BUSSTEPP LECTURES ON SUPERSYMMETRY

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Abstract. This is the written version of the supersymmetry lectures delivered at the 30th and 31st British Universities Summer Schools in Theoretical Elementary Particle Physics (BUSSTEPP) held in Oxford in September 2000 and in Manchester in August-September 2001.

(Version of 21 September 2001)

Contents

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Introduction

The aim of these lectures is to introduce supersymmetry to graduate students in Physics having completed the first year of their PhD studies in a British university. No previous exposure to supersymmetry is expected, but familiarity with topics normally covered in an introductory course in relativistic field theory will be assumed. These include, but are not restricted to, the following: lagrangian formulation of relativistic field theories, Lie symmetries, representations of the Poincaré group, gauge theories and spontaneous symmetry breaking. I have adopted a conservative approach to the subject, discussing exclusively four-dimensional rigid $N=1$ supersymmetry.

The lecture notes are accompanied by a series of Exercises and Problems. Exercises are meant to fill in the details of the lectures. They are relatively easy and require little else than following the logic flow of the lectures. Problems are more involved (although none are really difficult) and make good topics for tutorials.

The written version of the lectures contains more material than can be comfortably covered during the School and certainly more exercises and problems than can possibly be completed by the student during this time. It is my hope, however, that the interested student can continue working on the problems and exercises after the School has ended and that the written version of these notes can be of help in this task.

STOP Throughout the written version of the lectures you will find paragraphs like this one with one of the following signs:

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indicating, respectively, caveats, exercises, scholia and the (very) occasional amusing comment.

These notes are organised as follows.

In Lecture I we will introduce the simplest field theoretical model exhibiting (linearly realised) supersymmetry: the Wess–Zumino model. It will serve to illustrate many of the properties found in more phenomenologically realistic models. We will prove that the Wess–Zumino model is invariant under a "super" extension of the Poincaré algebra, known as the $N=1$ Poincaré superalgebra. The tutorial problem for this lecture investigates the superconformal invariance of the massless Wess–Zumino model.

In Lecture II we will study another simple four-dimensional supersymmetric field theory: supersymmetric Yang–Mills. This is obtained by coupling pure Yang–Mills theory to adjoint fermions. We will show that the action is invariant under the Poincaré superalgebra, and that the algebra closes on-shell and up to gauge transformations. This theory is also classically superconformal invariant, and this is the topic of the tutorial problem for this lecture.

In Lecture III we will study the representations of the $N{=}1$ Poincaré superalgebra. We will see that representations of this superalgebra consist of mass-degenerate multiplets of irreducible representations of the Poincaré algebra. We will see that unitary representations of the Poincaré superalgebra have non-negative energy and that they consist of an equal number of bosonic and fermionic fields. We will discuss the most important multiplets: the chiral multiplet, the gauge multiplet and the supergravity multiplet. Constructing supersymmetric field theories can be understood as finding field-theoretical realisations of these multiplets. The tutorial problem introduces the extended Poincaré superalgebra, the notion of central charges and the "BPS" bound on the mass of any state in a unitary representation.

In Lecture IV we will introduce superspace and superfields. Superspace does for the Poincaré superalgebra what Minkowski space does for the Poincaré algebra; namely it provides a natural arena in which to discuss the representations and in which to build invariant actions. We will learn how to construct invariant actions and we will recover the Wess–Zumino model as the simplest possible action built out of a chiral superfield. The tutorial problem discusses more general models built out of chiral superfields: we will see that the most general renormalisable model consists of N chiral multiplets with a cubic superpotential and the most general model consists of a supersymmetric sigma model on a Kähler manifold and a holomorphic function on the manifold (the superpotential).

In Lecture V we continue with our treatment of superspace, by studying supersymmetric gauge theories in superspace. We will see that supersymmetric Yang–Mills is the natural theory associated to a vector superfield. We start by discussing the abelian theory, which is easier to motivate and then generalise to the nonabelian case after a brief discussion of the coupling of gauge fields to matter (in the form of chiral superfields). This is all that is needed to construct the most general renormalisable supersymmetric lagrangian in four dimensions. In the tutorial problem we introduce the Kähler quotient in the simple context of the $\mathbb{C}P^N$ model.

In Lecture VI we will discuss the spontaneous breaking of supersymmetry. We will discuss the relation between spontaneous supersymmetry breaking and the vacuum energy and the vacuum expectation values of auxiliary fields. We discuss the O'Raifeartaigh model, Fayet–Iliopoulos terms and the Witten index. In the tutorial problem we discuss an example of Higgs mechanism in an SU(5) supersymmetric gauge theory.

Lastly, there are two appendices. Appendix A includes the basic mathematical definitions needed in the lectures. More importantly, it also includes our conventions. It should probably be skimmed first for notation and then revisited as needed. It is aimed to be self-contained. Appendix B is a "reference card" containing formulas which I have found very useful in calculations. I hope you do too.

Enjoy!

Notes for lecturers

The format of the School allocated six one-hour lectures to this topic. With this time constraint I was forced to streamline the presentation. This meant among other things that many of the Exercises were indeed left as exercises; although I tried to do enough to illustrate the different computational techniques.

The six lectures in the School did not actually correspond to the six lectures in the written version of the notes. (In fact, since the conventions must be introduced along the way, the written version really has seven lectures.) The first lecture was basically Lecture I, only that there was only enough time to do the kinetic term in detail. The second lecture did correspond to Lecture II with some additional highlights from Lecture III: the notion of supermultiplet, the balance between bosonic and fermionic degrees of freedom, and the positivity of the energy in a unitary representation. This allowed me to devote the third lecture to introducing superspace, roughly speaking the first three sections in Lecture IV, which was then completed in the fourth lecture. The fifth lecture covered the abelian part of Lecture V and all too briefly mentioned the extension to nonabelian gauge theories. The sixth and final lecture was devoted to Lecture VI.

It may seem strange to skip a detailed analysis of the representation theory of the Poincaré superalgebra, but this is in fact not strictly speaking necessary in the logical flow of the lectures, which are aimed at supersymmetric field theory model building. Of course, they are an essential part of the topic itself, and this is why they have been kept in the written version.

Acknowledgements

Supersymmetry is a vast subject and trying to do justice to it in just a few lectures is a daunting task and one that I am not sure I have accomplished to any degree. In contrast, I had the good fortune to learn supersymmetry from Peter van Nieuwenhuizen and at a much more leisurely pace. His semester-long course on supersymmetry and supergravity was memorable. At the time I appreciated the course from the perspective of the student. Now, from that of the lecturer, I appreciate the effort that went into it. I can think of no better opportunity than this one to thank him again for having given it.

The first version of these lectures were started while on a visit to CERN and finished, shortly before the School, while on a visit to the Spinoza Institute. I would like to extend my gratitude to both institutions, and Bernard de Wit in particular, for their hospitality.

I would like to thank Sonia Stanciu for finding several critical minus signs that I had misplaced; although I remain solely responsible for any which have yet to find their right place.

On a personal note, I would like to extend a warm "Mulţumesc frumos" to Veronica Rădulescu for providing a very pleasant atmosphere and for so gracefully accommodating my less-than-social schedule for two weeks during the writing of the first version of these lectures.

Last, but by no means least, I would also like to thank John Wheater and Jeff Forshaw for organising the Summer Schools, and all the other participants (students, tutors and lecturers alike) for their questions and comments, some of which have been incorporated in the updated version of these notes. Thank you all!

References

The amount of literature in this topic can be overwhelming to the beginner. Luckily there are not that many books to choose from and I have found the following references to be useful in the preparation of these notes:

- *Supersymmetry and Supergravity*
  J Wess and J Bagger
  (Princeton University Press, 1983) (Second Edition, 1992)
- *Supersymmetric gauge field theory and string theory*
  D Bailin and A Love
  (IoP Publishing, 1994)
- *Superspace*
  SJ Gates, Jr., MT Grisaru, M Roček and W Siegel
  (Benjamin/Cummings, 1983)
- *Fields*
  W Siegel
  (arXiv:hep-th/9912205)
- *Dynamical breaking of supersymmetry*
  E Witten
  Nucl.Phys. **B188** (1981) 513–554.

Of course, they use different conventions to the ones in these notes and thus care must be exercised when importing/exporting any formulas. As mentioned in the Introduction, my aim has been to make these notes self-contained, at least as far as calculations are concerned. Please let me know if I have not succeeded in this endeavour so that I can correct this in future versions. Similarly, I would appreciate any comments or suggestions, as well as pointers to errors of any kind: computational, conceptual, pedagogical,... (My email is written in the cover page of these notes.)

The latest version of these notes can always be found at the following URL:

`http://www.maths.ed.ac.uk/~jmf/BUSSTEPP.html`

I. The Wess–Zumino model

We start by introducing supersymmetry in the context of a simple four-dimensional field theory: the Wess–Zumino model. This is arguably the simplest supersymmetric field theory in four dimensions. We start by discussing the free massless Wess–Zumino model and then we make the model more interesting by adding masses and interactions.

I.1. **The free massless Wess–Zumino model.**

The field content of the Wess–Zumino model consists of a real scalar field S, a real pseudoscalar field P and a real (i.e., Majorana) spinor ψ. (See the Appendix for our conventions.) Of course, ψ is anticommuting. The (free, massless) lagrangian for these fields is:

$$\mathcal{L}_{\text{kin}} = -\tfrac{1}{2}\left(\partial S\right)^2 - \tfrac{1}{2}\left(\partial P\right)^2 - \tfrac{1}{2}\bar{\psi}\,\partial\!\!\!/\,\psi\ , \tag{1}$$

where $\partial\!\!\!/ = \partial_\mu \gamma^\mu$ and $\bar{\psi} = \psi^t C = \psi^\dagger i\gamma^0$. The signs have been chosen in order to make the hamiltonian positive-semidefinite in the chosen (mostly plus) metric. The action is defined as usual by

$$I_{\text{kin}} = \int d^4x\,\mathcal{L}_{\text{kin}} \tag{2}$$

To make the action have the proper dimension, the bosonic fields S and P must have dimension 1 and the fermionic field ψ must have dimension $\frac{3}{2}$, in units where ∂_μ has dimension 1.

STOP You may wonder why it is that P is taken to be a pseudoscalar, since the above action is clearly symmetric in S and P. The pseudoscalar nature of P will manifest itself shortly when we discuss supersymmetry, and at the end of the lecture when we introduce interactions: the Yukawa coupling between P and ψ will have a γ_5. Since changing the orientation changes the sign in γ_5, the action would not be invariant unless P also changed sign. This means that it is a pseudoscalar.

Exercise I.1. *Check that the action I_{kin} is real and that the equations of motion are*

$$\Box\, S = \Box\, P = \partial\!\!\!/\,\psi = 0\ , \tag{3}$$

where $\Box = \partial_\mu \partial^\mu$.

We now discuss the symmetries of the action I_{kin}. It will turn out that the action is left invariant by a "super" extension of the Poincaré algebra, so we briefly remind ourselves of the Poincaré invariance of the above action. The Poincaré algebra is the Lie algebra of the group of isometries of Minkowski space. As such it is isomorphic to the semidirect product of the algebra of Lorentz transformations and the algebra of translations. Let $\mathsf{M}_{\mu\nu} = -\mathsf{M}_{\nu\mu}$ be a basis for the (six-dimensional) Lorentz algebra and let P_μ be a basis for the (four-dimensional) translation algebra. The form of the algebra in this basis is recalled in (A-2) in the Appendix.

Let τ^μ and $\lambda^{\mu\nu} = -\lambda^{\nu\mu}$ be constant parameters. Then for any field $\varphi = S$, P or ψ we define infinitesimal Poincaré transformations by

$$\begin{aligned} \delta_\tau \varphi &= \tau^\mu \mathsf{P}_\mu \cdot \varphi \\ \delta_\lambda \varphi &= \tfrac{1}{2}\lambda^{\mu\nu} \mathsf{M}_{\mu\nu} \cdot \varphi \ , \end{aligned} \tag{4}$$

where

$$\begin{aligned} \mathsf{P}_\mu \cdot S &= -\partial_\mu S & \mathsf{M}_{\mu\nu} \cdot S &= -(x_\mu \partial_\nu - x_\nu \partial_\mu) S \\ \mathsf{P}_\mu \cdot P &= -\partial_\mu P & \mathsf{M}_{\mu\nu} \cdot P &= -(x_\mu \partial_\nu - x_\nu \partial_\mu) P \\ \mathsf{P}_\mu \cdot \psi &= -\partial_\mu \psi & \mathsf{M}_{\mu\nu} \cdot \psi &= -(x_\mu \partial_\nu - x_\nu \partial_\mu) \psi - \Sigma_{\mu\nu}\psi \ , \end{aligned} \tag{5}$$

and $\Sigma_{\mu\nu} = \frac{1}{2}\gamma_{\mu\nu}$.

The reason for the minus signs is that the action on functions is inverse to that on points. More precisely, let G be a group of transformations on a space X: every group element $g \in G$ sends a point $x \in X$ to another point $g \cdot x \in X$. Now suppose that $f : X \to \mathbb{R}$ is a function. How does the group act on it? The physically meaningful quantity is the value $f(x)$ that the function takes on a point; hence this is what should be invariant. In other words, the transformed function on the transformed point $(g \cdot f)(g \cdot x)$ should be the same as the original function on the original point $f(x)$. This means that $(g \cdot f)(x) = f(g^{-1} \cdot x)$ for all $x \in X$.
As an illustration, let's apply this to the translations on Minkowski space, sending x^μ to $x^\mu + \tau^\mu$. Suppose φ is a scalar field. Then the action of the translations is $\varphi \mapsto \varphi'$ where $\varphi'(x^\mu) = \varphi(x^\mu - \tau^\mu)$. For infinitesimal τ^μ we have $\varphi'(x^\mu) = \varphi(x^\mu) - \tau^\mu \partial_\mu \varphi(x^\mu)$, or equivalently

$$\tau^\mu \mathsf{P}_\mu \cdot \varphi = \varphi' - \varphi = -\tau^\mu \partial_\mu \varphi \ ,$$

which agrees with the above definition.

Exercise I.2. *Show that the above operators satisfy the Poincaré algebra* (A-2) *and show that*

$$\begin{aligned} \delta_\tau \mathcal{L}_{kin} &= \partial_\mu \left(-\tau^\mu \mathcal{L}_{kin}\right) \\ \delta_\lambda \mathcal{L}_{kin} &= \partial_\mu \left(\lambda^{\mu\nu} x_\nu \mathcal{L}_{kin}\right) \ . \end{aligned} \tag{6}$$

Conclude that the action I_{kin} is Poincaré invariant.

I should issue a word of warning when computing the algebra of operators such as P_μ and $\mathsf{M}_{\mu\nu}$. These operators are defined *only on fields*, where by "fields" we mean products of S, P and ψ. For instance, $\mathsf{P}_\mu \cdot (x^\nu S) = x^\nu \mathsf{P}_\mu \cdot S$: it does *not* act on the coordinate x^ν. Similarly, $\mathsf{M}_{\mu\nu} \cdot \partial_\rho S = \partial_\rho (\mathsf{M}_{\mu\nu} \cdot S)$, and of course the ∂_ρ *does* act on the coordinates which appear in $\mathsf{M}_{\mu\nu} \cdot S$.

I.2. **Invariance under supersymmetry.** More interestingly the action is invariant under the following "supersymmetry" transformations:

$$\begin{aligned} \delta_\varepsilon S &= \bar\varepsilon \psi \\ \delta_\varepsilon P &= \bar\varepsilon \gamma_5 \psi \\ \delta_\varepsilon \psi &= \partial\!\!\!/ (S + P\gamma_5)\varepsilon \ , \end{aligned} \tag{7}$$

where ε is a constant Majorana spinor. Notice that because transformations of any kind should not change the Bose–Fermi parity of a field, we are forced to take ε anticommuting, just like ψ. Notice also that for the above transformations to preserve the dimension of the fields, ε must have dimension $-\frac{1}{2}$. Finally notice that they preserve the reality properties of the fields.

Exercise I.3. *Show that under the above transformations the free lagrangian changes by a total derivative:*

$$\delta_\varepsilon \mathcal{L}_{kin} = \partial_\mu \left(-\tfrac{1}{2} \bar{\varepsilon} \gamma^\mu \partial\!\!\!/ \left(S + P\gamma_5 \right) \psi \right) \ , \tag{8}$$

and conclude that the action is invariant.

The supersymmetry transformations are generated by a spinorial *supercharge* Q of dimension $\frac{1}{2}$ such that for all fields φ,

$$\delta_\varepsilon \varphi = \bar{\varepsilon} \mathsf{Q} \cdot \varphi \ . \tag{9}$$

The action of Q on the bosonic fields is clear:

$$\mathsf{Q} \cdot S = \psi \qquad \text{and} \qquad \mathsf{Q} \cdot P = \gamma_5 \, \psi \ . \tag{10}$$

To work out the action of Q on ψ it is convenient to introduce indices.

First of all notice that $\bar{\varepsilon}\mathsf{Q} = \varepsilon^b C_{ba} \mathsf{Q}^a = \varepsilon_a \mathsf{Q}^a = -\varepsilon^a \mathsf{Q}_a$, whereas

$$\delta_\varepsilon \psi^a = \left((\gamma^\mu)^a{}_b \partial_\mu S + (\gamma^\mu \gamma_5)^a{}_b \partial_\mu P \right) \varepsilon^b \ . \tag{11}$$

Equating the two, and taking into account that $\psi_a = \psi^b C_{ba}$ and similarly for Q, one finds that

$$\mathsf{Q}_a \cdot \psi_b = - \left(\gamma^\mu \right)_{ab} \partial_\mu S + \left(\gamma^\mu \gamma_5 \right)_{ab} \partial_\mu P \ , \tag{12}$$

where we have lowered the indices of γ^μ and $\gamma^\mu\gamma_5$ with C and used respectively the symmetry and antisymmetry of the resulting forms.

I.3. **On-shell closure of the algebra.** We now check the closure of the algebra generated by P_μ, $\mathsf{M}_{\mu\nu}$ and Q_a. We have already seen that P_μ and $\mathsf{M}_{\mu\nu}$ obey the Poincaré algebra, so it remains to check the brackets involving Q_a. The supercharges Q_a are spinorial and hence transform nontrivially under Lorentz transformations. We therefore expect their bracket with the Lorentz generators $\mathsf{M}_{\mu\nu}$ to reflect this. Also the dimension of Q_a is $\frac{1}{2}$ and the dimension of the translation generators P_μ is 1, whence their bracket would have dimension $\frac{3}{2}$. Since there is no generator with the required dimension, we expect that their bracket should vanish. Indeed, we have the following.

Exercise I.4. *Show that*

$$\begin{aligned} [\mathsf{P}_\mu, \mathsf{Q}_a] \cdot \varphi &= 0 \\ [\mathsf{M}_{\mu\nu}, \mathsf{Q}_a] \cdot \varphi &= - \left(\Sigma_{\mu\nu} \right)_a{}^b \mathsf{Q}_b \cdot \varphi \ , \end{aligned} \tag{13}$$

where φ is any of the fields S, P or ψ.

We now compute the bracket of two supercharges. The first thing we notice is that, because Q_a anticommutes with the parameter ε, it is the *anticommutator* of the generators which appears in the commutator of transformations. More precisely,

$$
\begin{aligned}
[\delta_{\varepsilon_1}, \delta_{\varepsilon_2}] \cdot \varphi &= \left[-\varepsilon_1^a \mathsf{Q}_a, -\varepsilon_2^b \mathsf{Q}_b\right] \cdot \varphi \\
&= \varepsilon_1^a \mathsf{Q}_a \cdot \varepsilon_2^b \mathsf{Q}_b \cdot \varphi - \varepsilon_2^b \mathsf{Q}_b \cdot \varepsilon_1^a \mathsf{Q}_a \cdot \varphi \\
&= -\varepsilon_1^a \varepsilon_2^b \left(\mathsf{Q}_a \cdot \mathsf{Q}_b + \mathsf{Q}_b \cdot \mathsf{Q}_a\right) \cdot \varphi \\
&= -\varepsilon_1^a \varepsilon_2^b \left[\mathsf{Q}_a, \mathsf{Q}_b\right] \cdot \varphi \ ,
\end{aligned} \tag{14}
$$

where we use the *same* notation $[-,-]$ for the bracket of any two elements in a Lie superalgebra. On dimensional grounds, the bracket of two supercharges, having dimension 1, must be a translation. Indeed, one can show the following.

Exercise I.5. *Show that*

$$[\mathsf{Q}_a, \mathsf{Q}_b] \cdot S = 2 \left(\gamma^\mu\right)_{ab} \mathsf{P}_\mu \cdot S$$

and similarly for P, whereas for ψ one has instead

$$[\mathsf{Q}_a, \mathsf{Q}_b] \cdot \psi = 2 \left(\gamma^\mu\right)_{ab} \mathsf{P}_\mu \cdot \psi + (\gamma^\mu)_{ab} \gamma_\mu \partial\!\!\!/ \psi \ .$$

If we use the classical equations of motion for ψ, the second term in the right-hand side of the last equation vanishes and we obtain an *on-shell* realisation of the extension of the Poincaré algebra (A-2) defined by the following extra brackets:

$$
\begin{aligned}
[\mathsf{P}_\mu, \mathsf{Q}_a] &= 0 \\
[\mathsf{M}_{\mu\nu}, \mathsf{Q}_a] &= -\left(\Sigma_{\mu\nu}\right)_a{}^b \mathsf{Q}_b \\
[\mathsf{Q}_a, \mathsf{Q}_b] &= 2 \left(\gamma^\mu\right)_{ab} \mathsf{P}_\mu \ .
\end{aligned} \tag{15}
$$

These brackets together with (A-2) define the *(N=1) Poincaré superalgebra.*

The fact that the commutator of two supersymmetries is a translation has a remarkable consequence. In theories where supersymmetry is local, so that the spinor parameter is allowed to depend on the point, the commutator of two local supersymmetries is an infinitesimal translation whose parameter is allowed to depend on the point; in other words, it is an infinitesimal general coordinate transformation or, equivalently, an infinitesimal diffeomorphism. This means that theories with local supersymmetry automatically incorporate gravity. This is why such theories are called supergravity theories.

A (N=1) supersymmetric field theory is by definition any field theory which admits a realisation of the (N=1) Poincaré superalgebra on the space of fields (maybe on-shell and up to gauge equivalence) which leaves the action invariant. In particular this means that supersymmetry transformations take solutions to solutions.

It may seem disturbing to find that supersymmetry is only realised on-shell, since in computing perturbative quantum corrections, it is necessary to consider virtual particles running along in loops. This problem is of course well-known, e.g., in gauge theories where the BRST symmetry is only realised provided the antighost equation of motion is satisfied. The solution, in both cases, is the introduction of non-propagating auxiliary fields. We will see the need for this when we study the representation theory of the Poincaré superalgebra. In general finding a complete set of auxiliary fields is a hard (sometimes unsolvable) problem; but we will see that in the case of $N{=}1$ Poincaré supersymmetry, the superspace formalism to be introduced in Lecture IV will automatically solve this problem.

I.4. **Adding masses and interactions.** There are of course other supersymmetric actions that can be built out of the same fields S, P and ψ by adding extra terms to the free action I_{kin}. For example, we could add mass terms:

$$\mathcal{L}_{\text{m}} = -\tfrac{1}{2}m_1^2 S^2 - \tfrac{1}{2}m_2^2 P^2 - \tfrac{1}{2}m_3\bar{\psi}\psi \ , \tag{16}$$

where m_i for $i = 1, 2, 3$ have units of mass.

Exercise I.6. *Show that the action*

$$\int d^4x \ (\mathcal{L}_{kin} + \mathcal{L}_m) \tag{17}$$

is invariant under a modified set of supersymmetry transformations

$$\begin{aligned}\delta_\varepsilon S &= \bar{\varepsilon}\psi\\ \delta_\varepsilon P &= \bar{\varepsilon}\gamma_5\psi\\ \delta_\varepsilon\psi &= (\partial\!\!\!/ - m)(S + P\gamma_5)\varepsilon \ ,\end{aligned} \tag{18}$$

provided that $m_1 = m_2 = m_3 = m$. More concretely, show that with these choices of m_i,

$$\delta_\varepsilon\left(\mathcal{L}_{kin} + \mathcal{L}_m\right) = \partial_\mu X^\mu \ , \tag{19}$$

where

$$X^\mu = -\tfrac{1}{2}\bar{\varepsilon}\gamma^\mu\left(S - P\gamma_5\right)\left(\overleftarrow{\partial\!\!\!/} - m\right)\psi \ , \tag{20}$$

where for any ζ, $\zeta\overleftarrow{\partial\!\!\!/} = \partial_\mu\zeta\gamma^\mu$. Moreover show that the above supersymmetry transformations close, up to the equations of motion of the fermions, to realise the Poincaré superalgebra.

