{\sqrt{-g}}\frac{\delta S_M}{\delta g_{\mu\nu}(x)}\,,$$

to find the energy E for a scalar field in flat spacetime. Show that the result agrees with what you would obtain using the canonical formalism.

[H2] *Conjugate Momenta* **[3 pts]**

Show that in flat spacetime P^μ as derived here from the energy momentum tensor $T^{\mu\mu}$, when interpreted as an operator in the canonical formalism, satisfies $[P^\mu, \varphi(x)] = -\mathrm{i}\partial^\mu\varphi(x)$. Thus, it behaves exactly as you would expect the energy and momentum operators to do, namely as the the conjugate momenta to time and space, hence represented by $-\mathrm{i}\partial^\mu$.

[H3] *Trace* **[3 pts]**

Consider a general action $S_M = \int \mathrm{d}^4x\sqrt{-q}(A + g^{\mu\nu}B_{\mu\nu} + g^{\mu\nu}g^{\lambda\rho}C_{\mu\nu\lambda\rho} + \ldots)$. Work in the linearized scenario, where $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, $-g = 1 + \eta^{\mu\nu}h_{\mu\nu}$, and $g^{\mu\nu} = \eta^{\mu\nu} + h^{\mu\nu}$ up to order $O(h^2)$. Show that

$$T_{\mu\nu} = 2(B_{\mu\nu} + 2C_{\mu\nu\lambda\rho}\eta^{\lambda\rho} + \ldots) - \eta_{\mu\nu}\mathcal{L}$$

in flat space time (i.e. for $h = 0$). Compute its trace to show that the C-term does not contribute,

$$T \equiv \eta^{\mu\nu}T_{\mu\nu} = -(4A + 2\eta^{\mu\nu}B_{\mu\nu} + 0\cdot\eta^{\mu\nu}\eta^{\lambda\rho}C_{\mu\nu\lambda\rho} + \ldots)\,.$$

[H4] *Traceless* **[2 pts]**

Show that for the Maxwell field the energy momentum tensor becomes $T_{ij} = -(E_iE_j + B_iB_j) + \frac{1}{2}\delta_{ij}(\vec{E}^2 + \vec{B}^2)$ and hence $T = T_\mu{}^\mu = 0$.