This result illustrates an important point: irreducible representations of the Poincaré superalgebra are mass degenerate; that is, all fields have the same mass. This actually follows easily from the Poincaré superalgebra itself. The (squared) mass is up to a sign the eigenvalue of the operator $\mathsf{P}^2 = \eta^{\mu\nu}\mathsf{P}_\mu\mathsf{P}_\nu$ which, from equations (A-2) and the first equation in (15), is seen to be a Casimir of the Poincaré superalgebra. Therefore on an irreducible representation P^2 must act as a multiple of the identity.

The action (17) is still free, hence physically not very interesting. It is possible to add interacting terms in such a way that Poincaré supersymmetry is preserved.

Indeed, consider the following interaction terms

$$\mathcal{L}_{\text{int}} = -\lambda \left(\bar{\psi} \left(S - P\gamma_5\right) \psi + \tfrac{1}{2}\lambda \left(S^2 + P^2\right)^2 + mS\left(S^2 + P^2\right) \right) \ . \tag{21}$$

The *Wess–Zumino model* is defined by the action

$$I_{\text{WZ}} = \int d^4x \left(\mathcal{L}_{\text{kin}} + \mathcal{L}_{\text{m}} + \mathcal{L}_{\text{int}}\right) \ . \tag{22}$$

Exercise I.7. *Prove that I_{WZ} is invariant under the following modified supersymmetry transformations:*

$$\begin{aligned} \delta_\varepsilon S &= \bar{\varepsilon}\psi \\ \delta_\varepsilon P &= \bar{\varepsilon}\gamma_5\psi \\ \delta_\varepsilon \psi &= \left[\partial\!\!\!/ - m - \lambda\left(S + P\gamma_5\right)\right]\left(S + P\gamma_5\right)\varepsilon \ , \end{aligned} \tag{23}$$

and verify that these transformations close on-shell to give a realisation of the Poincaré superalgebra. More concretely, show that

$$\delta_\varepsilon \mathcal{L}_{WZ} = \partial_\mu Y^\mu \ , \tag{24}$$

where

$$Y^\mu = -\tfrac{1}{2}\bar{\varepsilon}\gamma^\mu(S - P\gamma_5)\left(\overleftarrow{\partial\!\!\!/} - m - \lambda(S - P\gamma_5)\right)\psi \ . \tag{25}$$

For future reference, we notice that the supersymmetry transformations in (23) can be rewritten in terms of the generator Q_a as follows:

$$\begin{aligned} \mathsf{Q}_a \cdot S &= \psi_a \\ \mathsf{Q}_a \cdot P &= -(\gamma_5)_a{}^b\psi_b \\ \mathsf{Q}_a \cdot \psi_b &= -\partial_\mu S(\gamma^\mu)_{ab} + \partial_\mu P(\gamma^\mu\gamma_5)_{ab} - mSC_{ab} - mP(\gamma_5)_{ab} \\ &\quad - \lambda(S^2 - P^2)C_{ab} - 2\lambda SP(\gamma_5)_{ab} \ . \end{aligned} \tag{26}$$

Problem 1 (Superconformal invariance, Part I)**.**

In this problem you are invited to show that the massless Wess–Zumino model is classically invariant under a larger symmetry than the Poincaré superalgebra: the conformal superalgebra.

The conformal algebra of Minkowski space contains the Poincaré algebra as a subalgebra, and in addition it has five other generators: the dilation D and the special conformal transformations K_μ. The conformal algebra has the following (nonzero) brackets in addition to those in (A-2):

$$\begin{aligned} [\mathsf{P}_\mu, \mathsf{D}] &= \mathsf{P}_\mu \\ [\mathsf{K}_\mu, \mathsf{D}] &= -\mathsf{K}_\mu \\ [\mathsf{P}_\mu, \mathsf{K}_\nu] &= 2\eta_{\mu\nu}\mathsf{D} - 2\mathsf{M}_{\mu\nu} \\ [\mathsf{M}_{\mu\nu}, \mathsf{K}_\rho] &= \eta_{\nu\rho}\mathsf{K}_\mu - \eta_{\mu\rho}\mathsf{K}_\nu \ . \end{aligned} \tag{27}$$

Any supersymmetric field theory which is in addition conformal invariant will be invariant under the smallest superalgebra generated by these two Lie (super)algebras. This superalgebra is called the conformal superalgebra. We will see that the massless Wess–Zumino model is classically conformal invariant. This will then show that it is also classically superconformal invariant. In the course of the problem you will also discover the form of the conformal superalgebra.

(1) Prove that the following, together with (5), define a realisation of the conformal algebra on the fields in the Wess–Zumino model:

$$\begin{aligned}
\mathsf{D}\cdot S &= -x^\mu \partial_\mu S - S\\
\mathsf{D}\cdot P &= -x^\mu \partial_\mu P - P\\
\mathsf{D}\cdot \psi &= -x^\mu \partial_\mu \psi - \tfrac{3}{2}\psi\\
\mathsf{K}_\mu\cdot S &= -2x_\mu x^\nu \partial_\nu S + x^2\partial_\mu S - 2x_\mu S\\
\mathsf{K}_\mu\cdot P &= -2x_\mu x^\nu \partial_\nu P + x^2\partial_\mu P - 2x_\mu P\\
\mathsf{K}_\mu\cdot \psi &= -2x_\mu x^\nu \partial_\nu \psi + x^2\partial_\mu \psi - 3x_\mu \psi + x^\nu\gamma_{\nu\mu}\psi
\end{aligned}$$

(2) Prove that the massless Wess–Zumino action with lagrangian

$$\begin{aligned}
\mathcal{L}_{\mathrm{mWZ}} = -\tfrac{1}{2}\left(\partial S\right)^2 - \tfrac{1}{2}\left(\partial P\right)^2 - \tfrac{1}{2}\bar\psi\,\partial\!\!\!/\,\psi\\
- \lambda\bar\psi\left(S - P\gamma_5\right)\psi - \tfrac{1}{2}\lambda^2\left(S^2+P^2\right)^2 \quad (28)
\end{aligned}$$

is conformal invariant. More precisely, show that

$$\mathsf{D}\cdot\mathcal{L}_{\mathrm{mWZ}} = \partial_\mu(-x^\mu\mathcal{L}_{\mathrm{mWZ}})$$

and that

$$\mathsf{K}_\mu\cdot\mathcal{L}_{\mathrm{mWZ}} = \partial_\nu\left[\left(-2x_\mu x^\nu + x^2\delta_\mu^\nu\right)\mathcal{L}_{\mathrm{mWZ}}\right]\ ,$$

and conclude that the action is invariant.

☞ It is actually enough to prove that the action is invariant under K_μ and P_μ, since as can be easily seen from the explicit form of the algebra, these two sets of elements generate the whole conformal algebra.

We now know that the massless Wess–Zumino model is invariant both under supersymmetry and under conformal transformations. It follows that it is also invariant under any transformation obtained by taking commutators of these and the resulting transformations until the algebra closes (at least on-shell). We will now show that this process results in an on-shell realisation of the conformal superalgebra. In addition to the conformal and superPoincaré generators, the conformal superalgebra has also a second spinorial generator S_a, generating conformal supersymmetries and a further bosonic generator R generating the so-called R-symmetry to be defined below.

Let κ_μ be a constant vector and let δ_κ denote an infinitesimal special conformal transformation, defined on fields φ by $\delta_\kappa\varphi = \kappa^\mu\mathsf{K}_\mu\cdot\varphi$. The commutator of an infinitesimal supersymmetry and an infinitesimal special conformal transformation is, by definition, a conformal supersymmetry. These are generated by a spinorial generator S_a defined by

$$[\mathsf{K}_\mu, \mathsf{Q}_a] = +(\gamma_\mu)_a{}^b \mathsf{S}_b \ .$$

Let ζ be an anticommuting Majorana spinor and define an infinitesimal conformal supersymmetry δ_ζ as $\delta_\zeta \varphi = \bar\zeta \mathsf{S} \cdot \varphi$.

3. Show that the infinitesimal conformal supersymmetries take the following form:

$$\begin{aligned}
\delta_\zeta S &= \bar\zeta x^\mu \gamma_\mu \psi \\
\delta_\zeta P &= \bar\zeta x^\mu \gamma_\mu \gamma_5 \psi \\
\delta_\zeta \psi &= -\left[\left(\not\partial - \lambda(S + P\gamma_5)\right)(S + P\gamma_5)\right] x^\mu \gamma_\mu \zeta - 2(S - P\gamma_5)\zeta \ .
\end{aligned} \tag{29}$$

4. Show that the action of S_a on fields is given by:

$$\begin{aligned}
\mathsf{S}_a \cdot S &= x^\mu (\gamma_\mu)_{ab}\, \psi^b \\
\mathsf{S}_a \cdot P &= x^\mu (\gamma_\mu \gamma_5)_{ab}\, \psi^b \\
\mathsf{S}_a \cdot \psi_b &= -(x^\mu \partial_\mu + 2) S C_{ab} + (x^\mu \partial_\mu + 2) P (\gamma_5)_{ab} \\
&\quad - (x_\mu \partial_\nu S + \tfrac{1}{2} \epsilon_{\mu\nu}{}^{\rho\sigma} x_\rho \partial_\sigma P)(\gamma^{\mu\nu})_{ab} \\
&\quad - \lambda(S^2 - P^2) x^\mu (\gamma_\mu)_{ab} - 2\lambda S P x^\mu (\gamma_\mu \gamma_5)_{ab} \ .
\end{aligned} \tag{30}$$

5. Show that the action of P_μ, $\mathsf{M}_{\mu\nu}$, K_μ, D, R, Q_a and S_a on the fields S, P and ψ defines an on-shell realisation of the *conformal superalgebra* defined by the following (nonzero) brackets in addition to those in (A-2), (15) and (27):

$$\begin{aligned}
[\mathsf{M}_{\mu\nu}, \mathsf{S}_a] &= -\left(\Sigma_{\mu\nu}\right)_a{}^b \mathsf{S}_b \\
[\mathsf{R}, \mathsf{Q}_a] &= +\tfrac{1}{2} (\gamma_5)_a{}^b \mathsf{Q}_b \\
[\mathsf{R}, \mathsf{S}_a] &= -\tfrac{1}{2} (\gamma_5)_a{}^b \mathsf{S}_b \\
[\mathsf{D}, \mathsf{Q}_a] &= -\tfrac{1}{2} \mathsf{Q}_a \\
[\mathsf{D}, \mathsf{S}_a] &= +\tfrac{1}{2} \mathsf{S}_a \\
[\mathsf{K}_\mu, \mathsf{Q}_a] &= +(\gamma_\mu)_a{}^b \mathsf{S}_b \\
[\mathsf{P}_\mu, \mathsf{S}_a] &= -(\gamma_\mu)_a{}^b \mathsf{Q}_b \\
[\mathsf{S}_a, \mathsf{S}_b] &= -2(\gamma^\mu)_{ab} \mathsf{K}_\mu \\
[\mathsf{Q}_a, \mathsf{S}_b] &= +2 C_{ab} \mathsf{D} - 2(\gamma_5)_{ab} \mathsf{R} + (\gamma^{\mu\nu})_{ab} \mathsf{M}_{\mu\nu} \ ,
\end{aligned} \tag{31}$$

where the action of the R-symmetry on fields is

$$\begin{aligned}
\mathsf{R} \cdot S &= P \\
\mathsf{R} \cdot P &= -S \\
\mathsf{R} \cdot \psi &= \tfrac{1}{2} \gamma_5\, \psi \ .
\end{aligned} \tag{32}$$

☞ This shows that the massless Wess–Zumino model is classically superconformal invariant. However several facts should alert us that this symmetry will be broken by quantum effects. First of all the R-symmetry acts via γ_5 and this sort of symmetries are usually quantum-mechanically anomalous. Similarly, we expect that the trace and conformal anomalies should break invariance under D and K_μ respectively. This is in fact the case. What is remarkable is that the Wess–Zumino model (with or without mass) is actually quantum mechanically supersymmetric to all orders in perturbation theory. Moreover the model only requires wave function renormalisation: the mass m and the coupling constant λ do not renormalise. This sort of *nonrenormalisation theorems* are quite common in supersymmetric theories. We will be able to explain why this is the case in a later lecture, although we will not have the time to develop the necessary formalism to prove it.

II. Supersymmetric Yang–Mills theory

In this section we introduce another simple model exhibiting supersymmetry: supersymmetric Yang–Mills. This model consists of ordinary Yang–Mills theory coupled to adjoint fermions. We will see that this model admits an on-shell realisation of the Poincaré superalgebra which however only closes up to gauge transformations. More is true and in the tutorial you will show that the theory is actually superconformal invariant, just like the massless Wess–Zumino model.

II.1. **Supersymmetric Yang–Mills.** The existence of a supersymmetric extension of Yang–Mills theory could be suspected from the study of the representations of the Poincaré superalgebra (see Lecture III), but this does not mean that it is an obvious fact. Indeed, the existence of supersymmetric Yang–Mills theories depends on the dimensionality and the signature of spacetime. Of course one can always write down the Yang–Mills action in any dimension and then couple it to fermions, but as we will see in the next lecture, supersymmetry requires a delicate balance between the bosonic and fermionic degrees of freedom. Let us consider only lorentzian spacetimes. A gauge field in d dimensions has $d-2$ physical degrees of freedom corresponding to the transverse polarisations. The number of degrees of freedom of a fermion field depends on what kind fermion it is, but it always a power of 2. An unconstrained Dirac spinor in d dimensions has $2^{d/2}$ or $2^{(d-1)/2}$ real degrees of freedom, for d even or odd respectively: a Dirac spinor has $2^{d/2}$ or $2^{(d-1)/2}$ complex components but the Dirac equation cuts this number in half. In even dimensions, one can further restrict the spinor by imposing that it be chiral (Weyl). This cuts the number of degrees of freedom by two. Alternatively, in some dimensions (depending on the signature of the metric) one can impose a reality (Majorana) condition which also halves the number of degrees of freedom. For a lorentzian metric of signature $(1, d-1)$, Majorana spinors exist for $d \equiv 1, 2, 3, 4 \mod 8$. When $d \equiv 2 \mod 8$ one can in fact impose that a spinor be both Majorana and Weyl, cutting the number of degrees of freedom in four. The next exercise asks you to determine in which dimensions can supersymmetric Yang–Mills theory exist based on the balance between bosonic and fermionic degrees of freedom.

Exercise II.1. *Verify via a counting of degrees of freedom that (N=1) supersymmetric Yang–Mills can exist only in the following dimensions and with the following types of spinors:*

d	*Spinor*
3	*Majorana*
4	*Majorana (or Weyl)*
6	*Weyl*
10	*Majorana–Weyl*

The fact that these dimensions are of the form $2+n$, where $n = 1, 2, 4, 8$ are the dimensions of the real division algebras is not coincidental. It is a further curious fact that these are precisely the dimensions in which the classical superstring exists. Unlike superstring theory, in which only the ten-dimensional theory survives quantisation, it turns out that supersymmetric Yang–Mills theory exists in each of these dimensions. Although we are mostly concerned with four-dimensional field theories in these notes, the six-dimensional and ten-dimensional theories are useful tools since upon dimensional reduction to four dimensions they yield N=2 and N=4 supersymmetric Yang–Mills, respectively.

We will now write down a supersymmetric Yang–Mills theory in four dimensions. We will show that the action is invariant under a supersymmetry algebra which closes on-shell and up to gauge transformations to a realisation of the Poincaré superalgebra.

II.2. **A brief review of Yang–Mills theory.** Let us start by reviewing Yang–Mills theory. We pick a gauge group G which we take to be a compact Lie group. We let $\mathfrak{g}$ denote its Lie algebra. We must also make the choice of an invariant inner product in the Lie algebra, which we will call Tr. Fix a basis $\{T_i\}$ for $\mathfrak{g}$ and let $G_{ij} = \operatorname{Tr} T_i T_j$ be the invariant inner product and ${f_{ij}}^k$ be the structure constants.

The gauge field is a one-form in Minkowski space with values in $\mathfrak{g}$: $A_\mu = A_\mu^i T_i$. Geometrically it represents a connection in a principal G bundle on Minkowski space. The field-strength $F_{\mu\nu} = F_{\mu\nu}^i T_i$ is the curvature two-form of that connection, and it is defined as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + g[A_\mu, A_\nu] ,$$

or relative to the basis $\{T_i\}$:

$$F_{\mu\nu}^i = \partial_\mu A_\nu^i - \partial_\nu A_\mu^i + g {f_{jk}}^i A_\mu^j A_\nu^k ,$$

where g is the Yang–Mills coupling constant. The lagrangian is then given by

$$\mathcal{L}_{\text{YM}} = -\tfrac{1}{4} \operatorname{Tr} F_{\mu\nu} F^{\mu\nu} ,$$

and the action is as usual the integral

$$I_{\text{YM}} = -\tfrac{1}{4} \int d^4x \, \operatorname{Tr} F_{\mu\nu} F^{\mu\nu} .$$

The sign has been chosen so that with our choice of spacetime metric, the hamiltonian is positive-semidefinite.

Exercise II.2. *Show that the action is invariant under the natural action of the Poincaré algebra:*

$$\begin{aligned} \mathsf{P}_\mu \cdot A_\nu &= -\partial_\mu A_\nu \\ \mathsf{M}_{\mu\nu} \cdot A_\rho &= -(x_\mu \partial_\nu - x_\nu \partial_\mu) A_\rho - \eta_{\nu\rho} A_\mu + \eta_{\mu\rho} A_\nu \ . \end{aligned} \tag{33}$$

The action is also invariant under gauge transformations. Let $U(x)$ be a G-valued function on Minkowski space. The gauge field A_μ transforms in such a way that the covariant derivative $\mathcal{D}_\mu = \partial_\mu + gA_\mu$ transforms covariantly:

$$\mathcal{D}_\mu^U = \partial_\mu + gA_\mu^U = U\mathcal{D}_\mu U^{-1} = U\left(\partial_\mu + gA_\mu\right)U^{-1} \ ,$$

whence the transformed gauge field is

$$A_\mu^U = UA_\mu U^{-1} - \tfrac{1}{g}(\partial_\mu U)U^{-1} \ .$$

The field-strength transforms covariantly

$$F_{\mu\nu}^U = UF_{\mu\nu}U^{-1} \ ,$$

which together with the invariance of the inner product (or equivalently, cyclicity of the trace) implies that the Lagrangian is invariant.

Suppose that $U(x) = \exp\omega(x)$ where $\omega(x) = \omega(x)^i T_i$ is a $\mathfrak{g}$-valued function. Keeping only terms linear in ω in the gauge transformation of the gauge field, we arrive at the infinitesimal gauge transformations:

$$\delta_\omega A_\mu = -\tfrac{1}{g}\mathcal{D}_\mu \omega \implies \delta_\omega F_{\mu\nu} = [\omega, F_{\mu\nu}] \ , \tag{34}$$

which is easily verified to be an invariance of the Yang–Mills lagrangian.

II.3. **Supersymmetric extension.** We will now find a supersymmetric extension of this action. Because supersymmetry exchanges bosons and fermions, we will add some fermionic fields. Since the bosons A_μ are $\mathfrak{g}$-valued, supersymmetry will require that so be the fermions. Therefore we will consider an adjoint Majorana spinor $\Psi = \Psi^i T_i$. The natural gauge invariant interaction between the spinors and the gauge fields is the minimally coupled lagrangian

$$-\tfrac{1}{2}\operatorname{Tr}\bar{\Psi}\not{\mathcal{D}}\Psi \ , \tag{35}$$

where $\bar{\Psi} = \Psi^t C$, $\not{\mathcal{D}} = \gamma^\mu \mathcal{D}_\mu$ and

$$\mathcal{D}_\mu \Psi = \partial_\mu \Psi + g[A_\mu, \Psi] \implies \mathcal{D}_\mu \Psi^i = \partial_\mu \Psi^i + g f_{jk}{}^i A_\mu^j \Psi^k \ .$$

Exercise II.3. *Prove that the minimal coupling interaction* (35) *is invariant under the infinitesimal gauge transformations* (34) *and*

$$\delta_\omega \Psi = [\omega, \Psi] \ . \tag{36}$$

The action

$$I_{\mathrm{SYM}} = \int d^4x\, \mathcal{L}_{\mathrm{SYM}} \ , \tag{37}$$

with

$$\mathcal{L}_{\mathrm{SYM}} = -\tfrac{1}{4}\operatorname{Tr} F_{\mu\nu}F^{\mu\nu} - \tfrac{1}{2}\operatorname{Tr}\bar{\Psi}\not{\mathcal{D}}\Psi \tag{38}$$

is therefore both Poincaré and gauge invariant. One can also verify that it is real. In addition, as we will now show, it is also invariant under supersymmetry.

Taking into account dimensional considerations, Bose–Fermi parity, and equivariance under the gauge group (namely that gauge transformations should commute with supersymmetry) we arrive at the following supersymmetry transformations rules:

$$\begin{aligned}\delta_\varepsilon A_\mu = \bar{\varepsilon}\gamma_\mu\Psi &\implies \mathsf{Q}_a \cdot A_\mu = -(\gamma_\mu)_a{}^b\Psi_b\\ \delta_\varepsilon\Psi = \tfrac{1}{2}\alpha F_{\mu\nu}\gamma^{\mu\nu}\varepsilon &\implies \mathsf{Q}_a\cdot\Psi_b = -\tfrac{1}{2}\alpha F_{\mu\nu}(\gamma^{\mu\nu})_{ab}\ ,\end{aligned}$$

where α is a parameter to be determined and ε in again an anticommuting Majorana spinor.

☞ The condition on gauge equivariance is essentially the condition that we should only have rigid supersymmetry. Suppose that supersymmetries and gauge transformations would not commute. Their commutator would be another type of supersymmetry (exchanging bosons and fermions) but the parameter of the transformation would be local, since gauge transformations have local parameters. This would imply the existence of a local supersymmetry. Since we are only considering rigid supersymmetries, we must have that supersymmetry transformations and gauge transformations commute.

✍ **Exercise II.4.** *Prove that the above "supersymmetries" commute with infinitesimal gauge transformations:*

$$[\delta_\epsilon, \delta_\omega]\varphi = 0\ ,$$

on any field $\varphi = A_\mu, \Psi$.

Now let us vary the lagrangian $\mathcal{L}_{\mathrm{SYM}}$. This task is made a little easier after noticing that for any variation δA_μ of the gauge field—including, of course, supersymmetries—the field strength varies according to

$$\delta F_{\mu\nu} = \mathcal{D}_\mu\delta A_\nu - \mathcal{D}_\nu\delta A_\mu\ .$$

Varying the lagrangian we notice that there are two types of terms in $\delta_\varepsilon\mathcal{L}_{\mathrm{SYM}}$: terms linear in Ψ and terms cubic in Ψ. Invariance of the action demands that they should vanish separately.

It is easy to show that the terms linear in Ψ cancel up a total derivative provided that $\alpha = -1$. This result uses equation (A-6) and the Bianchi identity

$$\mathcal{D}_{[\mu}F_{\nu\rho]} = 0\ .$$

On the other hand, the cubic terms vanish on their own using the Fierz identity (A-10) and the identities (A-7).

✍ **Exercise II.5.** *Prove the above claims; that is, prove that under the supersymmetry transformations:*

$$\begin{aligned} \delta_\varepsilon A_\mu &= \bar{\varepsilon}\gamma_\mu \Psi \\ \delta_\varepsilon \Psi &= -\tfrac{1}{2} F_{\mu\nu}\gamma^{\mu\nu}\varepsilon \end{aligned} \tag{39}$$

the lagrangian $\mathcal{L}_{SYM}$ transforms into a total derivative

$$\delta_\varepsilon \mathcal{L}_{SYM} = \partial_\mu \left(-\tfrac{1}{4}\bar{\varepsilon}\gamma^\mu \gamma^{\rho\sigma} F_{\rho\sigma}\Psi \right) ,$$

and conclude that the action I_{SYM} is invariant.

II.4. **Closure of the supersymmetry algebra.** We have called the above transformations "supersymmetries" but we have still to show that they correspond to a realisation of the Poincaré superalgebra (15). We saw in the Wess–Zumino model that the algebra only closed up to the equations of motion of the fermions. In this case we will also have to allow for gauge transformations. The reason is the following: although supersymmetries commute with gauge transformations, it is easy to see that translations do not. Therefore the commutator of two supersymmetries could not simply yield a translation. Instead, and provided the equations of motion are satisfied, it yields a translation and an infinitesimal gauge transformation.

✍ **Exercise II.6.** *Prove that*

$$[\mathsf{Q}_a, \mathsf{Q}_b] \cdot A_\mu = 2(\gamma^\rho)_{ab}\mathsf{P}_\rho \cdot A_\mu + 2(\gamma^\rho)_{ab}\mathcal{D}_\mu A_\rho \ .$$

Notice that the second term in the above equation has the form of an infinitesimal gauge transformation with (field-dependent) parameter $-2g\gamma^\rho A_\rho$, whereas the first term agrees with the Poincaré superalgebra (15).

✍ **Exercise II.7.** *Prove that up to terms involving the equation of motion of the fermion ($\not{\mathcal{D}}\Psi = 0$),*

$$[\mathsf{Q}_a, \mathsf{Q}_b] \cdot \Psi = 2(\gamma^\rho)_{ab}\mathsf{P}_\rho \cdot \Psi - 2g(\gamma^\rho)_{ab}[A_\rho, \Psi] \ .$$

Again notice that the second term has the form of an infinitesimal gauge transformation with the *same* parameter $-2g\gamma^\rho A_\rho$, whereas again the first term agrees with the Poincaré superalgebra (15).

The fact that the gauge transformation is the same one in both cases allows us to conclude that the Poincaré superalgebra is realised on-shell and up to gauge transformations on the fields A_μ and Ψ.

☞ There is a geometric picture which serves to understand the above result. One can understand infinitesimal symmetries as vector fields on the (infinite-dimensional) space of fields $\mathcal{F}$. Each point in this space corresponds to a particular field configuration (A_μ, Ψ). An infinitesimal symmetry $(\delta A_\mu, \delta\Psi)$ is a particular kind of vector field on $\mathcal{F}$; in other words, the assignment of a small displacement (a tangent vector field) to every field configuration.

Now let $\mathcal{F}_0 \subset \mathcal{F}$ be the subspace corresponding to those field configurations which obey the classical equations of motion. A symmetry of the action preserves the equations of motion, and hence sends solutions to solutions. Therefore symmetries preserve $\mathcal{F}_0$ and infinitesimal symmetries are vector fields which are tangent to $\mathcal{F}_0$. The group $\mathcal{G}$ of gauge transformations, since it acts by symmetries, preserves the subspace $\mathcal{F}_0$ and in fact foliates it into gauge orbits: two configurations being in the same orbit if there is a gauge transformation that relates them. Unlike other symmetries, gauge-related configurations are physically indistinguishable. Therefore the space of physical configurations is the space $\mathcal{F}_0/\mathcal{G}$ of gauge orbits.

Now, any vector field on $\mathcal{F}_0$ defines a vector field on $\mathcal{F}_0/\mathcal{G}$: one simply throws away the components tangent to the gauge orbits. The result we found above can be restated as saying that in the space of physical configurations we have a realisation of the Poincaré superalgebra.

We have proven that the theory defined by the lagrangian (38) is a supersymmetric field theory. It is called *(N=1) (pure) supersymmetric Yang–Mills*. This is the simplest four-dimensional supersymmetric gauge theory, but by no means the only one. One can add matter coupling in the form of Wess–Zumino multiplets. Some of these theories have extended supersymmetry (in the sense of Problem 3). Extended supersymmetry constrains the dynamics of the gauge theory. In the last five years there has been much progress made on gauge theories with extended supersymmetry, including for the first time the exact (nonperturbative) solution of nontrivial interacting four-dimensional quantum field theories.

Problem 2 (SUPERCONFORMAL INVARIANCE, PART II).

This problem does for supersymmetric Yang–Mills what Problem 1 did for the Wess–Zumino model: namely, it invites you to show that supersymmetric Yang–Mills is classically invariant under the conformal superalgebra. As with the Wess–Zumino model the strategy will be to show that the theory is conformal invariant and hence that it is invariant under the smallest superalgebra generated by the Poincaré supersymmetry and the conformal transformations. This superalgebra will be shown to be (on-shell and up to gauge transformations) the conformal superalgebra introduced in Problem 1.

(1) Show that supersymmetric Yang–Mills theory described by the action I_{SYM} with lagrangian (38) is invariant under the conformal transformations:

$$\begin{aligned} \mathsf{D} \cdot A_\rho &= -x^\mu \partial_\mu A_\rho - A_\rho \\ \mathsf{D} \cdot \Psi &= -x^\mu \partial_\mu \Psi - \tfrac{3}{2}\Psi \\ \mathsf{K}_\mu \cdot A_\rho &= -2x_\mu x^\nu \partial_\nu A_\rho + x^2 \partial_\mu A_\rho - 2x_\mu A_\rho + 2x_\rho A_\mu - 2\eta_{\mu\rho} x^\nu A_\nu \\ \mathsf{K}_\mu \cdot \Psi &= -2x_\mu x^\nu \partial_\nu \Psi + x^2 \partial_\mu \Psi - 3x_\mu \Psi + x^\nu \gamma_{\nu\mu} \Psi \ . \end{aligned}$$

More precisely show that

$$\mathsf{D} \cdot \mathcal{L}_{\text{SYM}} = \partial_\mu \left(-x^\mu \mathcal{L}_{\text{SYM}}\right)$$
$$\mathsf{K}_\mu \cdot \mathcal{L}_{\text{SYM}} = \partial_\nu \left[\left(-2x_\mu x^\nu + x^2 \delta_\mu^\nu\right) \mathcal{L}_{\text{SYM}}\right] ,$$

and conclude that the action I_{SYM} is invariant.

(2) Show that I_{SYM} is invariant under the R-symmetry:

$$\mathsf{R} \cdot A_\mu = 0 \qquad \text{and} \qquad \mathsf{R} \cdot \Psi = \tfrac{1}{2}\gamma_5 \Psi .$$

(3) Referring to the discussion preceding Part 3 in Problem 1, show that the infinitesimal conformal supersymmetry of supersymmetric Yang–Mills takes the form:

$$\delta_\zeta A_\mu = \bar{\zeta} x^\nu \gamma_\nu \gamma_\mu \Psi$$
$$\delta_\zeta \Psi = \tfrac{1}{2} x^\rho F_{\mu\nu} \gamma^{\mu\nu} \gamma^\rho \zeta .$$

(4) Defining the generator S_a by

$$\delta_\zeta \varphi = \bar{\zeta} \mathsf{S} \cdot \varphi = -\zeta^a \mathsf{S}_a \cdot \varphi$$

show that the action of S_a is given by

$$\mathsf{S}_a \cdot A_\mu = \left(x^\nu \gamma_\nu \gamma_\mu\right)_{ab} \Psi^b$$
$$\mathsf{S}_a \cdot \Psi = -\tfrac{1}{2} x_\rho F_{\mu\nu} \left(\gamma^{\mu\nu}\gamma^\rho\right)_{ba}$$
$$= \tfrac{1}{2} \epsilon^{\mu\nu\rho\sigma} x_\rho F_{\mu\nu} \left(\gamma_\sigma \gamma_5\right)_{ab} + x^\mu F_{\mu\nu} \left(\gamma^\nu\right)_{ab} .$$

(5) Finally show that S_a, together with $\mathsf{M}_{\mu\nu}$, P_μ, K_μ, D, R and Q_a, define an on-shell (and up to gauge transformations) realisation of the conformal superalgebra defined by the brackets (A-2), (15), (27) and (31).

☞ Again we expect that the classical superconformal symmetry of supersymmetric Yang–Mills will be broken by quantum effects: again the R-symmetry acts by chiral transformations which are anomalous, and as this theory has a nonzero beta function, conformal invariance will also fail at the quantum level. Nevertheless Poincaré supersymmetry will be preserved at all orders in perturbation theory.

Remarkably one can couple supersymmetric Yang–Mills to supersymmetric matter in such a way that the resulting quantum theory is still superconformal invariant. One such theory is the so-called N=4 supersymmetric Yang–Mills. This theory has vanishing beta function and is in fact superconformally invariant to all orders. It is not a realistic quantum field theory for phenomenological purposes, but it has many nice properties: it is maximally supersymmetric (having 16 supercharges), it exhibits electromagnetic (Montonen–Olive) duality and it has been conjectured (Maldacena) to be equivalent at strong coupling to type IIB string theory on a ten-dimensional background of the form $\text{adS}_5 \times S^5$, where S^5 is the round 5-sphere and adS_5, five-dimensional anti-de Sitter space, is the lorentzian analogue of hyperbolic space in that dimension.

III. Representations of the Poincaré superalgebra

In the first two lectures we met the Poincaré superalgebra and showed that it is a symmetry of the Wess–Zumino model (in Lecture I) and of Yang–Mills theories with adjoint fermions (in Lecture II). In the present lecture we will study the representations of this algebra. We will see that irreducible representations of the Poincaré superalgebra consist of multiplets of irreducible representations of the Poincaré algebra containing fields of different spins (or helicities) but of the same mass. This degeneracy in the mass is not seen in nature and hence supersymmetry, if a symmetry of nature at all, must be broken. In Lecture VI we will discuss spontaneous supersymmetry breaking.

III.1. **Unitary representations.** It will prove convenient both in this lecture and in later ones, to rewrite the Poincaré superalgebra in terms of two-component spinors. (See the Appendix for our conventions.) The supercharge Q^a, being a Majorana spinor decomposes into two Weyl spinors

$$\mathsf{Q}^a = \begin{pmatrix} \mathsf{Q}^\alpha \\ \bar{\mathsf{Q}}_{\dot\alpha} \end{pmatrix} \ , \tag{40}$$

in terms of which, the nonzero brackets in (15) now become

$$\begin{aligned} [\mathsf{M}_{\mu\nu}, \mathsf{Q}_\alpha] &= -\tfrac{1}{2} \left(\sigma_{\mu\nu}\right)_\alpha{}^\beta \mathsf{Q}_\beta \\ \left[\mathsf{M}_{\mu\nu}, \bar{\mathsf{Q}}_{\dot\alpha}\right] &= \tfrac{1}{2} \left(\bar\sigma_{\mu\nu}\right)_{\dot\alpha}{}^{\dot\beta} \bar{\mathsf{Q}}_{\dot\beta} \\ \left[\mathsf{Q}_\alpha, \bar{\mathsf{Q}}_{\dot\beta}\right] &= 2i \left(\sigma^\mu\right)_{\alpha\dot\beta} \mathsf{P}_\mu \ . \end{aligned} \tag{41}$$

For the purposes of this lecture we will be interested in unitary representations of the Poincaré superalgebra. This means that representations will have a positive-definite invariant hermitian inner product and the generators of the algebra will obey the following hermiticity conditions:

$$\mathsf{M}_{\mu\nu}^\dagger = -\mathsf{M}_{\mu\nu} \qquad \mathsf{P}_\mu^\dagger = -\mathsf{P}_\mu \qquad \mathsf{Q}_\alpha^\dagger = \bar{\mathsf{Q}}_{\dot\alpha} \ . \tag{42}$$

✍ **Exercise III.1.** *Show that these hermiticity conditions are consistent with the Poincaré superalgebra.*

Notice that P_μ is antihermitian, hence its eigenvalues will be imaginary. Indeed, we have seen that P_μ acts like $-\partial_\mu$ on fields. For example, acting on a plane wave $\varphi = \exp(ip \cdot x)$, $\mathsf{P}_\mu \cdot \varphi = -ip_\mu \varphi$. Therefore on a momentum eigenstate $|p\rangle$, the eigenvalue of P_μ is $-ip_\mu$.

A remarkable property of supersymmetric theories is that the energy is positive-semidefinite in a unitary representation. Indeed, acting on a momentum eigenstate $|p\rangle$ the supersymmetry algebra becomes

$$\left[\mathsf{Q}_\alpha, \bar{\mathsf{Q}}_{\dot\beta}\right] |p\rangle = 2 \begin{pmatrix} -p_0 + p_3 & p_1 - ip_2 \\ p_1 + ip_2 & -p_0 - p_3 \end{pmatrix} |p\rangle \ .$$

Recalling that the energy is given by $p^0 = -p_0$, we obtain

$$p^0|p\rangle = \tfrac{1}{4}\left([\mathsf{Q}_1, \mathsf{Q}_1^\dagger] + [\mathsf{Q}_2, \mathsf{Q}_2^\dagger]\right)|p\rangle \ .$$

In other words, the hamiltonian can be written in the following manifestly positive-semidefinite way:

$$H = \tfrac{1}{4}\left(\mathsf{Q}_1\mathsf{Q}_1^\dagger + \mathsf{Q}_1^\dagger\mathsf{Q}_1 + \mathsf{Q}_2\mathsf{Q}_2^\dagger + \mathsf{Q}_2^\dagger\mathsf{Q}_2\right) \ . \tag{43}$$

This shows that energy of any state is positive unless the state is annihilated by all the supercharges, in which case it is zero. Indeed, if $|\psi\rangle$ is *any* state, we have that the expectation value of the hamiltonian (the energy) is given by a sum of squares:

$$\langle\psi|H|\psi\rangle = \tfrac{1}{4}\left(\|\mathsf{Q}_1|\psi\rangle\|^2 + \|\mathsf{Q}_1^\dagger|\psi\rangle\|^2 + \|\mathsf{Q}_2|\psi\rangle\|^2 + \|\mathsf{Q}_2^\dagger|\psi\rangle\|^2\right) \ .$$

This is a very important fact of supersymmetry and one which plays a crucial role in many applications, particularly in discussing the spontaneous breaking of supersymmetry.

III.2. **Induced representations in a nutshell.** The construction of unitary representations of the Poincaré superalgebra can be thought of as a mild extension of the construction of unitary representations of the Poincaré algebra. This method is originally due to Wigner and was greatly generalised by Mackey. The method consists of inducing the representation from a finite-dimensional unitary representation of some compact subgroup. Let us review this briefly as it will be the basis for our construction of irreducible representations of the Poincaré superalgebra.

The Poincaré algebra has two Casimir operators: P^2 and W^2, where $\mathsf{W}^\mu = \frac{1}{2}\epsilon^{\mu\nu\lambda\rho}\mathsf{P}_\nu\mathsf{M}_{\lambda\rho}$ is the Pauli–Lubansky vector. By Schur's lemma, on an irreducible representation they must both act as multiplication by scalars. Let's focus on P^2. On an irreducible representation $\mathsf{P}^2 = m^2$, where m is the "rest-mass" of the particle described by the representation. Remember that on a state with momentum p, P_μ has eigenvalue $-ip_\mu$, hence P^2 has eigenvalues $-p^2$, which equals m^2 with our choice of metric. Because physical masses are real, we have $m^2 \geq 0$, hence we can distinguish two kinds of physical representations: *massless* for which $m^2 = 0$ and *massive* for which $m^2 > 0$.

Wigner's method starts by choosing a nonzero momentum p on the mass-shell: $p^2 = -m^2$. We let G_p denote the subgroup of the Lorentz group (or rather of $\mathrm{SL}(2,\mathbb{C})$) which leaves p invariant. G_p is known as the *little group*. Wigner's method, which we will not describe in any more detail than this, consists in the following. First one chooses a unitary finite-dimensional irreducible representation of the little group. Doing this for every p in the mass shell defines a family of representations indexed by p. The representation is carried by functions assigning to a momentum p in the mass shell, a state $\phi(p)$ in this representation.

Finally, one Fourier transforms to obtain fields on Minkowski spacetime subject to their classical equations of motion.

☞ In more mathematical terms, the construction can be described as follows. The mass shell $\mathcal{M}_{m^2} = \{p^\mu \mid p^2 = -m^2\}$ is acted on transitively by the Lorentz group L. Fix a vector $p \in \mathcal{M}_{m^2}$ and let G_p be the little group. Then $\mathcal{M}_{m^2}$ can be seen as the space of right cosets of G_p in L; that is, it is a homogeneous space L/G_p. Any representation $\mathbb{V}$ of G_p defines a homogeneous vector bundle on $\mathcal{M}_{m^2}$ whose space of sections carries a representation of the Poincaré group. This representation is said to be *induced* from $\mathbb{V}$. If $\mathbb{V}$ is unitary and irreducible, then so will be the induced representation. The induced representation naturally lives in momentum space, but for field theoretical applications we would like to work with fields in Minkowski space. This is easily achieved by Fourier transform, but since the momenta on the mass-shell obey $p^2 = -m^2$, it follows that the Fourier transform $\varphi(x)$ of a function $\tilde{\varphi}(p)$ automatically satisfies the Klein–Gordon equation. More is true, however, and the familiar classical relativistic equations of motion: Klein–Gordon, Dirac, Rarita–Schwinger,... can be understood group theoretically simply as projections onto irreducible representations of the Poincaré group.

In extending this method to the Poincaré superalgebra all that happens is that now the Lie algebra of the little group gets extended by the supercharges, since these commute with P_μ and hence stabilise the chosen 4-vector p_μ. Therefore we now induce from a unitary irreducible representations of the little (super)group. This representation will be reducible when restricted to the little group and will at the end of the day generate a *supermultiplet* of fields.

Before applying this procedure we will need to know about the structure of the little groups. The little group happens to be different for massive and for massless representations, as the next exercise asks you to show.

✍ **Exercise III.2.** *Let p_μ be a momentum obeying $p^0 > 0$, $p^2 = -m^2$. Prove that the little group of p_μ is isomorphic to:*

- $\mathrm{SU}(2)$*, for $m^2 > 0$;*
- $\widetilde{\mathrm{E}}_2$*, for $m^2 = 0$,*

where $\mathrm{E}_2 \cong \mathrm{SO}(2) \ltimes \mathbb{R}^2$*, is the two-dimensional euclidean group and* $\widetilde{\mathrm{E}}_2 \cong \mathrm{Spin}(2) \ltimes \mathbb{R}^2$ *is its double cover.*
(Hint: Argue that two momenta which are Lorentz-related have isomorphic little groups and then choose a convenient p_μ in each case.)

III.3. **Massless representations.** Let us start by considering massless representations. As shown in Exercise III.2, the little group for the momentum p^μ of a massless particle is noncompact. Therefore its finite-dimensional unitary representations must all come from its maximal compact subgroup $\mathrm{Spin}(2)$ and be trivial on the translation subgroup $\mathbb{R}^2$. The unitary representations of $\mathrm{Spin}(2)$ are one-dimensional and indexed by a number $\lambda \in \frac{1}{2}\mathbb{Z}$ called the *helicity*. Since CPT reverses the helicity, it may be necessary to include both helicities $\pm\lambda$ in order to obtain a CPT-self-conjugate representation.

Let's choose $p^\mu = (E, 0, 0, -E)$, with $E > 0$. Then the algebra of the supercharges

$$\left[\mathsf{Q}_\alpha, \bar{\mathsf{Q}}_{\dot\beta}\right] = 4E \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} .$$

Exercise III.3. *Show that as a consequence of the above algebra,* $\mathsf{Q}_1 = 0$ *in any unitary representation.*

Let us now define $\mathsf{q} \equiv (1/2\sqrt{E})\, \mathsf{Q}_2$, in terms of which the supersymmetry algebra becomes the fermionic oscillator algebra:

$$\mathsf{q}\,\mathsf{q}^\dagger + \mathsf{q}^\dagger\,\mathsf{q} = 1 .$$

This algebra has a unique irreducible representation of dimension 2. If $|\Omega\rangle$ is a state annihilated by q, then the representation has as basis $\{|\Omega\rangle, \mathsf{q}^\dagger|\Omega\rangle\}$. Actually, $|\Omega\rangle$ carries quantum numbers corresponding to the momentum p and also to the helicity λ, so that $|\Omega\rangle = |p, \lambda\rangle$.

Exercise III.4. *Paying close attention to the helicity of the supersymmetry charges, prove that* q *lowers the helicity by* $\frac{1}{2}$*, and that* $\mathsf{q}^\dagger$ *raises it by the same amount. Deduce that the massless supersymmetry multiplet of helicity* λ *contains two irreducible representations of the Poincaré algebra with helicities* λ *and* $\lambda + \frac{1}{2}$*.*

For example, if we take $\lambda = 0$, then we have two irreducible representations of the Poincaré algebra with helicities 0 and $\frac{1}{2}$. This representation cannot be realised on its own in a quantum field theory, because of the CPT invariance of quantum field theories. Since CPT changes the sign of the helicity, if a representation with helicity s appears, so will the representation with helicity $-s$. That means that representations which are not CPT-self-conjugate appear in CPT-conjugate pairs. The CPT-conjugate representation to the one discussed at the head of this paragraph has helicities $-\frac{1}{2}$ and 0. Taking both representations into account we find two states with helicity 0 and one state each with helicity $\pm\frac{1}{2}$. This is precisely the helicity content of the massless Wess–Zumino model: the helicity 0 states are the scalar and the pseudoscalar fields and the states of helicities $\pm\frac{1}{2}$ correspond to the physical degrees of freedom of the spinor.

If instead we start with helicity $\lambda = \frac{1}{2}$, then the supermultiplet has helicities $\frac{1}{2}$ and 1 and the CPT-conjugate supermultiplet has helicities -1 and $-\frac{1}{2}$. These are precisely the helicities appearing in supersymmetric Yang–Mills. The multiplet in question is therefore called the *gauge multiplet*.

Now take the $\lambda = \frac{3}{2}$ supermultiplet and add its CPT-conjugate. In this way we obtain a CPT-self-conjugate representation with helicities $-2, -\frac{3}{2}, \frac{3}{2}, 2$. This has the degrees of freedom of a graviton (helicities ±2) and a *gravitino* (helicities $\pm\frac{3}{2}$). This multiplet is realised field theoretically in supergravity, and not surprisingly it is called the *supergravity multiplet*.

III.4. **Massive representations.** Let us now discuss massive representations. As shown in Exercise III.2, the little group for the momentum p_μ of a massive particle is SU(2). Its finite-dimensional irreducible unitary representations are well-known: they are indexed by the *spin* s, where $2s$ is a non-negative integer, and have dimension $2s+1$.

A massive particle can always be boosted to its rest frame, so that we can choose a momentum $p^\mu = (m, 0, 0, 0)$ with $m > 0$. The supercharges now obey

$$\left[\mathsf{Q}_\alpha, \bar{\mathsf{Q}}_{\dot{\beta}}\right] = 2m \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} .$$

Thus we can introduce $\mathsf{q}_\alpha \equiv (1/\sqrt{2m})\mathsf{Q}_\alpha$, in terms of which the supersymmetry algebra is now the algebra of two identical fermionic oscillators:

$$\mathsf{q}_\alpha \, (\mathsf{q}_\beta)^\dagger + (\mathsf{q}_\beta)^\dagger \, \mathsf{q}_\alpha = \delta_{\alpha\beta} \ . \tag{44}$$

This algebra has a unique irreducible representation of dimension 4 with basis

$$\{|\Omega\rangle, (\mathsf{q}_1)^\dagger|\Omega\rangle, (\mathsf{q}_2)^\dagger|\Omega\rangle, (\mathsf{q}_1)^\dagger(\mathsf{q}_2)^\dagger|\Omega\rangle\} \ ,$$

where $|\Omega\rangle$ is a nonzero state obeying

$$\mathsf{q}_1|\Omega\rangle = \mathsf{q}_2|\Omega\rangle = 0 \ .$$

However unlike the case of massless representations, $|\Omega\rangle$ is now degenerate, since it carries spin: for spin s, $|\Omega\rangle$ is really a $(2s+1)$-dimensional SU(2) multiplet. Notice that $(\mathsf{q}_\alpha)^\dagger$ transforms as an SU(2)-doublet of spin $\frac{1}{2}$. This must be taken into account when determining the spin content of the states in the supersymmetry multiplet. Instead of simply adding the helicities like in the massless case, now we must use the Clebsch–Gordon series to add the spins. On the other hand, massive representations are automatically CPT-self-conjugate so we don't have to worry about adding the CPT-conjugate representation.

For example, if we take $s = 0$, then we find the following spectrum: $|p, 0\rangle$ with spin 0, $(\mathsf{q}_\alpha)^\dagger|p, 0\rangle$ with spin $\frac{1}{2}$ and $(\mathsf{q}_1)^\dagger(\mathsf{q}_2)^\dagger|p, 0\rangle$ which has spin 0 too. The field content described by this multiplet is then a scalar field, a pseudo-scalar field, and a Majorana fermion, which is precisely the field content of the Wess–Zumino model. The multiplet is known as the *scalar* or *Wess–Zumino multiplet*.

Exercise III.5. *What is the spin content of the massive supermultiplet with $s = \frac{1}{2}$? What would be the field content of a theory admitting this representation of the Poincaré superalgebra?*

All representations of the Poincaré superalgebra share the property that the number of fermionic and bosonic states match. For the massless representations this is clear because the whatever the Bose–Fermi parity of $|p, \lambda\rangle$, it is opposite that of $|p, \lambda + \frac{1}{2}\rangle$.

For the massive representations we see that whatever the Bose–Fermi parity of the $2s + 1$ states $|p, s\rangle$, it is opposite that of the $2(2s + 1)$ states $(\mathsf{q}_\alpha)^\dagger|p, s\rangle$ and the same as that of the $2s+1$ states $(\mathsf{q}_1)^\dagger(\mathsf{q}_2)^\dagger|p, s\rangle$. Therefore there are $2(2s + 1)$ bosonic and $2(2s + 1)$ fermionic states.

Problem 3 (Supersymmetry and the BPS bound)**.**

Here we introduce the extended Poincaré superalgebra and study its unitary representations. In particular we will see the emergence of central charges, the fact that the mass of a unitary representation satisfies a bound, called the BPS bound, and that the sizes of representations depends on whether the bound is or is not saturated.

The extended Poincaré superalgebra is the extension of the Poincaré algebra by N supercharges Q_I for $I = 1, 2, \ldots, N$. The nonzero brackets are now

$$\begin{aligned} [\mathsf{Q}_{\alpha\,I}, \mathsf{Q}_{\beta\,J}] &= 2\epsilon_{\alpha\beta}\mathsf{Z}_{IJ} \\ \left[\mathsf{Q}_{\alpha\,I}, \bar{\mathsf{Q}}^J_{\dot\alpha}\right] &= 2i\delta_I^J\,(\sigma^\mu)_{\alpha\dot\alpha}\mathsf{P}_\mu\ , \end{aligned} \tag{45}$$

where Z_{IJ} commute with all generators of the algebra and are therefore known as the *central charges*. Notice that $\mathsf{Z}_{IJ} = -\mathsf{Z}_{JI}$, whence central charges requires $N \geq 2$. The hermiticity condition on the supercharges now says that

$$(\mathsf{Q}_{\alpha\,I})^\dagger = \bar{\mathsf{Q}}^I_{\dot\alpha}\ .$$

We start by considering massless representations. Choose a lightlike momentum $p^\mu = (E, 0, 0, -E)$ with $E > 0$. The supercharges obey

$$[\mathsf{Q}_{\alpha\,I}, (\mathsf{Q}_{\beta\,J})^\dagger] = 4E\delta_I^J \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\ .$$

(1) Prove that all $\mathsf{Q}_{1\,I}$ must act trivially on any unitary representation, and conclude that the central charges must vanish for massless unitary representations.
(2) Consider a massless multiplet with lowest helicity λ. Which helicities appear and with what multiplicities?
(3) Prove that CPT-self-conjugate multiplets exist only for even N. Discuss the CPT-self-conjugate multiplets for $N = 2$, $N = 4$ and $N = 8$. These are respectively the N=2 *hypermultiplet*, the N=4 *gauge multiplet* and N=8 *supergravity multiplet*.

Now we consider massive representations without central charges. The situation is very similar to the N=1 case discussed in lecture.

4. Work out the massive N=2 multiplets without central charges and with spin s=0 and s=$\frac{1}{2}$. Show that for s=0 the spin content is $(0^5, \frac{1}{2}^4, 1)$ in the obvious notation, and for s=$\frac{1}{2}$ it is given by $(\frac{3}{2}, 1^4, \frac{1}{2}^6, 0^4)$.

Now consider massive $N=2$ multiplets with central charges. In this case $\mathsf{Z}_{IJ} = \mathsf{z}\epsilon_{IJ}$, where there is only one central charge z. Since z is central, it acts as a multiple of the identity, say z, in any irreducible representation. The algebra of supercharges is now:

$$[\mathsf{Q}_{\alpha I}, \mathsf{Q}_{\alpha J}] = z\,\epsilon_{IJ}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$
$$\left[\mathsf{Q}_{\alpha I}, (\mathsf{Q}_{\alpha J})^\dagger\right] = 2m\,\delta_I^J\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} .$$

5. Show that for a unitary massive representation of mass m, the following bound is always satisfied: $m \geq |z|$. *(Hint: Consider the algebra satisfied by the linear combination of supercharges* $\mathsf{Q}_{\alpha 1} \pm \bar{\epsilon}^{\dot{\alpha}\dot{\beta}}(\mathsf{Q}_{\beta 2})^\dagger$*.)*
6. Show that representations where the bound is not saturated—that is, $m > |z|$—have the same multiplicities as massive representations without central charge.
7. Show that massive representations where the bound is saturated have the same multiplicities as massless representations.

☞ The bound in Part 5 above is called the *BPS bound* since it generalises the Bogomol'nyi bound for the Prasad–Sommerfield limit of Yang–Mills–Higgs theory. In fact, in the context of $N=2$ supersymmetric Yang–Mills it is *precisely* the Bogomol'nyi bound.

The result in Part 7 above explains why BPS saturated multiplets are also called *short multiplets*. The difference in multiplicity between ordinary massive multiplets and those which are BPS saturated underlies the rigidity of the BPS-saturated condition under deformation: either under quantum corrections or under other continuous changes in the parameters of the model.

IV. Superspace and Superfields

In the previous lectures we have studied the representations of the Poincaré superalgebra and we have seen some of its field theoretical realisations. In both the Wess–Zumino model and supersymmetric Yang–Mills, proving the supersymmetry of the action was a rather tedious task, and moreover the superalgebra was only realised on-shell and, in the case of supersymmetric Yang–Mills, up to gauge transformations.

It would be nice to have a formulation in which supersymmetry was manifest, just like Poincaré invariance is in usual relativistic field theories. Such theories must have in addition to the physical fields, so-called auxiliary fields in just the right number to reach the balance between bosonic and fermionic fields which supersymmetry demands. For example, in the Wess–Zumino model this balance is present on the physical degree of freedoms: 2 bosonic and 2 fermionic. In order to have a manifestly supersymmetric formulation this balance in the degrees of freedom must be present without the need to go on-shell. For example, in the Wess–Zumino model, the bosons are defined by 2 real functions S and P, whereas the fermions are defined by 4: ψ^a. We conclude therefore that a manifestly supersymmetric formulation must contain at least two additional bosonic fields. The superfield formulation will do just that.

Superfields are fields in superspace, and superspace is to the Poincaré superalgebra what Minkowski space is to the Poincaré algebra. Just like we can easily write down manifestly Poincaré invariant models as theories of fields on Minkowski space, we will be able to (almost) effortlessly write down models invariant under the Poincaré superalgebra as theories of superfields in superspace.

In this lecture we will introduce the notions of superspace and superfields. We will discuss the scalar superfields and will rewrite the Wess–Zumino model in superspace. Unpacking the superspace action, we will recover a version of this model with the requisite number of auxiliary fields for the off-shell closure of the Poincaré superalgebra. The auxiliary fields are essential not only in the manifestly supersymmetric formulation of field theories but, as we will see in Lecture VI, also play an important role in the breaking of supersymmetry.

IV.1. **Superspace.** For our purposes the most important characteristic of Minkowski space is that, as discussed in the Appendix, it is acted upon transitively by the Poincaré group. We would now like to do something similar with the "Lie supergroup" corresponding to the Poincaré superalgebra.

We will not give the precise mathematical definition of a Lie supergroup in these lectures. Morally speaking a Lie supergroup is what one obtains by exponentiating elements of a Lie superalgebra. We will formally work with exponentials of elements of the superalgebra keeping in mind that the parameters associated to odd elements are themselves anticommuting.

By analogy with the treatment of Minkowski space in the Appendix, we will define Minkowski superspace (or superspace for short) as the space of right cosets of the Lorentz group. Notice that the Poincaré superalgebra has the structure of a semidirect product, just like the Poincaré algebra, where the translation algebra is replaced by the superalgebra generated by P_μ and Q_a. Points in superspace are then in one-to-one correspondence with elements of the Poincaré supergroup of the form

$$\exp(x^\mu \mathsf{P}_\mu)\exp(\bar\theta \mathsf{Q}) ,$$

where θ is an anticommuting Majorana spinor and $\bar\theta\mathsf{Q} = -\theta^a \mathsf{Q}_a$ as usual.

The Poincaré group acts on superspace by left multiplication with the relevant group element. However as we discussed in the Appendix, this action generates an antirepresentation of the Poincaré superalgebra. In order to generate a representation of the Poincaré superalgebra we must therefore start with the opposite superalgebra—the superalgebra where all brackets are multiplied by -1. In the case of the Poincaré superalgebra, the relevant brackets are now

$$\begin{aligned}[\mathsf{P}_\mu, \mathsf{Q}_a] &= 0\\ [\mathsf{Q}_a, \mathsf{Q}_b] &= -2\,(\gamma^\mu)_{ab}\,\mathsf{P}_\mu \ .\end{aligned} \tag{46}$$

Translations act as expected:

$$\exp(\tau^\mu \mathsf{P}_\mu)\exp(x^\mu \mathsf{P}_\mu)\exp(\bar\theta\mathsf{Q}) = \exp\left((x^\mu+\tau^\mu)\mathsf{P}_\mu\right)\exp(\bar\theta\mathsf{Q}) ,$$

so that the point (x,θ) gets sent to the point $(x+\tau,\theta)$.

The action of the Lorentz group is also as expected: x^μ transforms as a vector and θ as a Majorana spinor. In particular, Lorentz transformations do not mix the coordinates.

On the other hand, the noncommutativity of the superalgebra generated by P_μ and Q_a has as a consequence that a supertranslation does not just shift θ but also x, as the next exercise asks you to show. This is the reason why supersymmetry mixes bosonic and fermionic fields.

Exercise IV.1. *With the help of the Baker–Campbell–Hausdorff formula* (A-1), *show that*

$$\exp(\bar\varepsilon\mathsf{Q})\exp(\bar\theta\mathsf{Q}) = \exp(-\bar\varepsilon\gamma^\mu\theta\mathsf{P}_\mu)\exp\left((\overline{\theta+\varepsilon})\mathsf{Q}\right) .$$

It follows that the action of a supertranslation on the point (x^μ,θ) is given by $(x^\mu - \bar\varepsilon\gamma^\mu\theta, \theta+\varepsilon)$.

We speak of points in superspace, but in fact, as in noncommutative geometry, of which superspace is an example (albeit a mild one), one is supposed to think of x and θ are coordinate functions. There are no points corresponding to θ, but rather nilpotent elements in the (noncommutative) algebra of functions. For simplicity of exposition we will continue to talk of (x, θ) as a point, although it is good to keep in mind that this is an oversimplification. Doing so will avoid "koans" like

What is the point with coordinates $x^\mu - \bar{\varepsilon}\gamma^\mu\theta$?

This question has no answer because whereas (for fixed μ) x^μ is an ordinary function assigning a real number to each point, the object $x^\mu - \bar{\varepsilon}\gamma^\mu\theta$ is quite different, since $\bar{\varepsilon}\gamma^\mu\theta$ is certainly not a number. What it is, is an even element in the "coordinate ring" of the superspace, which is now a Grassmann algebra: with generators θ and $\bar{\theta}$ and coefficients which are honest functions of x^μ. This is to be understood in the sense of noncommutative geometry, as we now briefly explain.

Noncommutative geometry starts from the observation that in many cases the (commutative) algebra of functions of a space determines the space itself, and moreover that many of the standard geometric concepts with which we are familiar, can be rephrased purely in terms of the algebra of functions, without ever mentioning the notion of a point. (This is what von Neumann called "pointless geometry".) In noncommutative geometry one simply starts with a noncommutative algebra and interprets it as the algebra of functions on a "noncommutative space." Of course, this space does not really exist. Any question for which this formalism is appropriate should be answerable purely in terms of the noncommutative algebra. Luckily this is the case for those applications of this formalism to supersymmetry with which we are concerned in these lectures.

In the case of superspace, the noncommutativity is mild. There are commuting coordinates, the x^μ, but also (mildly) noncommuting coordinates θ and $\bar{\theta}$. More importantly, these coordinates are nilpotent: big enough powers of them vanish. In some sense, superspace consists of ordinary Minkowski space with some "nilpotent fuzz" around each point.

IV.2. **Superfields.** A superfield $\Phi(x, \theta)$ is by definition a (differentiable) function of x and θ. By linearising the geometric action on points, and recalling that the action on functions is inverse to that on points, we can work out the infinitesimal actions of P_μ and Q_a on superfields:

$$\begin{aligned} \mathsf{P}_\mu \cdot \Phi &= -\partial_\mu \Phi \\ \mathsf{Q}_a \cdot \Phi &= \left(\partial_a + (\gamma^\mu)_{ab}\theta^b\partial_\mu\right)\Phi \ , \end{aligned} \tag{47}$$

where by definition $\partial_a \theta^b = \delta_a^b$.

Exercise IV.2. *Verify that the above derivations satisfy the opposite superalgebra* (46).

Since both P_μ and Q_a act as derivations, they obey the Leibniz rule and hence products of superfields transform under (super)translations in the same way as a single superfield. Indeed, if f is any differentiable function, $f(\Phi)$ transforms under P_μ and Q_a as in equation (47). Similarly, if Φ^i for $i = 1, 2, \ldots, n$ transform as in (47), so will any differentiable function $f(\Phi^i)$.

The derivations $-\partial_\mu$ and $Q_a := \partial_a + (\gamma^\mu)_{ab}\theta^b\partial_\mu$ are the vector fields generating the infinitesimal *left* action of the Poincaré supergroup. The infinitesimal *right* action is also generated by vector fields which, because left and right multiplications commute, will (anti)commute with them. Since ordinary translations commute, right translations are also generated by $-\partial_\mu$. On the other hand, the noncommutativity of the supertranslations means that the expression for the right action of Q_a is different. In fact, from Exercise IV.1 we read off

$$\exp(\bar\theta\mathsf{Q})\exp(\bar\varepsilon\mathsf{Q}) = \exp(+\bar\varepsilon\gamma^\mu\theta\mathsf{P}_\mu)\exp\left((\overline{\theta+\varepsilon})\mathsf{Q}\right) ,$$

whence the infinitesimal generator (on superfields) is given by the *supercovariant derivative*

$$D_a := \partial_a - (\gamma^\mu)_{ab}\theta^b\partial_\mu .$$

Exercise IV.3. *Verify that the derivations Q_a and D_a anticommute and that*

$$[D_a, D_b] = -2(\gamma^\mu)_{ab}\partial_\mu . \tag{48}$$

We are almost ready to construct supersymmetric lagrangians. Recall that a lagrangian $\mathcal{L}$ is supersymmetric if it is Poincaré invariant and such that its supersymmetric variation is a total derivative:

$$\delta_\varepsilon\mathcal{L} = \partial_\mu\left(\bar\varepsilon K^\mu\right) .$$

It is very easy to construct supersymmetric lagrangians using superfields.

To explain this let us make several crucial observations. First of all notice that because the odd coordinates θ are anticommuting, the dependence on θ is at most polynomial, and because θ has four real components, the degree of the polynomial is at most 4.

Exercise IV.4. *Show that a superfield $\Phi(x,\theta)$ has the following θ-expansion*

$$\begin{aligned}\Phi(x,\theta) = \phi(x) + \bar\theta\chi(x) + \bar\theta\theta\, F(x) + \bar\theta\gamma_5\theta\, G(x) \\ + \bar\theta\gamma^\mu\gamma_5\theta\, v_\mu(x) + \bar\theta\theta\,\bar\theta\xi(x) + \bar\theta\theta\,\bar\theta\theta\, E(x) ,\end{aligned}$$

where ϕ, E, F, G, v_μ, χ and ξ are fields in Minkowski space.
(Hint: You may want to use the Fierz-like identities (A-11).)

Now let $L(x,\theta)$ be any Lorentz-invariant function of x and θ which transforms under supertranslations according to equation (47). For example, any function built out of superfields, their derivatives and their supercovariant derivatives transforms according to equation (47). The next exercise asks you to show that under a supertranslation, the component of L with the highest power of θ transforms into a total derivative. Its integral is therefore invariant under supertranslations, Lorentz invariant (since L is) and, by the Poincaré superalgebra, also invariant under translations. In other words, it is invariant under supersymmetry!

✍ **Exercise IV.5.** *Let $\Phi(x,\theta)$ be a superfield and let $E(x)$ be its $(\bar{\theta}\theta)^2$ component, as in Exercise IV.4. Show that $E(x)$ transforms into a total derivative under supertranslations:*

$$\delta_\varepsilon E = \partial_\mu \left(-\tfrac{1}{4}\bar{\varepsilon}\gamma^\mu\xi\right) .$$

(Hint: As in Exercise IV.4, you may want to use the identities (A-11).)

We will see how this works in practice in two examples: the Wess–Zumino model presently and in the next lecture the case of supersymmetric Yang–Mills.

IV.3. **Superfields in two-component formalism.** The cleanest superspace formulation of the Wess–Zumino model requires us to describe superspace in terms of two-component spinors. Since θ is a Majorana spinor, it can be written as $\theta^a = (\theta^\alpha, \bar{\theta}_{\dot{\alpha}})$. Taking into account equation (A-20), a point in superspace can be written as

$$\exp(x^\mu \mathsf{P}_\mu)\, \exp\left(-(\theta\mathsf{Q} + \bar{\theta}\bar{\mathsf{Q}})\right) .$$

The two-component version of the opposite superalgebra (46) is now

$$\left[\mathsf{Q}_\alpha, \bar{\mathsf{Q}}_{\dot{\beta}}\right] = -2i(\sigma^\mu)_{\alpha\dot{\beta}}\mathsf{P}_\mu , \tag{49}$$

with all other brackets vanishing.

✍ **Exercise IV.6.** *Show that under left multiplication by* $\exp(\varepsilon\mathsf{Q})$ *the point* $(x^\mu, \theta, \bar{\theta})$ *gets sent to the point* $(x^\mu - i\varepsilon\sigma^\mu\bar{\theta}, \theta - \varepsilon, \bar{\theta})$*. Similarly, show that under left multiplication by* $\exp(\bar{\varepsilon}\bar{\mathsf{Q}})$*,* $(x^\mu, \theta, \bar{\theta})$ *gets sent to* $(x^\mu - i\bar{\varepsilon}\bar{\sigma}^\mu\theta, \theta, \bar{\theta} - \bar{\varepsilon})$*.*

This means that action on superfields (recall that the action on functions is inverse to that on points) is generated by the following derivations:

$$Q_\alpha = \partial_\alpha + i(\sigma^\mu)_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_\mu \qquad \text{and} \qquad \bar{Q}_{\dot{\alpha}} = \bar{\partial}_{\dot{\alpha}} + i(\bar{\sigma}^\mu)_{\dot{\alpha}\beta}\theta^\beta\partial_\mu . \tag{50}$$

Repeating this for the right action, we find the following expressions for the supercovariant derivatives:

$$D_\alpha = \partial_\alpha - i(\sigma^\mu)_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_\mu \qquad \text{and} \qquad \bar{D}_{\dot{\alpha}} = \bar{\partial}_{\dot{\alpha}} - i(\bar{\sigma}^\mu)_{\dot{\alpha}\beta}\theta^\beta\partial_\mu . \tag{51}$$

Exercise IV.7. *Verify that* $\bar{Q}_{\dot\alpha} = (Q_\alpha)^*$ *and* $\bar{D}_{\dot\alpha} = (D_\alpha)^*$*. Also show that any of* Q_α *and* $\bar{Q}_{\dot\alpha}$ *anticommute with any of* D_α *and* $\bar{D}_{\dot\alpha}$*, and that they obey the following brackets:*

$$[Q_\alpha, \bar{Q}_{\dot\beta}] = +2i\sigma^\mu_{\alpha\dot\beta}\partial_\mu \qquad \text{and} \qquad [D_\alpha, \bar{D}_{\dot\beta}] = -2i\sigma^\mu_{\alpha\dot\beta}\partial_\mu \ . \tag{52}$$

IV.4. **Chiral superfields.** Let $\Phi(x,\theta,\bar\theta)$ be a complex superfield. Expanding it as a series in θ we obtain

$$\begin{aligned}\Phi(x,\theta,\bar\theta) &= \phi(x) + \theta\chi(x) + \bar\theta\bar\chi'(x) + \bar\theta\bar\sigma^\mu\theta v_\mu(x) \\ &\quad + \theta^2 F(x) + \bar\theta^2 \bar{F}'(x) + \bar\theta^2\theta\xi(x) + \theta^2\bar\theta\bar\xi'(x) + \theta^2\bar\theta^2 D(x) \ ,\end{aligned} \tag{53}$$

where ϕ, χ, $\bar\chi'$, v_μ, ξ, $\bar\xi'$, F, F' and D are all different complex fields.

Therefore an unconstrained superfield Φ gives rise to a large number of component fields. Taking ϕ, the lowest component of the superfield, to be a complex scalar we see that the superfield contains too many component fields for it to yield an irreducible representation of the Poincaré superalgebra. Therefore we need to impose constraints on the superfield in such a way as to cut down the size of the representation. We now discuss one such constraint and in the following lecture will discuss another.

Let us define a *chiral superfield* as a superfield Φ which satisfies the condition

$$\bar{D}_{\dot\alpha}\Phi = 0 \ . \tag{54}$$

Similarly we define an *antichiral superfield* as one satisfying

$$D_\alpha\Phi = 0 \ . \tag{55}$$

Chiral superfields behave very much like holomorphic functions. Indeed, notice that a real (anti)chiral superfield is necessarily constant. Indeed, the complex conjugate of a chiral field is antichiral. If Φ is real and chiral, then it also antichiral, whence it is annihilated by both D_α and $\bar{D}_{\dot\alpha}$ and hence by their anticommutator, which is essentially ∂_μ, whence we would conclude that Φ is constant.

It is very easy to solve for the most general (anti)chiral superfield. Indeed, notice that the supercovariant derivatives admit the following operatorial decompositions

$$D_\alpha = e^{iU}\partial_\alpha e^{-iU} \qquad \text{and} \qquad \bar{D}_{\dot\alpha} = e^{-iU}\bar\partial_{\dot\alpha} e^{iU} \ , \tag{56}$$

where $U = \theta\sigma^\mu\bar\theta\partial_\mu$ is real.

Exercise IV.8. *Use this result to prove that the most general chiral superfield takes the form*

$$\Phi(x,\theta,\bar\theta) = \phi(y) + \theta\chi(y) + \theta^2 F(y)\ ,$$

where $y^\mu = x^\mu - i\theta\sigma^\mu\bar\theta$. *Expand this to obtain*

$$\begin{aligned}\Phi(x,\theta,\bar\theta) = \phi(x) &+ \theta\chi(x) + \theta^2 F(x) + i\bar\theta\bar\sigma^\mu\theta\partial_\mu\phi(x)\\ &- \tfrac{i}{2}\theta^2\bar\theta\bar\sigma^\mu\partial_\mu\chi(x) + \tfrac{1}{4}\theta^2\bar\theta^2\,\Box\,\phi(x)\ . \end{aligned}\tag{57}$$

It is possible to project out the different component fields in a chiral superfields by taking derivatives. One can think of this as Taylor expansions in superspace.

Exercise IV.9. *Let* Φ *be a chiral superfield. Show that*

$$\begin{aligned}\phi(x) &= \Phi\big|\\ \chi_\alpha(x) &= D_\alpha\Phi\big|\\ F(x) &= -\tfrac{1}{4}D^2\Phi\big|\ ,\end{aligned}$$

where $D^2 = D^\alpha D_\alpha$ *and where* $\big|$ *denotes the operation of setting* $\theta = \bar\theta = 0$ *in the resulting expressions.*

IV.5. **The Wess–Zumino model revisited.** We will now recover the Wess–Zumino model in superspace. The lagrangian couldn't be simpler.

Let Φ be a chiral superfield. Its dimension is equal to that of its lowest component $\Phi\big|$, which in this case, being a complex scalar, has dimension 1.

Since θ has dimension $-\frac{1}{2}$, the highest component of any superfield (the coefficient of $\theta^2\bar\theta^2$) has dimension two more than that of the superfield. Therefore if we want to build a lagrangian out of Φ we need to take a quadratic expression. Since Φ is complex and but the action should be real, we have essentially one choice: $\bar\Phi\Phi$, where $\bar\Phi = (\Phi)^*$. The highest component of $\bar\Phi\Phi$ is real, has dimension 4, is Poincaré invariant and transforms into a total derivative under supersymmetry. It therefore has all the right properties to be a supersymmetric lagrangian.

Exercise IV.10. *Let* Φ *be a chiral superfield and let* $\bar\Phi = (\Phi)^*$ *be its (antichiral) complex conjugate. Show that the highest component of* $\bar\Phi\Phi$ *is given by*

$$-\partial_\mu\phi\partial^\mu\bar\phi + F\bar F + \tfrac{i}{4}\left(\chi\sigma^\mu\partial_\mu\bar\chi + \bar\chi\bar\sigma^\mu\partial_\mu\chi\right) + \tfrac{1}{4}\partial_\mu\left(\phi\partial^\mu\bar\phi + \bar\phi\partial^\mu\phi\right)\ .$$

Rewrite the lagrangian of the free massless Wess–Zumino model (given in (1)*) in terms of two-component spinors and show that it agrees (up to total derivatives and after using the equation of motion of* F*) with* $2\bar\Phi\Phi$ *where* $\phi = \frac{1}{2}(S + iP)$ *and* $\psi^a = (\chi^\alpha, \bar\chi_{\dot\alpha})$.

A complex scalar field is not really a scalar field in the strict sense. Because of CPT-invariance, changing the orientation in Minkowski space complex conjugates the complex scalar. This means that the real part is indeed a scalar, but that the imaginary part is a pseudoscalar. This is consistent with the identification $\phi = \frac{1}{2}(S + iP)$ in the above exercise.

Let us now recover the supersymmetry transformations of the component fields from superspace. By definition, $\delta_\varepsilon \Phi = -(\varepsilon Q + \bar{\varepsilon}\bar{Q})\Phi$. In computing the action of Q_α and $\bar{Q}_{\dot\alpha}$ on a chiral superfield Φ, it is perhaps easier to write Φ as

$$\Phi = e^{-iU}\left(\phi + \theta\chi + \theta^2 F\right) ,$$

and the supercharges as

$$Q_\alpha = e^{-iU}\partial_\alpha e^{iU} \qquad \text{and} \qquad \bar{Q}_{\dot\alpha} = \bar{D}_{\dot\alpha} + 2i(\bar\sigma^\mu)_{\dot\alpha\beta}\theta^\beta\partial_\mu ,$$

with $U = \theta\sigma^\mu\bar\theta\partial_\mu$.

Exercise IV.11. *Doing so, or the hard way, show that*

$$\begin{aligned} \delta_\varepsilon\phi &= -\varepsilon\chi \\ \delta_\varepsilon\chi_\alpha &= -2\varepsilon_\alpha F + 2i\bar\varepsilon^{\dot\alpha}(\bar\sigma^\mu)_{\dot\alpha\alpha}\partial_\mu\phi \\ \delta_\varepsilon F &= i\bar\varepsilon\bar\sigma^\mu\partial_\mu\chi . \end{aligned} \tag{58}$$

Now rewrite the supersymmetry transformations (7) *of the free massless Wess–Zumino model in terms of two-component spinors and show that they agree with the ones above after using the F equations of motion and under the identification $\phi = \frac{1}{2}(S + iP)$ and $\psi^a = (\chi^\alpha, \bar\chi_{\dot\alpha})$.*

The above result illustrates why in the formulation of the Wess–Zumino model seen in Lecture I, the Poincaré superalgebra only closes on-shell. In that formulation the auxiliary field F has been eliminated using its equation of motion $F = 0$. However for this to be consistent, its variation under supersymmetry has to vanish as well, and as we have just seen F varies into the equation of motion of the fermion.

Let us introduce the following notation:

$$\int d^2\theta d^2\bar\theta \leftrightarrow \text{the coefficient of } \theta^2\bar\theta^2.$$

The notation is supposed to be suggestive of integration in superspace. Of course this integral is purely formal and has not measure-theoretic content. It is an instance of the familiar Berezin integral in the path integral formulation of theories with fermions; only that in this case the definition is not given in this way, since the Grassmann algebra in quantum field theory has to be infinitely generated so that correlation functions of an arbitrary number of fermions are not automatically zero. Therefore it makes no sense to extract the "top" component of an element of the Grassmann algebra.

In this notation, the (free, massless) Wess–Zumino model is described by the following action:

$$\int d^4x d^2\theta d^2\bar{\theta}\, 2\bar{\Phi}\Phi \ . \tag{59}$$

A convenient way to compute superspace integrals of functions of chiral superfields is to notice that

$$\int d^4x d^2\theta d^2\bar{\theta} K(\Phi, \bar{\Phi}) = \int d^4x \tfrac{1}{16} D^2 \bar{D}^2 K(\Phi, \bar{\Phi})\big| \ . \tag{60}$$

This is true even if Φ is not a chiral superfield, but it becomes particularly useful if it is, since we can use chirality and Exercise IV.9 to greatly simplify the computations.

Exercise IV.12. *Take $K(\Phi, \bar{\Phi}) = \bar{\Phi}\Phi$ and, using the above expression for $\int d^4x d^2\theta d^2\bar{\theta} K(\Phi, \bar{\Phi})$, rederive the result in the first part of Exercise IV.10.*

In Problem 1 we saw that the free massless Wess–Zumino model is invariant under the R-symmetry (32). This symmetry can also be realised geometrically in superspace. Notice that the infinitesimal R-symmetry acts on the component fields of the superfield as

$$\mathsf{R} \cdot \phi = i\phi \qquad \mathsf{R} \cdot \chi = -\tfrac{i}{2}\chi \qquad \text{and} \qquad \mathsf{R} \cdot \bar{\chi} = \tfrac{i}{2}\bar{\chi} \ .$$

Since $\phi = \Phi|$ we are forced to set $\mathsf{R} \cdot \Phi = i\Phi$, which is consistent with the R-symmetry transformation properties of the fermions provided that θ and $\bar{\theta}$ transform according to

$$\mathsf{R} \cdot \theta = \tfrac{3i}{2}\theta \qquad \text{and} \qquad \mathsf{R} \cdot \bar{\theta} = -\tfrac{3i}{2}\bar{\theta} \ . \tag{61}$$

This forces the superspace "measures" $d^2\theta$ and $d^2\bar{\theta}$ to transform as well:

$$\mathsf{R} \cdot d^2\theta = -3i d^2\theta \qquad \text{and} \qquad \mathsf{R} \cdot d^2\bar{\theta} = 3i d^2\bar{\theta} \ , \tag{62}$$

and this shows that the lagrangian $\int d^2\theta d^2\bar{\theta}\Phi\bar{\Phi}$ is manifestly invariant under the R-symmetry.

IV.6. **The superpotential.** We now add masses and interactions to the theory with superspace lagrangian $\bar{\Phi}\Phi$.

The observation that allows us to do this is the following. It follows from the supersymmetry transformation properties (58) of a chiral superfield, that its θ^2 component transforms as a total derivative. Now suppose that Φ is a chiral superfield. Then so is any power of Φ and in fact any differentiable function $W(\Phi)$. Therefore the θ^2 component of $W(\Phi)$ is supersymmetric. However it is not real, so we take its real part. The function $W(\Phi)$ is called the *superpotential*. In the case of the Wess–Zumino model it is enough to take W to be a cubic polynomial. In fact, on dimensional grounds, a renormalisable superpotential is at most cubic. This follows because the θ^2 component of $W(\Phi)$ has dimension 1 more than that of $W(\Phi)$. Since the dimension of a lagrangian term must be at most four, the dimension of $W(\Phi)$ must be at most three. Since Φ has dimension 1 and renormalisability does not allow coupling constants of negative dimension, we see that $W(\Phi)$ must be at most cubic.

Let us introduce the notation

$$\int d^2\theta \leftrightarrow \text{the coefficient of } \theta^2$$
$$\int d^2\bar{\theta} \leftrightarrow \text{the coefficient of } \bar{\theta}^2,$$

with the same caveat about superspace integration as before. A convenient way to compute such chiral superspace integrals is again to notice that

$$\begin{aligned}\int d^4x d^2\theta\, W(\Phi) &= -\int d^4x\, \tfrac{1}{4} D^2 W(\Phi)\big| \ ,\\ \int d^4x d^2\bar{\theta}\, \overline{W}(\bar{\Phi}) &= -\int d^4x\, \tfrac{1}{4} \bar{D}^2 \overline{W}(\bar{\Phi})\big| \ .\end{aligned} \tag{63}$$

Exercise IV.13. *Let $W(\Phi)$ be given by*

$$W(\Phi) = \mu\Phi^2 + \nu\Phi^3 \ .$$

Determine μ and ν in such a way that the action obtained by adding to the action (59) *the superpotential term*

$$\int d^2\theta\, W(\Phi) + \int d^2\bar{\theta}\, \overline{W(\Phi)} \tag{64}$$

and eliminating the auxiliary field via its equation of motion we recover the Wess–Zumino model, under the identification $\phi = \frac{1}{2}(S + iP)$ and $\psi^a = (\chi^\alpha, \bar{\chi}_{\dot{\alpha}})$.
(Hint: I get $\mu = m$ and $\nu = \frac{4}{3}\lambda$.)

R-symmetry can help put constraints in the superpotential. Notice that the R-symmetry transformation properties of the superspace measures $d^2\theta$ and $d^2\bar{\theta}$ in (62) says that an R-invariant superpotential must transform as $\mathsf{R}\cdot W(\Phi) = 3iW(\Phi)$. This means that only the cubic term is invariant and in particular that the model must be massless. This is consistent with the results of Problem 1: the conformal superalgebra contains the R-symmetry, yet it is not a symmetry of the model unless the mass is set to zero.
It is nevertheless possible to redefine the action of the R-symmetry on the fields in such a way that the mass terms are R-invariant. For example, we could take $\mathsf{R}\cdot\Phi = \frac{3i}{2}\Phi$, but this then prohibits the cubic term in the superpotential and renders the theory free. Of course the massive theory, even if free, is not (super)conformal invariant.

In other words, we see that the Wess–Zumino model described by the action (22) can be succinctly written in superspace as

$$\int d^4x\, d^2\theta\, d^2\bar{\theta}\, 2\bar{\Phi}\Phi + \left[\int d^4x\, d^2\theta \left(m\Phi^2 + \tfrac{4}{3}\lambda\Phi^3\right) + \text{c.c.}\right] \ . \tag{65}$$

Using equation (63) and Exercise IV.9 it is very easy to read off the contribution of the superpotential to the the lagrangian:

$$\frac{dW(\phi)}{d\phi} F - \tfrac{1}{4}\frac{d^2W(\phi)}{d\phi^2}\chi\chi + \text{c.c} \ ,$$

and hence immediately obtain the Yukawa couplings and the fermion mass. The scalar potential (including the masses) is obtained after eliminating the auxiliary field.

We leave the obvious generalisations of the Wess–Zumino model to the tutorial problem. It is a pleasure to contemplate how much simpler it is to write these actions down in superspace than in components, and furthermore the fact that we know a priori that the resulting theories will be supersymmetric.

The power of superfields is not restricted to facilitating the construction of supersymmetric models. There is a full-fledged superspace approach to supersymmetric quantum field theories, together with Feynman rules for "supergraphs" and manifestly supersymmetry regularisation schemes. This formalism has made it possible to prove certain powerful "nonrenormalisation" theorems which lie at the heart of the attraction of supersymmetric theories. A simple consequence of superspace perturbation theory is that in a theory of chiral superfields, any counterterm is of the form of an integral over all of superspace (that is, of the form $\int d^4x\, d^2\theta d^2\bar{\theta}$). This means that in a renormalisable theory, the superpotential terms—being integrals over chiral superspace (that is, $\int d^4x d^2\theta$ or $\int d^4x d^2\bar{\theta}$)—are not renormalised. Since the superpotential contains both the mass and the couplings of the chiral superfields, it means that the tree level masses and couplings receive no perturbative loop corrections. In fact, "miraculous cancellations" at the one-loop level were already observed in the early days of supersymmetry, which suggested that there was only need for wave-function renormalisation. The nonrenormalisation theorem (for chiral superfields) is the statement that this persists to all orders in perturbation theory. More importantly, the absence of mass renormalisation provides a solution of the gauge hierarchy problem, since a hierarchy of masses fixed at tree-level will receive no further radiative corrections. From a phenomenological point of view, this is one of the most attractive features of supersymmetric theories.

Problem 4 (Models with chiral superfields)**.**

In this tutorial problem we discuss the most general supersymmetric models which can be constructed out of chiral superfields. Let Φ^i, for $i = 1, 2, \ldots, N$, be chiral superfields, and let $(\Phi^i)^* = \bar{\Phi}^{\bar{\imath}}$ be the conjugate antichiral fields.

(1) Show that the most general supersymmetric renormalisable lagrangian involving these fields is given by the sum of a kinetic term

$$\int d^2\theta d^2\bar{\theta}\, K_{i\bar{\jmath}} \Phi^i \bar{\Phi}^{\bar{\jmath}}$$

and a superpotential term (64) with

$$W(\Phi) = a_i \Phi^i + \tfrac{1}{2} m_{ij} \Phi^i \Phi^j + \tfrac{1}{3} \lambda_{ijk} \Phi^i \Phi^j \Phi^k \ ,$$

where a_i, m_{ij} and λ_{ijk} are totally symmetric real constants, and $K_{i\bar{j}}$ is a constant hermitian matrix. Moreover unitarity of the model forces $K_{i\bar{j}}$ to be positive definite.

(2) Argue that via a complex change of variables $\Phi^i \mapsto M^i{}_j \Phi^j$, where M is a matrix in $\mathrm{GL}(N,\mathbb{C})$, we can take $K_{i\bar{j}} = \delta_{i\bar{j}}$ without loss of generality. Moreover we we still have the freedom to make a unitary transformation $\Phi^i \mapsto U^i{}_j \Phi^j$, where U is a matrix in $\mathrm{U}(N)$ with which to diagonalise the mass matrix m_{ij}. Conclude that the most general supersymmetric renormalisable lagrangian involving N chiral superfields is given by the sum of a kinetic term

$$\int d^2\theta d^2\bar{\theta} \sum_{i=1}^{N} \Phi^i \bar{\Phi}_i \ ,$$

where $\bar{\Phi}_i = \delta_{i\bar{j}} \bar{\Phi}^{\bar{j}}$, and a superpotential term (64) with

$$W = a_i \Phi^i + \sum_{i=1}^{N} m_i (\Phi^i)^2 + \tfrac{1}{3} \lambda_{ijk} \Phi^i \Phi^j \Phi^k \ .$$

(3) Expand the above action into components and eliminate the auxiliary fields via their equations of motion.

If we don't insist on renormalisability, we can generalise the above model in two ways. First of all we can consider more general superpotentials, but we can also contemplate more complicated kinetic terms. Let $K(\Phi, \bar{\Phi})$ be a real function of Φ^i and $\bar{\Phi}^{\bar{\imath}}$ and consider the kinetic term

$$\int d^4x \, d^2\theta d^2\bar{\theta} \, K(\Phi, \bar{\Phi}) \ . \tag{66}$$

4. Show that the above action is invariant under the transformations

$$K(\Phi, \bar{\Phi}) \mapsto K(\Phi, \bar{\Phi}) + \Lambda(\Phi) + \overline{\Lambda(\Phi)} \ . \tag{67}$$

5. Expand the above kinetic term and show that it gives rise to a supersymmetric extension of the "hermitian sigma model"

$$-\int d^4x \, g_{i\bar{j}}(\phi, \bar{\phi}) \partial_\mu \phi^i \partial^\mu \bar{\phi}^{\bar{j}} \ ,$$

with metric

$$g_{i\bar{j}}(\phi, \bar{\phi}) = \partial_i \partial_{\bar{j}} K(\phi, \bar{\phi}) \ ,$$

where $\partial_i = \partial / \partial \phi^i$ and $\partial_{\bar{\imath}} = \partial / \partial \bar{\phi}^{\bar{\imath}}$.

☞ Such a metric $g_{i\bar{j}}$ is called *Kähler*. Notice that it is the metric which is physical even though the superspace action is written in terms of the *Kähler potential* K. This is because the action is invariant under the *Kähler gauge transformations* (67) which leave the metric invariant.

6. Eliminate the auxiliary fields via their equations of motion and show that the resulting lagrangian becomes (up to a total derivative)
$$-g_{i\bar{\jmath}}\partial_\mu\phi^i\partial^\mu\bar{\phi}^{\bar{\jmath}} + \tfrac{i}{2}g_{i\bar{\jmath}}\chi^i\sigma^\mu\nabla_\mu\bar{\chi}^{\bar{\jmath}} + \tfrac{1}{16}R_{ij\bar{k}\bar{\ell}}\chi^i\chi^j\bar{\chi}^{\bar{k}}\bar{\chi}^{\bar{\ell}}\ ,$$
where
$$\begin{aligned}\nabla_\mu\bar{\chi}^{\bar{\imath}} &= \partial_\mu\bar{\chi}^{\bar{\imath}} + \Gamma_{\bar{\jmath}\bar{k}}{}^{\bar{\imath}}\partial_\mu\bar{\phi}^{\bar{\jmath}}\bar{\chi}^{\bar{k}}\\ \Gamma_{\bar{\jmath}\bar{k}}{}^{\bar{\imath}} &= g^{i\bar{\imath}}\partial_{\bar{k}}g_{i\bar{\jmath}} \qquad \big(\text{and } \Gamma_{jk}{}^{i} = g^{i\bar{\imath}}\partial_k g_{\bar{\imath}j}\big)\\ R_{ij\bar{k}\bar{\ell}} &= \partial_i\partial_{\bar{k}}g_{j\bar{\ell}} - g^{m\bar{m}}\partial_i g_{j\bar{m}}\partial_{\bar{k}}g_{m\bar{\ell}}\ ,\end{aligned}$$
where $g^{i\bar{\jmath}}$ is the inverse of $g_{i\bar{\jmath}}$, which is assumed invertible due to the positive-definiteness (or more generally, nondegeneracy) of the kinetic term.
7. Finally, consider an arbitrary differentiable function $W(\Phi)$ and add to the kinetic term (66) the corresponding superpotential term (64). Expand the resulting action in components and eliminate the auxiliary fields using their field equations to arrive at the most general supersymmetric action involving only scalar multiplets:
$$\begin{aligned}- g_{i\bar{\jmath}}\partial_\mu\phi^i\partial^\mu\bar{\phi}^{\bar{\jmath}} &+ \tfrac{i}{2}g_{i\bar{\jmath}}\chi^i\sigma^\mu\nabla_\mu\bar{\chi}^{\bar{\jmath}} + \tfrac{1}{16}R_{ij\bar{k}\bar{\ell}}\chi^i\chi^j\bar{\chi}^{\bar{k}}\bar{\chi}^{\bar{\ell}}\\ &- g^{i\bar{\jmath}}\partial_i W\partial_{\bar{\jmath}}\overline{W} - \tfrac{1}{4}\chi^i\chi^j H_{ij}(W) - \tfrac{1}{4}\bar{\chi}^{\bar{\imath}}\bar{\chi}^{\bar{\jmath}}H_{\bar{\imath}\bar{\jmath}}(\overline{W})\ , \quad (68)\end{aligned}$$
where
$$\begin{aligned}H_{ij}(W) &= \nabla_i\partial_j W = \partial_i\partial_j W - \Gamma_{ij}{}^{k}\partial_k W\\ H_{\bar{\imath}\bar{\jmath}}(\overline{W}) &= \nabla_{\bar{\imath}}\partial_{\bar{\jmath}}\overline{W} = \partial_{\bar{\imath}}\partial_{\bar{\jmath}}\overline{W} - \Gamma_{\bar{\imath}\bar{\jmath}}{}^{\bar{k}}\partial_{\bar{k}}\overline{W}\end{aligned}$$
is the Hessian of W.

☞ Models such as (68) are known as supersymmetric sigma models. The scalar fields can be understood as maps from the spacetime to a riemannian manifold. Not every riemannian manifold admits a supersymmetric sigma model and indeed this problem shows that supersymmetry requires the metric to be Kähler. The data of a supersymmetric sigma model is thus geometric in nature: a Kähler manifold (M, g) and a holomorphic function W on M. This and similar results underlie the deep connections between supersymmetry and geometry.

V. Supersymmetric Yang–Mills revisited

The general supersymmetric renormalisable models in four dimensions can be built out of the chiral superfields introduced in the previous lecture and the vector superfields to be introduced presently. In terms of components, chiral superfields contain complex scalar fields (parametrising a Kähler manifold, which must be flat in renormalisable models) and Majorana fermions. This is precisely the field content of the Wess–Zumino model discussed in Lecture I and in the previous lecture we saw how to write (and generalise) this model in superspace. In contrast, the vector superfield is so called because it contains a vector boson as well as a Majorana fermion. This is precisely the field content of the supersymmetric Yang–Mills theory discussed in Lecture II and in the present lecture we will learn how to write this theory down in superspace. By the end of this lecture we will know how to write down the most general renormalisable supersymmetric theory in four dimensions. The tutorial problem will introduce the Kähler quotient, in the context of the $\mathbb{C}P^N$ supersymmetric sigma model. Apart from its intrinsic mathematical interest, this construction serves to illustrate the fact that in some cases, the low energy effective theory of a supersymmetric gauge theory is a supersymmetric sigma model on the space of vacua.

V.1. **Vector superfields.** In the component expansion (53) of a general scalar superfield one finds a vector field v_μ. If we wish to identify this field with a vector boson we must make sure that it is real. Complex conjugating the superfield sends v_μ to its complex conjugate $\bar{v}_\mu$, hence reality of v_μ implies the reality of the superfield. I hope this motivates the following definition.

A *vector superfield* V is a scalar superfield which satisfies the reality condition $\bar{V} = V$.

Exercise V.1. *Show that the general vector superfield V has the following component expansion:*

$$V(x,\theta,\bar\theta) = C(x) + \theta\xi(x) + \bar\theta\bar\xi(x) + \bar\theta\bar\sigma^\mu\theta v_\mu(x) \\ + \theta^2 G(x) + \bar\theta^2\bar{G}(x) + \bar\theta^2\theta\eta(x) + \theta^2\bar\theta\bar\eta(x) + \theta^2\bar\theta^2 E(x) \ , \quad (69)$$

where C, v_μ and E are real fields.

The real part of a chiral superfield Λ is a particular kind of vector superfield, where the vector component is actually a derivative:

$$\Lambda + \bar\Lambda = (\phi + \bar\phi) + \theta\chi + \bar\theta\bar\chi + \theta^2 F + \bar\theta^2\bar F + i\bar\theta\bar\sigma^\mu\theta\partial_\mu(\phi - \bar\phi) \\ - \tfrac{i}{2}\theta^2\bar\theta\bar\sigma^\mu\partial_\mu\chi - \tfrac{i}{2}\bar\theta^2\theta\sigma^\mu\partial_\mu\bar\chi + \tfrac{1}{4}\theta^2\bar\theta^2\,\Box(\phi + \bar\phi) \ . \quad (70)$$

This suggests that the transformation

$$V \mapsto V - (\Lambda + \bar\Lambda) \ , \quad (71)$$

where V is a vector superfield and Λ is a chiral superfield, should be interpreted as the superspace version of a U(1) gauge transformation.

Exercise V.2. *Show that the transformation* (71) *has the following effect on the components of the vector superfield:*

$$
\begin{aligned}
C &\mapsto C - (\phi + \bar{\phi}) \\
\xi &\mapsto \xi - \chi \\
G &\mapsto G - F \\
v_\mu &\mapsto v_\mu - i\partial_\mu(\phi - \bar{\phi}) \\
\eta_\alpha &\mapsto \eta_\alpha + \tfrac{i}{2}(\sigma^\mu)_{\alpha\dot{\beta}}\partial_\mu\bar{\chi}^{\dot{\beta}} \\
E &\mapsto E - \tfrac{1}{4}\,\square(\phi + \bar{\phi})\ .
\end{aligned}
$$

This result teaches us two things. First of all, we see that the combinations

$$
\begin{aligned}
\lambda_\alpha &= \eta_\alpha - \tfrac{i}{2}(\sigma^\mu)_{\alpha\dot{\beta}}\partial_\mu\bar{\xi}^{\dot{\beta}} \\
D &:= E - \tfrac{1}{4}\,\square\, C
\end{aligned}
\tag{72}
$$

are gauge invariant.

I hope that the gauge-invariant field D will not be confused with the supercovariant derivative. This abuse of notation has become far too ingrained in the supersymmetry literature for me to even attempt to correct it here.

Of these gauge-invariant quantities, it is λ_α which is the lowest component in the vector superfield. This suggests that we try to construct a gauge-invariant lagrangian out of a superfield having λ_α as its lowest component. Such a superfield turns out to be easy to construct, as we shall see in the next section.

The second thing we learn is that because the fields C, G and ξ transform by shifts, we can choose a special gauge in which they vanish. This gauge is called the *Wess–Zumino gauge* and it of course breaks supersymmetry. Nevertheless it is a very convenient gauge for calculations, as we will have ample opportunity to demonstrate. For now, let us merely notice that in the Wess–Zumino gauge the vector superfield becomes

$$
V = \bar{\theta}\bar{\sigma}^\mu\theta v_\mu + \bar{\theta}^2\theta\lambda + \theta^2\bar{\theta}\bar{\lambda} + \theta^2\bar{\theta}^2 D\ , \tag{73}
$$

and that powers are very easy to compute:

$$
V^2 = -\tfrac{1}{2}\theta^2\bar{\theta}^2 v_\mu v^\mu\ ,
$$

with all higher powers vanishing. This is not a gratuitous comment. We will see that in coupling to matter and indeed already in the nonabelian case, it will be necessary to compute the exponential of the vector superfield e^V, which in the Wess–Zumino gauge becomes simply

$$
e^V = 1 + \bar{\theta}\bar{\sigma}^\mu\theta v_\mu + \bar{\theta}^2\theta\lambda + \theta^2\bar{\theta}\bar{\lambda} + \theta^2\bar{\theta}^2\left(D - \tfrac{1}{4}v_\mu v^\mu\right)\ . \tag{74}
$$

Furthermore gauge transformations with imaginary parameter $\phi = -\bar{\phi}$ and $\chi = F = 0$ still preserve the Wess–Zumino gauge and moreover induce in the vector field v_μ the expected U(1) gauge transformations

$$v_\mu \mapsto v_\mu - i\partial_\mu(\phi - \bar{\phi}) \ . \tag{75}$$

V.2. **The gauge-invariant action.** Define the following spinorial superfields

$$W_\alpha := -\tfrac{1}{4}\bar{D}^2 D_\alpha V \qquad \text{and} \qquad \overline{W}_{\dot\alpha} := -\tfrac{1}{4}D^2 \bar{D}_{\dot\alpha} V \ . \tag{76}$$

Notice that reality of V implies that $(W_\alpha)^* = \overline{W}_{\dot\alpha}$. To show that the lowest component of W_α is λ_α it will be convenient to compute it in the Wess–Zumino gauge (73). This is allowed because W_α is actually gauge invariant, so it does not matter in which gauge we compute it.

Exercise V.3. *Prove that the supercovariant derivatives satisfy the following identities:*

$$\begin{aligned} \left[\bar{D}_{\dot\alpha}, \left[\bar{D}_{\dot\beta}, D_\gamma\right]\right] &= 0 \\ \bar{D}_{\dot\alpha}\bar{D}^2 &= 0 \ , \end{aligned} \tag{77}$$

and use them to prove that W_α is both chiral:

$$\bar{D}_{\dot\beta} W_\alpha = 0 \ ,$$

and gauge invariant. Use complex conjugation to prove that $\overline{W}_{\dot\alpha}$ is antichiral and gauge invariant. Finally, show that the following "real" equation is satisfied:

$$D^\alpha W_\alpha = \bar{D}^{\dot\alpha}\overline{W}_{\dot\alpha} \ . \tag{78}$$

In the Wess–Zumino gauge, the vector superfield V can be written as

$$V = e^{-iU}\left[\bar\theta\bar\sigma^\mu\theta v_\mu + \bar\theta^2\theta\lambda + \theta^2\bar\theta\bar\lambda + \theta^2\bar\theta^2\left(D + \tfrac{i}{2}\partial^\mu v_\mu\right)\right] \ , \tag{79}$$

where as usual $U = \theta\sigma^\mu\bar\theta\partial_\mu$.

Exercise V.4. *Using this fact show that*

$$\bar{D}^{\dot\alpha} V = e^{-iU}\left[-\theta_\alpha(\sigma^\mu)^{\alpha\dot\alpha}v_\mu + 2\bar\theta^{\dot\alpha}\theta\lambda + \theta^2\bar\lambda^{\dot\alpha} + 2\theta^2\bar\theta^{\dot\alpha}\left(D + \tfrac{i}{2}\partial^\mu v_\mu\right)\right]$$

and that

$$-\tfrac{1}{4}\bar{D}^2 V = e^{-iU}\left[\theta\lambda + \theta^2\left(D + \tfrac{i}{2}\partial^\mu v_\mu\right)\right] \ ,$$

and conclude that W_α takes the following expression

$$W_\alpha = e^{-iU}\left[\lambda_\alpha + 2\theta_\alpha D + \tfrac{i}{2}\theta_\beta(\sigma^{\mu\nu})^\beta{}_\alpha f_{\mu\nu} + i\theta^2(\bar\sigma^\mu)_{\dot\beta\alpha}\partial_\mu\bar\lambda^{\dot\beta}\right] \ , \tag{80}$$

where $f_{\mu\nu} = \partial_\mu v_\nu - \partial_\nu v_\mu$ is the field-strength of the vector v_μ.
(Hint: You may want to use the expressions (56) for the supercovariant derivatives.)

Since W_α is chiral, so is $W^\alpha W_\alpha$, which is moreover Lorentz invariant. The θ^2 component is also Lorentz invariant and transforms as a total derivative under supersymmetry. Its real part can therefore be used as a supersymmetric lagrangian.

Exercise V.5. *Show that*

$$\int d^2\theta\, W^\alpha W_\alpha = 2i\lambda\sigma^\mu\partial_\mu\bar\lambda + 4D^2 - \tfrac{1}{2}f_{\mu\nu}f^{\mu\nu} + \tfrac{i}{4}\epsilon^{\mu\nu\rho\sigma}f_{\mu\nu}f_{\rho\sigma}\ , \tag{81}$$

and hence that its real part is given by

$$i\left(\lambda\sigma^\mu\partial_\mu\bar\lambda + \bar\lambda\bar\sigma^\mu\partial_\mu\lambda\right) - \tfrac{1}{2}f_{\mu\nu}f^{\mu\nu} + 4D^2\ . \tag{82}$$

It may seem from this expression that the supersymmetric Yang–Mills lagrangian involves an integral over chiral superspace, and perhaps that a similar nonrenormalisation theorem to the one for chiral superfields would prevent the Yang–Mills coupling constant to renormalise. This is *not* true. In fact, a closer look at the expression for the supersymmetric Yang–Mills reveals that it can be written as an integral over all of superspace, since the $\bar{D}^2$ in the definition of W_α acts like a $\int d^2\bar\theta$. In other words, counterterms can *and do* arise which renormalise the supersymmetric Yang–Mills action.

Now consider the supersymmetric Yang–Mills action with lagrangian (38) for the special case of the abelian group $G = \mathrm{U}(1)$. The resulting theory is free. Let $\Psi^a = (\psi^\alpha, \bar\psi_{\dot\alpha})$. Expanding the lagrangian we obtain

$$\mathcal{L}_{\mathrm{SYM}} = \tfrac{i}{4}\left(\psi\sigma^\mu\partial_\mu\bar\psi + \bar\psi\bar\sigma^\mu\partial_\mu\psi\right) - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}\ , \tag{83}$$

which agrees with half the lagrangian (82) provided that we eliminate the auxiliary field D and identify $A_\mu = v_\mu$ and $\psi_\alpha = \lambda_\alpha$. Actually, this last field identification has a phase ambiguity, and we will fix it by matching the supersymmetry transformation properties (39) with the ones obtained in superspace: $-(\varepsilon Q + \bar\varepsilon\bar{Q})V$.

V.3. **Supersymmetry transformations.** We can (and will) simplify the computation by working in the Wess–Zumino gauge. However it should be noticed that this gauge breaks supersymmetry; that is, the supersymmetry variation of a vector superfield in the Wess–Zumino gauge will not remain in the Wess–Zumino gauge. In order to get it back to this gauge it will be necessary to perform a compensating gauge transformation. This is a common trick in supersymmetry and it's worth doing it in some detail.

Exercise V.6. *Compute the supersymmetry transformation of a vector superfield V in the Wess–Zumino gauge (73) to obtain*

$$\begin{aligned}
-(\varepsilon Q + \bar\varepsilon \bar Q)V = \theta\sigma^\mu\bar\varepsilon v_\mu - \bar\theta\bar\sigma^\mu\varepsilon v_\mu - \theta^2\bar\varepsilon\bar\lambda - \bar\theta^2\varepsilon\lambda \\
+ \bar\theta\bar\sigma^\mu\theta\left(\bar\varepsilon\bar\sigma_\mu\lambda - \varepsilon\sigma_\mu\bar\lambda\right) \\
- 2\theta^2\bar\theta\bar\varepsilon\left(D - \tfrac{i}{4}\partial^\mu v_\mu\right) - 2\bar\theta^2\theta\varepsilon\left(D + \tfrac{i}{4}\partial^\mu v_\mu\right) \\
- \tfrac{i}{4}\theta^2\bar\theta\bar\sigma^{\mu\nu}\bar\varepsilon f_{\mu\nu} - \tfrac{i}{4}\bar\theta^2\theta\sigma^{\mu\nu}\varepsilon f_{\mu\nu} \\
+ \tfrac{i}{2}\theta^2\bar\theta^2\left(\varepsilon\sigma^\mu\partial_\mu\bar\lambda + \bar\varepsilon\bar\sigma^\mu\partial_\mu\lambda\right)\ .
\end{aligned} \tag{84}$$

As advertised, the resulting variation is not in the Wess–Zumino gauge. Nevertheless we can gauge transform it back to the Wess–Zumino gauge. Indeed, we can find a chiral superfield Λ with component fields ϕ, χ and F such that

$$\delta_\varepsilon V = -(\varepsilon Q + \bar\varepsilon\bar Q)V - (\Lambda + \bar\Lambda) \tag{85}$$

is again in the Wess–Zumino gauge. To do this notice that the first four terms in the expansion (84) of $-(\varepsilon Q + \bar\varepsilon\bar Q)V$ have to vanish in the Wess–Zumino gauge. This is enough to fix Λ up to the imaginary part of ϕ, which simply reflects the gauge invariance of the component theory.

Exercise V.7. *Show that the parameters of the compensating gauge transformation are given by (where we have chosen the imaginary part of ϕ to vanish)*

$$\begin{aligned}
\phi &= 0 \\
\chi^\alpha &= -(\sigma^\mu)^{\alpha\dot\alpha}\bar\varepsilon_{\dot\alpha} v_\mu \\
F &= -\bar\varepsilon\bar\lambda\ ,
\end{aligned} \tag{86}$$

and hence that

$$\begin{aligned}
\delta_\varepsilon V &= -(\varepsilon Q + \bar\varepsilon\bar Q)V - (\Lambda + \bar\Lambda) \\
&= \theta\sigma^\mu\bar\theta\delta_\varepsilon v_\mu + \bar\theta^2\theta\delta_\varepsilon\lambda + \theta^2\bar\theta\delta_\varepsilon\bar\lambda + \theta^2\bar\theta^2\delta_\varepsilon D\ ,
\end{aligned}$$

with

$$\begin{aligned}
\delta_\varepsilon v_\mu &= \varepsilon\sigma_\mu\bar\lambda - \bar\varepsilon\bar\sigma^\mu\lambda \\
\delta_\varepsilon\lambda_\alpha &= -2\varepsilon_\alpha D + \tfrac{i}{2}(\sigma^{\mu\nu})_{\alpha\beta}\epsilon^\beta f_{\mu\nu} \\
\delta_\varepsilon D &= \tfrac{i}{2}\left(\varepsilon\sigma^\mu\partial_\mu\bar\lambda + \bar\varepsilon\bar\sigma^\mu\partial_\mu\lambda\right)\ .
\end{aligned} \tag{87}$$

Rewriting the supersymmetry transformations (39) of supersymmetric Yang–Mills (for $G = \mathrm{U}(1)$) in terms of $\Psi^a = (\psi^\alpha, \bar\psi_{\dot\alpha})$ we obtain

$$\begin{aligned}
\delta_\varepsilon A_\mu &= -i(\bar\varepsilon\bar\sigma_\mu\psi + \varepsilon\sigma_\mu\bar\psi) \\
\delta_\varepsilon\psi_\alpha &= -\tfrac{1}{2}F_{\mu\nu}(\sigma^{\mu\nu})_{\alpha\beta}\varepsilon^\beta\ .
\end{aligned}$$

Therefore we see that they agree with the transformations (87) provided that as before we identify $v_\mu = A_\mu$, but now $\psi_\alpha = i\lambda_\alpha$.

In summary, supersymmetric Yang–Mills theory (38) with gauge group $\mathrm{U}(1)$ can be written in superspace in terms of a vector superfield

V which in the Wess–Zumino gauge has the expansion

$$V = \bar{\theta}\bar{\sigma}^{\mu}\theta A_{\mu} - i\bar{\theta}^2\theta\psi + i\theta^2\bar{\theta}\bar{\psi} + \theta^2\bar{\theta}^2 D \ ,$$

with lagrangian given by

$$\mathcal{L}_{\text{SYM}} = \int d^2\theta \, \tfrac{1}{4} W^{\alpha} W_{\alpha} + \text{c.c.} \ ,$$

with W_{α} given by (76).

V.4. **Coupling to matter.** Let us couple the above theory to matter in the form in one chiral superfield. We will postpone discussing more general matter couplings until we talk about nonabelian gauge theories.

Consider a chiral superfield Φ in a one-dimensional representation of the group U(1) with charge e. That is to say, if $\exp(i\varphi) \in \mathrm{U}(1)$ then its action on Φ is given by

$$\exp(i\varphi) \cdot \Phi = e^{ie\varphi}\Phi \qquad \text{and} \qquad \exp(i\varphi) \cdot \bar{\Phi} = e^{-ie\varphi}\bar{\Phi} \ .$$

The kinetic term $\bar{\Phi}\Phi$ is clearly invariant. If we wish to promote this symmetry to a gauge symmetry, we need to consider parameters $\varphi(x)$ which are functions on Minkowski space. However, $e^{ie\varphi(x)}\Phi$ is not a chiral superfield and hence this action of the gauge group does not respect supersymmetry. To cure this problem we need to promote φ to a full chiral superfield Λ, so that the gauge transformation now reads

$$\Phi \mapsto e^{ie\Lambda}\Phi \ . \tag{88}$$

Now the gauge transformed superfield remains chiral, but we pay the price that the kinetic term $\bar{\Phi}\Phi$ is no longer invariant. Indeed, it transforms as

$$\bar{\Phi}\Phi \mapsto \bar{\Phi}\Phi e^{ie(\Lambda - \bar{\Lambda})} \ .$$

However, we notice that $i(\Lambda - \bar{\Lambda})$ is a real superfield and hence can be reabsorbed in the gauge transformation of a vector superfield V:

$$V \mapsto V - \tfrac{i}{2}(\Lambda - \bar{\Lambda}) \ , \tag{89}$$

in such a way that the expression

$$\bar{\Phi}e^{2eV}\Phi$$

is gauge invariant under (88) and (89).

The coupled theory is now defined by the lagrangian

$$\int d^2\theta d^2\bar{\theta} \, \bar{\Phi}e^{2eV}\Phi + \left[\int d^2\theta \, \tfrac{1}{4} W^{\alpha} W_{\alpha} + \text{c.c}\right] \ , \tag{90}$$

which can be understood as the supersymmetric version of scalar QED.

The coupling term might look nonpolynomial (and hence nonrenormalisable), but since it is gauge invariant it can be computed in the Wess–Zumino gauge where $V^3 = 0$.

Exercise V.8. *Show that the component expansion of the lagrangian* (90)*, with* Φ *given by* (57)*,* V *in the Wess–Zumino gauge by* (73) *and having eliminated the auxiliary fields, is given by*

$$\begin{aligned} -\tfrac{1}{4} f_{\mu\nu} f^{\mu\nu} + \tfrac{i}{2}\left(\lambda\sigma^\mu\partial_\mu\bar\lambda + \bar\lambda\bar\sigma^\mu\partial_\mu\lambda\right) + \tfrac{i}{4}\left(\chi\sigma^\mu\overline{\mathcal{D}_\mu\chi} + \bar\chi\bar\sigma^\mu\mathcal{D}\chi\right) \\ -\mathcal{D}^\mu\phi\overline{\mathcal{D}_\mu\phi} - e\left(\bar\phi\lambda\chi + \phi\bar\lambda\bar\chi\right) - \tfrac{1}{2}e^2\left(|\phi|^2\right)^2 \ , \end{aligned} \tag{91}$$

where $\mathcal{D}_\mu\phi = \partial_\mu\phi - iev_\mu\phi$ *and similarly for* $\mathcal{D}_\mu\chi$*.*

The above model does not allow massive charged matter, since the mass term in the superpotential is not gauge invariant. In order to consider massive matter, and hence supersymmetric QED, it is necessary to include two oppositely charged chiral superfields $\Phi_\pm$, transforming under the U(1) gauge group as

$$\Phi_\pm \mapsto e^{\pm ie\Lambda}\Phi_\pm \ .$$

Then the supersymmetric QED lagrangian in superspace is given by

$$\begin{aligned} \int d^2\theta d^2\bar\theta \, \left(\bar\Phi_+ e^{2eV}\Phi_+ + \bar\Phi_- e^{-2eV}\Phi_-\right) \\ + \left[\int d^2\theta \, \left(\tfrac{1}{4}W^\alpha W_\alpha + m\Phi_+\Phi_-\right) + \text{c.c.}\right] \ . \end{aligned} \tag{92}$$

Exercise V.9. *Expand the supersymmetric QED lagrangian in components and verify that it describes a massless gauge boson (the* photon*) and a charged massive fermion (the* electron*), as well as a a massless neutral fermion (the* photino*) and a a massive charged scalar (the* selectron*).*

Detractors often say, with some sarcasm, that supersymmetry is doing well: already half the particles that it predicts have been found.

The coupling of supersymmetric gauge fields to supersymmetric matter suggests that the fundamental object is perhaps not the vector superfield V itself but its exponential $\exp V$, which in the Wess–Zumino gauge is not too different an object—compare equations (73) and (74). One might object that the supersymmetric field-strength W_α actually depends on V and not on its exponential, but this is easily circumvented by rewriting it thus:

$$W_\alpha = -\tfrac{1}{4}\bar D^2 e^{-V} D_\alpha e^V \ . \tag{93}$$

It turns out that this observation facilitates enormously the construction of nonabelian supersymmetric Yang–Mills theory in superspace.

V.5. **Nonabelian gauge symmetry.** As in Lecture II, let G be a compact Lie group with Lie algebra $\mathfrak{g}$ and fix an invariant inner product, denoted by Tr in the Lie algebra. The vector superfield V now takes values in $\mathfrak{g}$. Relative to a fixed basis $\{T_i\}$ for $\mathfrak{g}$ we can write

$$V = iV^i T_i \ , \tag{94}$$

where as we will see, the factor of i will guarantee that the superfields V^i are real.

The expression (93) for the field-strength makes sense for a Lie algebra valued V, since the only products of generators T_i appearing in the expression are in the form of commutators. The form of the gauge transformations can be deduced by coupling to matter.

Suppose that $\mathbf{\Phi}$ is a chiral superfield taking values in a unitary representation of G. This means that under a gauge transformation, $\mathbf{\Phi}$ transforms as

$$\mathbf{\Phi} \mapsto e^{\Lambda}\mathbf{\Phi} ,$$

where Λ is an antihermitian matrix whose entries are chiral superfields. The conjugate superfield $\bar{\mathbf{\Phi}}$ takes values in the conjugate dual representation; this means that now $\bar{\Phi}$ denotes the conjugate transpose. Under a gauge transformation, it transforms according to

$$\bar{\mathbf{\Phi}} \mapsto \bar{\mathbf{\Phi}}\, e^{\bar{\Lambda}} ,$$

where $\bar{\Lambda}$ is now the *hermitian* conjugate of Λ. Consider the coupling

$$\bar{\mathbf{\Phi}} e^{V} \mathbf{\Phi} . \tag{95}$$

Reality imposes that V be hermitian,

$$\bar{V} = V \tag{96}$$

where $\bar{V}$ is now the hermitian conjugate of V. Since the T_i are antihermitian, this means that the components V^i in (94) are vector superfields: $\bar{V}^i = V^i$. Gauge invariance implies that V should transform according to

$$e^{V} \mapsto e^{-\bar{\Lambda}} e^{V} e^{-\Lambda} . \tag{97}$$

We can check that the field-strength (93) transforms as expected under gauge transformations.

✍ **Exercise V.10.** *Show that the field-strength* (93) *transforms covariantly under the gauge transformation* (97)*:*

$$W_\alpha \mapsto e^{\Lambda} W_\alpha e^{-\Lambda} ,$$

and conclude that

$$\int d^2\theta \ \mathrm{Tr}\, W^\alpha W_\alpha$$

is gauge invariant

In order to compare this to the component version of supersymmetric Yang–Mills we would like to argue that we can compute the action in the Wess–Zumino gauge, but this requires first showing the existence of this gauge. The nonabelian gauge transformations (97) are hopelessly complicated in terms of V, but using the Baker–Campbell–Hausdorff formula (A-1) we can compute the first few terms and argue that the Wess–Zumino gauge exists.

Exercise V.11. *Using the Baker–Campbell–Hausdorff formula* (A-1)*, show that the nonabelian gauge transformations* (97) *takes the form*

$$V \mapsto V - (\Lambda + \bar\Lambda) - \tfrac{1}{2}[V, \Lambda - \bar\Lambda] + \cdots \ ,$$

and conclude that V can be put in the Wess–Zumino gauge (73) *by a judicious choice of $\Lambda + \bar\Lambda$.*

Notice that in the Wess–Zumino gauge, infinitesimal gauge transformations simplify tremendously. In fact, since $V^3 = 0$, the gauge transformation formula (97) for infinitesimal Λ, reduces to

$$V \mapsto V - (\Lambda + \bar\Lambda) - \tfrac{1}{2}[V, \Lambda - \bar\Lambda] - \tfrac{1}{12}[V, [V, \Lambda + \bar\Lambda]] \ . \tag{98}$$

Notice that an infinitesimal gauge transformation which preserves the Wess–Zumino gauge has the form

$$\Lambda = \omega + i\bar\theta\bar\sigma^\mu\theta\partial_\mu\omega + \tfrac{1}{4}\theta^2\bar\theta^2 \,\Box\, \omega \ , \tag{99}$$

for some Lie algebra-valued scalar field ω obeying $\bar\omega = -\omega$. In this case, the term in V^2 in the transformation law (98) is absent, as it has too many θ's.

Exercise V.12. *Show that the infinitesimal gauge transformation*

$$V \mapsto V - (\Lambda + \bar\Lambda) - \tfrac{1}{2}[V, \Lambda - \bar\Lambda]$$

for V in the Wess–Zumino gauge and with parameter Λ given by (99)*, induces the following transformation of the component fields:*

$$\begin{aligned} \delta_\omega v_\mu &= -2i\partial_\mu\omega - [v_\mu, \omega] \\ \delta_\omega \chi &= -[\chi, \omega] \\ \delta_\omega D &= -[D, \omega] \ . \end{aligned}$$

Conclude that $A_\mu = \frac{1}{2gi} v_\mu$, where g is the Yang–Mills coupling constant, obeys the transformation law (34) *of a gauge field.*

This result suggests that in order to identify the fields in the component formulation of supersymmetric Yang–Mills, we have to rescale the nonabelian vector superfield by $2g$, with g the Yang–Mills coupling constant. In order to obtain a lagrangian with the correct normalisation for the kinetic term, we also rescale the spinorial field strength by $1/(2g)$:

$$W_\alpha := -\tfrac{1}{8g}\bar D^2 e^{-2g\,V} D_\alpha e^{2g\,V} \ . \tag{100}$$

V.6. **Nonabelian gauge-invariant action.** We now construct the nonabelian gauge-invariant action. We will do this in the Wess–Zumino gauge, but we should realise that the nonabelian field-strength is no longer gauge invariant. Nevertheless we are after the superspace lagrangian $\operatorname{Tr} W^\alpha W_\alpha$, which is gauge invariant.

Exercise V.13. *Show that in Wess–Zumino gauge*

$$e^{-V} D_\alpha e^V = D_\alpha V - \tfrac{1}{2}[V, D_\alpha V] \ , \tag{101}$$

and use this to find the following expression for the nonabelian field-strength W_α *in* (100)*:*

$$W_\alpha = e^{-iU}\left[\lambda_\alpha + 2\theta_\alpha D + \tfrac{i}{2}\theta_\beta (\sigma^{\mu\nu})^\beta{}_\alpha f_{\mu\nu} + i\theta^2 \mathcal{D}_\mu \bar\lambda^{\dot\alpha} (\bar\sigma^\mu)_{\dot\alpha\alpha}\right] \ , \tag{102}$$

where

$$\begin{aligned} f_{\mu\nu} &= \partial_\mu v_\nu - \partial_\nu v_\mu - ig[v_\mu, v_\nu] \\ \mathcal{D}_\mu \lambda &= \partial_\mu \lambda - ig[v_\mu, \lambda] \ . \end{aligned}$$

The factors of i have to do with the fact that $v_\mu = v_\mu i T_i$. In terms of $A_\mu = -i v_\mu$ these expressions are standard:

$$\begin{aligned} f_{\mu\nu} &= i\left(\partial_\mu A_\nu - \partial_\nu A_\mu + g[A_\mu, A_\nu]\right) \\ \mathcal{D}_\mu \lambda &= \partial_\mu \lambda + g[A_\mu, \lambda] \ . \end{aligned}$$

Comparing (102) with the abelian version (80), we can use the results of Exercise V.5 to arrive at the component expansion for the lagrangian

$$\mathcal{L}_{\text{SYM}} = \int d^2\theta \tfrac{1}{4} \operatorname{Tr} W^\alpha W_\alpha + \text{c.c} \tag{103}$$

for (pure, nonabelian) supersymmetric Yang–Mills. Expanding in components, we obtain

$$\mathcal{L}_{\text{SYM}} = \tfrac{i}{2} \operatorname{Tr}\left(\lambda \sigma^\mu \mathcal{D}_\mu \bar\lambda + \bar\lambda \bar\sigma^\mu \mathcal{D}_\mu \lambda\right) - \tfrac{1}{4} \operatorname{Tr} f_{\mu\nu} f^{\mu\nu} + 2 \operatorname{Tr} D^2 \ . \tag{104}$$

In order to fix the correspondence with the component theory discussed in Lecture II, we need again to compare the supersymmetry transformations. As in the abelian theory this is once again easiest to do in the Wess–Zumino gauge, provided that we then perform a compensating gauge transformation to get the result back to that gauge. In other words, we define the supersymmetry transformation of the nonabelian vector superfield V in the Wess–Zumino gauge by

$$\begin{aligned} \delta_\varepsilon V &= \theta\sigma^\mu\bar\theta \delta_\varepsilon v_\mu + \bar\theta^2 \theta \delta_\varepsilon \lambda + \theta^2 \bar\theta \delta_\varepsilon \bar\lambda + \theta^2 \bar\theta^2 \delta_\varepsilon D \\ &= -(\varepsilon Q + \bar\varepsilon \bar Q) V - (\Lambda + \bar\Lambda) - \tfrac{1}{2}\left[V, \Lambda - \bar\Lambda\right] - \tfrac{1}{12}\left[V, \left[V, \Lambda + \bar\Lambda\right]\right] \ , \end{aligned}$$

where Λ is chosen in such a way that the right hand side in the second line above is again in the Wess–Zumino gauge. This calculation has been done already in the abelian case in Exercise V.6 and we can use much of that result. The only difference in the nonabelian case are the commutator terms in the expression of the gauge transformation: compare the above expression for $\delta_\varepsilon V$ and equation (85).

Exercise V.14. *Let V be a nonabelian vector superfield in the Wess–Zumino gauge. Follow the procedure outlined above to determine the supersymmetry transformation laws of the component fields. In other words, compute*

$$\delta_\varepsilon V := -(\varepsilon Q + \bar\varepsilon \bar Q)V - (\Lambda + \bar\Lambda) - \tfrac{1}{2}\left[V, \Lambda - \bar\Lambda\right] - \tfrac{1}{12}\left[V, \left[V, \Lambda + \bar\Lambda\right]\right]$$

for an appropriate Λ and show that, after rescaling the vector superfield $V \mapsto 2gV$, one obtains

$$\begin{aligned} \delta_\varepsilon v_\mu &= i\left(\varepsilon\sigma_\mu\bar\lambda + \bar\varepsilon\bar\sigma_\mu\lambda\right)\\ \delta_\varepsilon\lambda_\alpha &= -2\varepsilon_\alpha D + \tfrac{i}{2}(\sigma^{\mu\nu})_{\alpha\beta}\varepsilon^\beta f_{\mu\nu}\\ \delta_\varepsilon D &= \tfrac{i}{2}\left(\bar\varepsilon\bar\sigma^\mu \mathcal{D}_\mu\lambda - \varepsilon\sigma^\mu\overline{\mathcal{D}_\mu\lambda}\right)\ . \end{aligned} \tag{105}$$

Now expand the supersymmetry transformation law (39) *with $\Psi = (\psi^\alpha, \bar\psi_{\dot\alpha})$ and show that the result agrees with* (105) *after eliminating the auxiliary field, and provided that we identify $A_\mu = -iv_\mu$ and $\psi^\alpha = i\lambda_\alpha$.*

In summary, the supersymmetric Yang–Mills theory discussed in Lecture II has a superspace description in terms of a vector superfield

$$V = i\bar\theta^2\bar\sigma^\mu\theta A_\mu - i\bar\theta^2\theta\psi + i\theta^2\bar\theta\bar\psi + \theta^2\bar\theta^2 D$$

with lagrangian

$$\int d^2\theta\ \mathrm{Tr}\,\tfrac{1}{4}W^\alpha W_\alpha + \text{c.c.}\ ,$$

where W_α is given by (100).

To be perfectly honest we have omitted one possible term in the action which is present whenever the center of the Lie algebra $\mathfrak{g}$ is nontrivial; that is, whenever there are U(1) factors in the gauge group. Consider the quantity $\mathrm{Tr}\,\kappa V$ where $\kappa = \kappa^i T_i$ is a constant element in the center of the Lie algebra. This yields a term in the action called a *Fayet–Iliopoulos* term and, as we will see in Lecture VI, it plays an important role in the spontaneous breaking of supersymmetry.

Exercise V.15. *Show that the Fayet–Iliopoulos term*

$$\int d^2\theta\, d^2\bar\theta\ \mathrm{Tr}\,\kappa V = \mathrm{Tr}\,\kappa D$$

is both supersymmetric and gauge-invariant.

V.7. **Gauge-invariant interactions.** Having constructed the gauge-invariant action for pure supersymmetric Yang–Mills and having already seen the coupling to matter

$$\int d^2\theta d^2\bar\theta\ \bar{\mathbf{\Phi}} e^{2gV}\mathbf{\Phi}\ , \tag{106}$$

there remains one piece of the puzzle in order to be able to construct the most general renormalisable supersymmetric field theory in four dimensions: a gauge-invariant superpotential. On dimensional grounds, we saw that the most general renormalisable superpotential is a cubic polynomial

$$W(\mathbf{\Phi}) = a_I \Phi^I + \tfrac{1}{2} m_{IJ} \Phi^I \Phi^J + \tfrac{1}{3} \lambda_{IJK} \Phi^I \Phi^J \Phi^K \tag{107}$$

where the $\{\Phi^I\}$ are chiral superfields—the components of $\mathbf{\Phi}$ relative to some basis $\{\boldsymbol{e}_I\}$ for the representation.

Exercise V.16. *Prove that $W(\mathbf{\Phi})$ is gauge invariant if and only if a_I, m_{IJ} and λ_{IJK} are (symmetric) invariant tensors in the representation corresponding to $\mathbf{\Phi}$.*

Let us end by summarising what we have learned in this lecture. The general renormalisable supersymmetric action is built out of vector superfields V taking values in the Lie algebra of a compact Lie group G and a chiral superfield $\mathbf{\Phi}$ taking values in a unitary representation, not necessarily irreducible. The lagrangian can be written as follows:

$$\int d^2\theta d^2\bar\theta \, \left(\bar{\mathbf{\Phi}} e^{2gV} \mathbf{\Phi} + \operatorname{Tr} \kappa V \right) + \left[\int d^2\theta \, \left(\tfrac{1}{4} \operatorname{Tr} W^\alpha W_\alpha + W(\mathbf{\Phi}) \right) + \text{c.c.} \right] , \tag{108}$$

with $W(\mathbf{\Phi})$ given in (107) where a_I, m_{IJ} and λ_{IJK} are (symmetric) G-invariant tensors in the matter representation.

Strictly speaking when the group is not simple, one must then restore the Yang–Mills coupling separately in each factor of the Lie algebra by rescaling the corresponding vector superfield by $2g$, where the coupling constant g can be different for each factor, and rescaling the spinorial field-strength accordingly. This is possible because the Lie algebra of a compact Lie group splits as the direct product of several simple Lie algebras and an abelian Lie algebra, itself the product of a number of U(1)'s. The Yang–Mills superfield breaks up into the different factors and neither the metric nor the Lie bracket couples them.

We end this lecture by mentioning the names of the particles associated with the dynamical fields in the different superfields. In the vector superfield, the vector corresponds to the gauge bosons, whereas its fermionic superpartner is the *gaugino*. The supersymmetric partner of the photon and the gluons are called the *photino* and *gluinos*, respectively. There are two kinds of chiral superfield in phenomenological models, corresponding to the Higgs scalars and the quarks and leptons. In the former case the scalars are the Higgs fields and their fermionic partners are the *Higgsinos*. In the latter case, the fermions correspond to either quarks or leptons and their bosonic partners are the *squarks* and *sleptons*.

Problem 5 (Kähler quotients and the $\mathbb{C}P^N$ model).

In this problem we will study the "moduli space of vacua" of a supersymmetric gauge theory and show that, in the absence of superpotential, it is given by a "Kähler quotient." The low-energy effective theory is generically a sigma model in the moduli space of vacua and we will illustrate this in the so-called $\mathbb{C}P^N$ model.

Let Φ^I, for $I = 1, \ldots, N$ be N chiral superfields, which we will assemble into an N-dimensional vector $\boldsymbol{\Phi}$. Let $\bar{\boldsymbol{\Phi}}$ denote the conjugate transpose vector. It is an N-dimensional vector of antichiral superfields.

(1) Check that the kinetic term

$$\int d^2\theta \, d^2\bar{\theta} \, \bar{\boldsymbol{\Phi}}\boldsymbol{\Phi} \ ,$$

is invariant under the natural action of $\mathrm{U}(N)$

$$\boldsymbol{\Phi} \mapsto e^X \boldsymbol{\Phi} \ ,$$

where X is a constant antihermitian matrix.

Let us gauge a subgroup $G \subset \mathrm{U}(N)$ in this model by introducing a nonabelian vector superfield $V = V^i(iT_i)$, where $\{T_i\}$ is a basis for the Lie algebra $\mathfrak{g}$ of G. Since G is a subgroup of the unitary group, the T_i are antihermitian matrices. As we have seen in this lecture, the coupled theory has the following lagrangian

$$\int d^2\theta \, d^2\bar{\theta} \, \left(\bar{\boldsymbol{\Phi}} e^{2g\,V} \boldsymbol{\Phi} - 2g\kappa^2 \operatorname{Tr} V\right) + \left[\int d^2\theta \, \operatorname{Tr} \tfrac{1}{4} W^\alpha W_\alpha + \text{c.c.}\right] \ ,$$

where W_α is given in (100), and where we have introduced a conveniently normalised Fayet–Iliopoulos term, since G may have an abelian factor.

A choice of vacuum expectation values of the dynamical scalars in the chiral superfield yields a point $z^I = \langle \phi^I \rangle$ in $\mathbb{C}^N$. Let $\mathcal{M}_0 \subset \mathbb{C}^N$ correspond to those points $\boldsymbol{z} = (z^I)$ which minimise the potential of the theory.

2. Show that $\mathcal{M}_0$ consists of those points $\boldsymbol{z}$ in $\mathbb{C}^N$ such that

$$\bar{\boldsymbol{z}} T_i \boldsymbol{z} = \kappa^2 \operatorname{Tr} T_i \qquad \text{for all } i,$$

and that the potential is identically zero there.

Notation: Let $\mathfrak{g}^*$ denote the dual vector space of the Lie algebra $\mathfrak{g}$. Let us define a *momentum map* $\mu : \mathbb{C}^N \to \mathfrak{g}^*$ as follows. If $\boldsymbol{z} \in \mathbb{C}^N$ then $\mu(\boldsymbol{z})$ is the linear functional on $\mathfrak{g}$ which sends $X \in \mathfrak{g}$ to the *real* number

$$\langle \mu(\boldsymbol{z}), X \rangle := i \left(\bar{\boldsymbol{z}} X \boldsymbol{z} - \kappa^2 \operatorname{Tr} X\right) \ .$$

3. Show that $\mathcal{M}_0$ agrees with $\mu^{-1}(0)$; in other words,

$$\boldsymbol{z} \in \mathcal{M}_0 \iff \mu(\boldsymbol{z}) = 0 \ .$$

Since we have identified $\mathbb{C}^N$ as the space of vacuum expectation values of the dynamical scalar fields, the action of G on the fields induces an action of G on $\mathbb{C}^N$:

$$\boldsymbol{z} \mapsto e^X \boldsymbol{z} \ ,$$

where $X \in \mathfrak{g}$ is an antihermitian matrix.

4. Show that $\mathcal{M}_0$ is preserved by the action of G, so that if $\boldsymbol{z} \in \mathcal{M}_0$ then so does $e^X \boldsymbol{z}$ for all $X \in \mathfrak{g}$.

Since in a gauge theory field configurations which are related by a gauge transformations are physically indistinguishable, we have to identify gauge related vacua $\boldsymbol{z} \in \mathcal{M}_0$. This means that the *moduli space of vacua* is the quotient

$$\mathcal{M} := \mathcal{M}_0/G \ ,$$

which by the above result is well-defined. It can be shown that $\mathcal{M}$ admits a natural Kähler metric. With this metric, $\mathcal{M}$ is called the *Kähler quotient* of $\mathbb{C}^N$ by G. It is often denoted $\mathbb{C}^N//G$.

☞ One of the beautiful things about supersymmetry is that it allows us to understand this fact in physical terms. At low energies, only the lightest states will contribute to the dynamics. The scalar content of the low-energy effective theory is in fact a sigma model on the moduli space of vacua. We will see in the next lecture that since the potential vanishes on the space of vacua, supersymmetry is unbroken. This means that the low-energy effective theory is supersymmetric; but by Problem 4 we know that the supersymmetric sigma models are defined on manifolds admitting Kähler metrics. Therefore $\mathcal{M}$ must have a Kähler metric. In fact, it is possible to work out the form of this metric exactly at least in one simple, but important, example: the $\mathbb{C}P^N$ model, the Kähler quotient of $\mathbb{C}^{N+1}$ by U(1).

Let us take $N+1$ chiral superfields $\boldsymbol{\Phi} = (\Phi^I)$ for $I = 0, 1, \ldots, N$ and gauge the natural U(1) action

$$\boldsymbol{\Phi} \mapsto e^{i\vartheta} \boldsymbol{\Phi} \ ,$$

with $\vartheta \in \mathbb{R}$. To simplify matters, let us take $2g = \kappa = 1$. We have one vector superfield $V = \bar{V}$. The lagrangian is given by

$$\int d^2\theta\, d^2\bar{\theta}\, \left(\bar{\boldsymbol{\Phi}} e^V \boldsymbol{\Phi} - V\right) + \left[\int d^2\theta \ \mathrm{Tr}\, \tfrac{1}{4} W^\alpha W_\alpha + \text{c.c.}\right] \ .$$

The space $\mathcal{M}_0$ of minima of the potential is the unit sphere in $\mathbb{C}^{N+1}$:

$$\bar{\boldsymbol{z}}\boldsymbol{z} = 1 \ .$$

The moduli space of vacua is obtained by identifying each $\boldsymbol{z}$ on the unit sphere with $e^{i\vartheta}\boldsymbol{z}$ for any $\vartheta \in \mathbb{R}$. The resulting space is a compact smooth manifold, denoted $\mathbb{C}P^N$ and called the complex projective space. It is the space of complex lines through the origin in $\mathbb{C}^{N+1}$. The natural Kähler metric on $\mathbb{C}P^N$ is the so-called Fubini–Study metric. Let us see how supersymmetry gives rise to this metric.

5. Choose a point in $\mathcal{M}_0$ and expanding around that point, show that the U(1) gauge symmetry is broken and that the photon acquires a mass.

Since supersymmetry is not broken (see the next lecture) its superpartner, the photino, also acquires a mass. For energies lower than the mass of these fields, we can disregard their dynamics. The low-energy effective action becomes then

$$\int d^2\theta\, d^2\bar{\theta}\, \left(\bar{\mathbf{\Phi}} e^V \mathbf{\Phi} - V\right) \ .$$

6. Eliminate V using its (algebraic) equations of motion to obtain the following action:

$$\int d^2\theta\, d^2\bar{\theta}\, \log(\bar{\mathbf{\Phi}}\mathbf{\Phi}) \ .$$

7. Show that this action is still invariant under the abelian gauge symmetry $\mathbf{\Phi} \mapsto e^{i\Lambda}\mathbf{\Phi}$, with Λ a chiral superfield.
8. Use the gauge symmetry to fix, $\Phi^0 = 1$, say, and arrive at the following action in terms of the remaining chiral superfields Φ^I, $I = 1, \ldots, N$:

$$\int d^2\theta\, d^2\bar{\theta}\, \log(1 + \sum_{I=1}^{N} \Phi^I \bar{\Phi}_I) \ .$$

This is only possible at those points where ϕ^0 is different from zero. This simply reflects the fact that $\mathbb{C}P^N$, like most manifolds, does not have global coordinates.

9. Expand the action in components to obtain

$$-g_{I\bar{J}}(\phi, \bar{\phi})\partial_\mu \phi^I \partial^\mu \bar{\phi}^{\bar{J}} + \cdots$$

where $g_{I\bar{J}}$ is the Fubini–Study metric for $\mathbb{C}P^N$. Find the metric explicitly.

VI. Spontaneous supersymmetry breaking

In the previous lecture we have learned how to write down renormalisable supersymmetric models in four dimensions. However if supersymmetry is a symmetry of nature, it must be broken, since we do not observe the mass degeneracy between bosons and fermions that unbroken supersymmetry demands. There are three common ways to break supersymmetry:

- Introducing symmetry breaking terms explicitly in the action (*soft*);
- Breaking tree-level supersymmetry by quantum effects, either perturbatively or nonperturbatively (*dynamical*); and
- Breaking supersymmetry due to a choice of non-invariant vacuum (*spontaneous*).

We will not discuss dynamical supersymmetry breaking in these lectures, except to note that nonrenormalisation theorems usually forbid the perturbative dynamical breaking of supersymmetry. Neither will we discuss soft supersymmetry breaking, except to say that this means that the supersymmetric current is no longer conserved, and this forbids the coupling to (super)gravity. We will concentrate instead on spontaneous supersymmetry breaking.

STOP I should emphasise, however, that from the point of view of supersymmetric field theories (that is, ignoring (super)gravity) the most realistic models do involve soft breaking terms. These terms are the low-energy manifestation of the spontaneous breaking (at some high energy scale) of local supersymmetry, in which the gravitino acquires a mass via the super-Higgs mechanism.

VI.1. **Supersymmetry breaking and vacuum energy.** We saw in Lecture III the remarkable fact that in supersymmetric theories the energy is positive-semidefinite. This means in particular that the lowest-energy state—the vacuum, denoted $|\text{vac}\rangle$—has non-negative energy. Indeed, applying the hamiltonian to the vacuum and using (43), we obtain

$$\begin{aligned}&\langle\text{vac}|H|\text{vac}\rangle\\&\qquad=\tfrac{1}{4}\left(\|\mathsf{Q}_1|\text{vac}\rangle\|^2+\|\mathsf{Q}_1^\dagger|\text{vac}\rangle\|^2+\|\mathsf{Q}_2|\text{vac}\rangle\|^2+\|\mathsf{Q}_2^\dagger|\text{vac}\rangle\|^2\right)\ ,\end{aligned}$$

from where we deduce that the vacuum has zero energy if and only if it is supersymmetric, that is, if and only if it is annihilated by the supercharges. This gives an elegant restatement of the condition for the spontaneous breaking of supersymmetry: *supersymmetry is spontaneously broken if and only if the vacuum energy is positive.* This is to be contrasted with the spontaneous breaking of gauge symmetries, which is governed by the shape of the potential of the dynamical scalar fields. Spontaneous breaking of supersymmetry is impervious to the shape of the potential, but only to the minimum value of the energy. Figure 1 illustrates this point. Whereas only potentials (b) and (d) break supersymmetry, the potentials breaking gauge symmetry are (c) and (d).

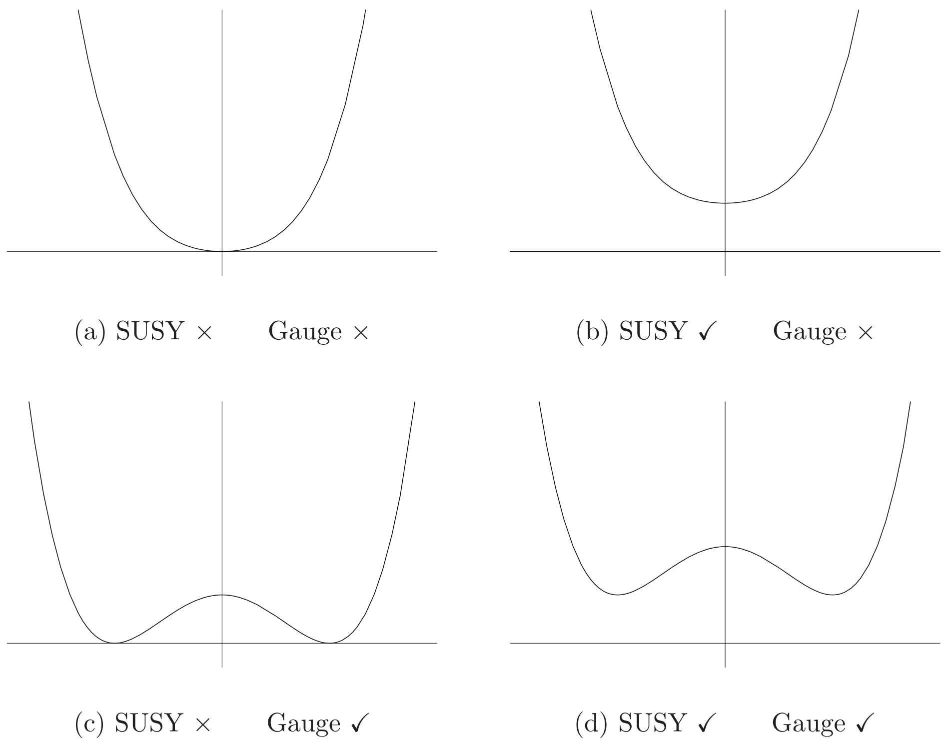


FIGURE 1. Generic forms of scalar potentials, indicating which symmetry is broken (denoted by a ✓) for each potential.

STOP You may ask whether one cannot simply shift the zero point energy in order to make it be precisely zero at the minimum of the potential. In contrast with nonsupersymmetric theories, the energy is now dictated by the symmetry, since the hamiltonian appears in the supersymmetry algebra.

VI.2. **Supersymmetry breaking and VEVs.** Another criterion of spontaneous supersymmetry breaking can be given in terms of vacuum expectation values of auxiliary fields.

We start with the observation that supersymmetry is spontaneously broken if and only if there is some field φ whose supersymmetry variation has a nontrivial vacuum expectation value:

$$\langle \mathrm{vac}|\, \delta_\varepsilon \varphi \,|\mathrm{vac}\rangle \neq 0 \;. \tag{109}$$

Indeed, notice that $\delta_\varepsilon\varphi = -\left[\varepsilon\mathsf{Q} + \bar\varepsilon\bar{\mathsf{Q}}, \varphi\right]$ as quantum operators, hence

$$\langle\text{vac}|\,\delta_\varepsilon\varphi\,|\text{vac}\rangle = -\varepsilon^\alpha\langle\text{vac}|\,[\mathsf{Q}_\alpha,\varphi]\,|\text{vac}\rangle - \bar\varepsilon^{\dot\alpha}\langle\text{vac}|\,[(\mathsf{Q}_\alpha)^\dagger,\varphi]\,|\text{vac}\rangle\ .$$

Because Lorentz invariance is sacred, no field which transform nontrivially under the Lorentz group is allowed to have a nonzero vacuum expectation value. Since supersymmetry exchanges bosons with fermions, and fermions always transform nontrivially under the Lorentz group, it means that the field φ in equation (109) must be fermionic. By examining the supersymmetry transformation laws for the fermionic fields in the different superfields we have met thus far, we can relate the spontaneous breaking of supersymmetry to the vacuum expectation values of auxiliary fields. This illustrates the importance of auxiliary fields beyond merely ensuring the off-shell closure of the supersymmetry algebra.

Let's start with the chiral superfields. Equation (58) describes how the fermions in the chiral superfield transform under supersymmetry. Only the dynamical scalar and the auxiliary field can have vacuum expectation values, and only the vacuum expectation value of the auxiliary field can give a nonzero contribution to equation (109). This sort of supersymmetry breaking is known as F-term (or O'Raifeartaigh) supersymmetry breaking.

In the case of the vector superfields, the transformation law of the fermion is now given by equation (105). Only the auxiliary field can have a nonzero vacuum expectation value and hence give a nonzero contribution to (109). This sort of supersymmetry breaking is known as D-term supersymmetry breaking and will be discussed in more detail below. Notice however that giving a nonzero vacuum expectation value to D breaks gauge invariance unless D, which is Lie algebra valued, happens to belong to the center; that is, to have vanishing Lie brackets with all other elements in the Lie algebra. This requires the gauge group to have abelian factors.

☞ Notice that when either the F or D auxiliary fields acquire nonzero vacuum expectation values, the transformation law of some fermion contains an inhomogeneous term:

$$\delta_\varepsilon\lambda_\alpha = -2\varepsilon_\alpha\,\langle D\rangle + \cdots \qquad \text{and} \qquad \delta_\varepsilon\chi_\alpha = -2\varepsilon_\alpha\,\langle F\rangle + \cdots$$

Such a fermion is called a *Goldstone fermion*, by analogy with the Goldstone boson which appears whenever a global continuous symmetry is spontaneously broken. Just like in the standard Higgs mechanism, wherein a vector boson "eats" the Goldstone boson to acquire mass, in a supergravity theory the gravitino acquires a mass by eating the Goldstone fermion, in a process known as the super-Higgs mechanism.

VI.3. **The O'Raifeartaigh model.** We now consider a model which breaks supersymmetry spontaneously because of a nonzero vacuum expectation value of the F field. Consider a theory of chiral superfields $\{\Phi^i\}$. The most general renormalisable lagrangian was worked out in

Problem 4. It consists of a positive-definite kinetic term

$$\int d^2\theta\, d^2\bar\theta \sum_i \Phi^i \bar\Phi_i$$

and a superpotential term

$$\int d^2\theta\, W(\mathbf{\Phi}) + \text{c.c.}\ ,$$

where $W(\mathbf{\Phi})$ is a cubic polynomial (for renormalisability)

$$W(\mathbf{\Phi}) = a_i \Phi^i + \tfrac{1}{2} m_{ij} \Phi^i \Phi^j + \tfrac{1}{3} \lambda_{ijk} \Phi^i \Phi^j \Phi^k\ .$$

In Problem 4 we found the component expression for the above lagrangian. From this one can read off the equations of motion of the auxiliary fields:

$$\bar F_i = -\frac{\partial W(\phi)}{\partial \phi^i} = -\left(a_i + m_{ij}\phi^j + \lambda_{ijk}\phi^j\phi^k\right)\ .$$

Substituting this back into the lagrangian, one gets the potential energy term:

$$\mathcal{V} = \sum_i \bar F_i F^i = \sum_i \left| -\frac{\partial W(\phi)}{\partial \phi^i}\right|^2 = \sum_i \left| a_i + m_{ij}\phi^j + \lambda_{ijk}\phi^j\phi^k\right|^2\ .$$

This potential is positive-semidefinite. It breaks supersymmetry if and only if there exist no vacuum expectation values $\langle\phi^i\rangle$ such that $\langle F^i\rangle = 0$ for all i. Notice that if $a_i = 0$, then $\langle\phi^i\rangle = 0$ always works, so that supersymmetry is not broken unless $a_i \neq 0$. Can we find superpotentials $W(\mathbf{\Phi})$ for which this is the case?

It turns out that one cannot find any interesting (i.e., interacting) such theories with less than three chiral superfields.

✍ **Exercise VI.1.** *Prove that if there is only one chiral superfield Φ, then the only cubic superpotential which breaks supersymmetry consists is $W(\Phi) = a\Phi$, so that the theory is free.*

In fact the same is true for two chiral superfields, although the proof is more involved. The simplest model needs three chiral superfields Φ_0, Φ_1 and Φ_2. This is the *O'Raifeartaigh model* and is described by the following superpotential:

$$W(\mathbf{\Phi}) = \mu\Phi_1\Phi_2 + \lambda\Phi_0\left(\Phi_1^2 - \alpha^2\right)\ ,$$

where α, μ and λ can be chosen to be real and positive by changing, if necessary, the overall phases of the chiral superfields and of W.

✍ **Exercise VI.2.** *Show that this superpotential is determined uniquely by the requirements of renormalisability, invariance under the R-symmetry*

$$\mathsf{R}\cdot\Phi_0 = \Phi_0 \qquad \mathsf{R}\cdot\Phi_1 = 0 \qquad \mathsf{R}\cdot\Phi_2 = \Phi_2 \ ,$$

and invariance under the discrete $\mathbb{Z}_2$ symmetry

$$\Phi_0 \mapsto \Phi_0 \qquad \Phi_1 \mapsto -\Phi_1 \qquad \Phi_2 \mapsto -\Phi_2 \ .$$

The equations of motion of the auxiliary fields are given by

$$\begin{aligned}\bar{F}_0 &= -\lambda\left(\phi_1^2 - \alpha^2\right)\\ \bar{F}_1 &= -\left(\mu\phi_2 - 2\lambda\phi_0\phi_1\right)\\ \bar{F}_2 &= -\mu\phi_1 \ .\end{aligned}$$

✍ **Exercise VI.3.** *Show that the above superpotential breaks supersymmetry spontaneously provided that λ, μ and α are nonzero.*

Let us introduce complex coordinates $z_i = \langle\phi_i\rangle$. The potential defines a function $\mathcal{V} : \mathbb{C}^3 \to \mathbb{R}$, which is actually positive:

$$\mathcal{V} = \lambda^2|z_1^2 - \alpha^2|^2 + \mu^2|z_1|^2 + |\mu z_2^2 - 2\lambda z_0 z_1|^2 \ .$$

To minimise the potential, notice that provided that $\mu \neq 0$, we can always set z_2 such that the last term vanishes for any values of z_0 or z_1. The other two terms depend only on z_1, hence the potential will have a flat direction along z_0.

✍ **Exercise VI.4.** *Show that provided $\mu^2 \geq 2\lambda^2\alpha^2$, the minimum of the potential $\mathcal{V}$ is at $z_1 = z_2 = 0$ and arbitrary values of z_0. Compute the spectrum of masses in this case and show that there is a massless fermion, which can be identified with the Goldstone fermion.*
(Hint: The masses will depend on z_0, but the fact there exists a massless fermion has to do with the vanishing of the determinant of the fermion mass matrix, and this is the case for all z_0.)

Notice that the existence of the Goldstone fermion was inferred from the vanishing of the determinant of the fermion mass matrix. This comes from the superpotential term and is protected from quantum corrections. But even if this were not the case, it is clear that under radiative corrections the condition that the vacuum energy is positive is stable under deformations, in the sense that this condition is preserved under small perturbations. In the language of (point set) topology, one would say that this is an *open* condition: meaning that in the relevant space of deformation parameters, every point for which the vacuum energy is positive has a neighbourhood consisting of points which share this property. This is illustrated in Figure 2 below, where the dashed lines indicate deformations of the potential, drawn with a solid line.

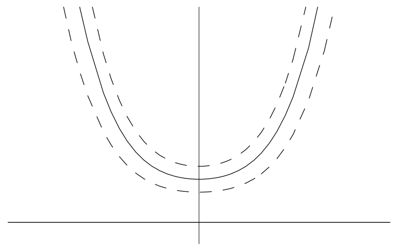


FIGURE 2. Potentials with positive vacuum energy are stable under deformations.

☞ How about chiral superfields coupled to gauge fields? Ignoring for the moment the Fayet–Iliopoulos terms, which will be the subject of the next section, let me just mention that it is possible to show that in the absence of Fayet–Iliopoulos terms, it is the O'Raifeartaigh mechanism again which governs the spontaneous breaking of supersymmetry, in the sense that if the F equations of motion ($F^i = 0$) are satisfied for some scalar vacuum expectation values, then it is possible to use the "global" gauge symmetry, which is a symmetry of the superpotential and hence of the F equations of motion, in order to find (possibly different) vacuum expectation values such that the D-equations of motion ($D^i = 0$) are also satisfied.

VI.4. **Fayet–Iliopoulos terms.** The O'Raifeartaigh model breaks supersymmetry because of the linear term in the superpotential (the F term), which gives a nonzero vacuum expectation value to the auxiliary field in the chiral superfield. It is also possible to break supersymmetry by giving a nonzero vacuum expectation value to the auxiliary field in the vector superfield. This is possible by adding a Fayet–Iliopoulos term to the action. Gauge invariance requires that the Fayet–Iliopoulos term belong to the center of the Lie algebra $\mathfrak{g}$ of the gauge group. Since the gauge group is compact, its Lie algebra is the direct product of a semisimple Lie algebra and an abelian Lie algebra. Semisimple Lie algebras have no center, hence for the Fayet–Iliopoulos term to exist, there has to be a nontrivial abelian factor. In other words, the gauge group must have at least one U(1) factor. To illustrate this phenomenon, we will actually consider an abelian Yang–Mills theory with gauge group U(1): supersymmetric QED, with superspace lagrangian (92), except that we also add a Fayet–Iliopoulos term κV to the superspace lagrangian:

$$\int d^2\theta d^2\bar\theta \, \left(\bar\Phi_+ e^{2eV}\Phi_+ + \bar\Phi_- e^{-2eV}\Phi_- + \kappa V\right) \\ + \left[\int d^2\theta \, \left(\tfrac{1}{4}W^\alpha W_\alpha + m\Phi_+\Phi_-\right) + \text{c.c.}\right] \; .$$

The potential energy terms are

$$2D^2 + \kappa D + 2eD\left(|\phi_+|^2 - |\phi_-|^2\right) \\ + |F_+|^2 + |F_-|^2 + m\left(F_+\phi_- + F_-\phi_+ + \bar F_+\bar\phi_- + \bar F_-\bar\phi_+\right) \; .$$

Eliminating the auxiliary fields via their equations of motion

$$\begin{aligned} F_\pm &= -m\bar{\phi}_\mp \\ D &= -\tfrac{1}{4}\left(\kappa + 2e\left(|\phi_+|^2 - |\phi_-|^2\right)\right) \end{aligned}$$

we obtain the potential energy

$$\mathcal{V} = \tfrac{1}{8}\left(\kappa + 2e\left(|\phi_+|^2 - |\phi_-|^2\right)\right)^2 + m^2\left(|\phi_-|^2 + |\phi_+|^2\right) \ .$$

Notice that for nonzero κ supersymmetry is spontaneously broken, since it is impossible to choose vacuum expectation values for the scalars such that $\langle F_\pm\rangle = \langle D\rangle = 0$.

Expanding the potential

$$\mathcal{V} = \tfrac{1}{8}\kappa^2 + (m^2 - \tfrac{1}{2}e\kappa)|\phi_-|^2 + (m^2 + \tfrac{1}{2}e\kappa)|\phi_+|^2 + \tfrac{1}{2}e^2\left(|\phi_+|^2 - |\phi_-|^2\right)^2$$

we notice that there are two regimes with different qualitative behaviours.

If $m^2 > \frac{1}{2}e\kappa$ the minimum of the potential occurs for $\langle\phi_+\rangle = \langle\phi_-\rangle = 0$ and the model describes two complex scalars with masses $m_\mp^2 = m^2 \pm \frac{1}{2}e\kappa$. The electron mass m does not change, and the photon and photino remain massless. Hence supersymmetry is spontaneously broken—the photino playing the rôle of the Goldstone fermion—and the gauge symmetry is unbroken. This is the situation depicted by the potential of the type (b) in Figure 1.

On the other hand if $m^2 < \frac{1}{2}e\kappa$, the minimum of the potential is no longer at $\langle\phi_+\rangle = \langle\phi_-\rangle = 0$. Instead we see that the minimum happens at $\langle\phi_+\rangle = 0$ but at $\langle\phi_-\rangle = z$ where

$$|z|^2 = \left(\frac{\kappa}{2e} - \frac{m^2}{e^2}\right) \ .$$

There is a circle of solutions corresponding to the phase of z. We can always choose the global phase so that z is real and positive:

$$z = \sqrt{\frac{\kappa}{2e} - \frac{m^2}{e^2}} \ .$$

Exercise VI.5. *Expand around $\langle\phi_+\rangle = 0$ and $\langle\phi_-\rangle = z$ and compute the mass spectrum. Show that the photon acquires a mass, signalling the spontaneous breaking of the* U(1) *gauge symmetry, but that there is a massless fermion in the spectrum, signalling the spontaneous breaking of supersymmetry.*

The situation is now the one depicted by the potential of type (d) in Figure 1.

VI.5. **The Witten index.** Finally let us introduce an extremely important concept in the determination of supersymmetry breaking. In theories with complicated vacuum structure it is often nontrivial to determine whether supersymmetry is broken. The Witten index is a quantity which can help determine when supersymmetry is *not* broken, provided that one can actually compute it. Its computation is facilitated by the fact that it is in a certain sense a "topological" invariant.

Suppose that we have a supersymmetric theory, by which we mean a unitary representation of the Poincaré superalgebra on some Hilbert space $\mathcal{H}$. We will furthermore assume that $\mathcal{H}$ decomposes as a direct sum (or more generally a direct integral) of energy eigenstates

$$\mathcal{H} = \bigoplus_{E \geq 0} \mathcal{H}_E \ ,$$

with each $\mathcal{H}_E$ finite-dimensional. (In practice the extension to the general case is usually straightforward.)

Let β be a positive real number and consider the following quantity

$$I(\beta) = \mathrm{STr}_{\mathcal{H}}\, e^{-\beta H} = \mathrm{Tr}_{\mathcal{H}}\, (-1)^F\, e^{-\beta H} \ ,$$

which defines the *supertrace* STr, and where H is the hamiltonian and F is the fermion number operator. In particular, this means that $(-1)^F$ is $+1$ on a bosonic state and -1 on a fermionic state. We will show that $I(\beta)$ is actually independent of β—the resulting integer is called the *Witten index* of the representation $\mathcal{H}$.

The crucial observation is that in a supersymmetric theory there are an equal number of bosonic and fermionic states with any given positive energy. Hence the Witten index only receives contributions from the zero energy states, if any. This means in particular that a nonzero value of the Witten index signals the existence of some zero energy state which, by the discussion at the start of this lecture, implies that supersymmetry is *not* broken. In contrast, a zero value for the Witten index does not allow us to conclude anything, since all this says is that there is an equal number of bosonic and fermionic zero energy states, but this number could either be zero (broken supersymmetry) or nonzero (unbroken supersymmetry).

By definition,

$$I(\beta) = \sum_{E \geq 0} e^{\beta E}\ \mathrm{Tr}_{\mathcal{H}_E}\, (-1)^F = \sum_{E \geq 0} e^{\beta E}\, n(E),$$

where

$$n(E) = \mathrm{Tr}_{\mathcal{H}_E}\, (-1)^F = n_+(E) - n_-(E)$$

is the difference between the number of bosonic states with energy E and the number of fermionic states with the same energy. It is here that we make use of the assumption that $\mathcal{H}_E$ is finite-dimensional: so that $n_\pm(E)$, and hence their difference, are well-defined.

✍ **Exercise VI.6.** *Show that for $E \neq 0$, $n(E) = 0$.*
(Hint: You may find of use the expression (43) for the hamiltonian in terms of the supercharges.)

☞ Alternatively, one can prove the β independence of $I(\beta)$ by taking the derivative of $I(\beta)$ and showing that the result vanishes as a consequence of the expression (43) for the Hamiltonian of a supersymmetric theory, the fact that H commutes with the supercharges, and that the supertrace of an (anti)commutator vanishes. This last result (which you are encouraged to prove) is the super-analogue of the well-known fact that the trace of a commutator vanishes.

This result implies that

$$I(\beta) = \mathrm{Tr}_{\mathcal{H}_0} (-1)^F = n_+(0) - n_-(0) \ ,$$

is independent of β. This means that it can be computed for any value of β, for example in the limit as $\beta \to \infty$, where the calculation may simplify enormously. In fact, the Witten index is a "topological" invariant of the supersymmetric theory. As such it does not depend on parameters, here illustrated by the independence on β. This means that one can take couplings to desired values, put the theory in a finite volume and other simplifications.

☞ The Witten index is defined in principle for any supersymmetric theory. As we saw in Problem 4, there are supersymmetric theories whose data is geometric and it is to be expected that the Witten index should have some geometric meaning in this case. In fact, the dimensional reduction to one dimension of the supersymmetric sigma model discussed in Problem 4 yields a supersymmetric quantum mechanical model whose Witten index equals the Euler characteristic. More is true, however, and the computation of the Witten index gives a proof of the well-known Gauss–Bonnet theorem relating the Euler characteristic of the manifold to the curvature. In fact, the Witten index underlies many of the topological applications of supersymmetry and in particular the simplest known proof of the Atiyah–Singer index theorem relating the analytic index of an elliptic operator on a manifold to the topology of that manifold.
There are many deep and beautiful connections like that one between supersymmetry and mathematics. Indeed, whatever the final verdict might be for the existence of supersymmetry (albeit broken) in nature, the impact of supersymmetry in mathematics will be felt for many years to come.

Problem 6 (The Higgs mechanism)**.**

In supersymmetric theories the issue of gauge symmetry breaking (Higgs mechanism) and supersymmetric breaking are intimately related. Although the topic of this lecture has been supersymmetry breaking, in this tutorial you are asked to study a simple example of Higgs mechanism which preserves supersymmetry. The model in question is an SU(5) gauge theory coupled to adjoint matter in the form of chiral superfields. In other words, the model consists of a non-abelian vector superfield $V = V^i(iT_i)$ and an adjoint chiral superfield $\mathbf{\Phi} = \Phi^i T_i$, where T_i are 5×5 traceless antihermitian matrices. Notice that Φ^i are chiral superfields, hence complex, and V^i are vector superfields, hence real.

The superspace lagrangian has the form

$$\int d^2\theta\, d^2\bar\theta\ \operatorname{Tr} \bar{\boldsymbol\Phi} e^{2g\ \operatorname{ad} V}\boldsymbol\Phi + \left[\int d^2\theta \left(\tfrac14 \operatorname{Tr} W^\alpha W_\alpha + W(\boldsymbol\Phi)\right) + \text{c.c.}\right] ,$$

where we are treating the $\boldsymbol\Phi$ as matrices in the fundamental representation, hence V acts on $\boldsymbol\Phi$ via the matrix commutator (denoted $\operatorname{ad} V$) and Tr is the matrix trace. Since SU(5) is a simple group, there is no Fayet–Iliopoulos term in this model. Notice that since the T_i are antihermitian, the trace form $\operatorname{Tr} T_i T_j = -K_{ij}$ where K_{ij} is positive-definite.

(1) Show that the most general renormalisable gauge-invariant superpotential takes the form

$$W(\boldsymbol\Phi) = \tfrac12 m \operatorname{Tr} \boldsymbol\Phi^2 + \tfrac13 \lambda \operatorname{Tr} \boldsymbol\Phi^3 ,$$

and argue that m and λ can be taken to be real by changing, if necessary, the overall phases of W and of $\boldsymbol\Phi$.

(2) Expanding the superspace action in components and eliminating the auxiliary fields F and D, show that the scalar potential takes the form

$$\mathcal{V} = -\tfrac12 g^2 \operatorname{Tr}[\bar{\boldsymbol\phi}, \boldsymbol\phi]^2 - \operatorname{Tr} \overline{\nabla W} \nabla W ,$$

where ∇W is defined by $\operatorname{Tr} \nabla W T_i = -\partial W/\partial \phi^i$.

Let us remark that since the trace form on antihermitian matrices is negative-definite, the above potential is actually positive-semidefinite—in fact, it is a sum of squares.

Notation: Let $A := \langle \boldsymbol\phi \rangle$ be the vacuum expectation value of $\boldsymbol\phi$. It is a 5×5 traceless antihermitian matrix.

3. Show that $A = 0$ is a minimum of the potential $\mathcal{V}$.

This solution corresponds to unbroken SU(5) gauge theory and, since the potential is zero for this choice of A, unbroken supersymmetry. The rest of the problem explores other supersymmetric minima for which SU(5) is broken down to smaller subgroups. As we saw in the lecture, a vacuum is supersymmetric if and only if it has zero energy, hence we are interested in vacuum expectation values A for which $\mathcal{V} = 0$. These vacua will be degenerate, since they are acted upon by the subgroup of the gauge group which remains unbroken.

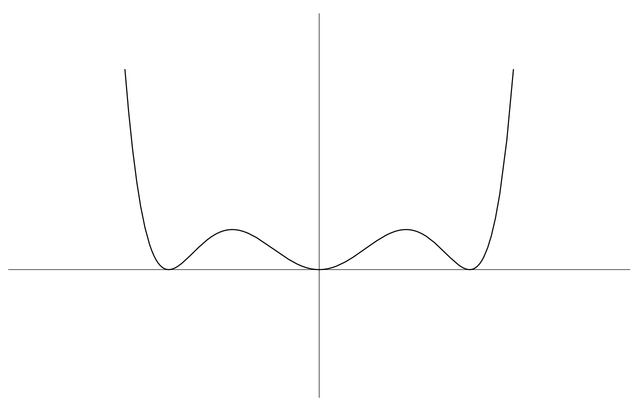
4. Show that the minima of the potential $\mathcal{V}$ correspond to those matrices A obeying the following two equations:

$$[A, \bar A] = 0 \qquad \text{and} \qquad mA + \lambda\left(A^2 - \tfrac15 \operatorname{Tr} A^2\right) = 0 ,$$

where $\bar A$ is the hermitian conjugate of A.

5. Conclude from the first equation that A can be diagonalised by a matrix in SU(5), hence we can assume that A takes the form
$$A = \begin{pmatrix} \mu_1 & & & & \\ & \mu_2 & & & \\ & & \mu_3 & & \\ & & & \mu_4 & \\ & & & & \mu_5 \end{pmatrix}$$
for complex numbers μ_i obeying $\sum_i \mu_i = 0$.
6. Assume that $\lambda \neq 0$ and show that both
$$\frac{3m}{\lambda}\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & -4 \end{pmatrix} \quad \text{and} \quad \frac{2m}{\lambda}\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & -\frac{3}{2} & \\ & & & & -\frac{3}{2} \end{pmatrix}$$
are possible choices for A which solve the equations. Which subgroup of SU(5) remains unbroken in each case?
(Answers: The groups are $\mathrm{S}\left(\mathrm{U}(4) \times \mathrm{U}(1)\right)$ and $\mathrm{S}\left(\mathrm{U}(3) \times \mathrm{U}(2)\right)$, which are locally isomorphic to $\mathrm{SU}(4) \times \mathrm{U}(1)$ and $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$, respectively; but you have to show this!)

It is possible to show that up to gauge transformations these are the only three minima of $\mathcal{V}$. Hence the situation in this problem corresponds to a potential which is a mixture of types (a) and (c) in Figure 1, and roughly sketched below:



Appendix A. Basic definitions and conventions

This appendix collects the basic definitions used in the lecture concerning Lie (super)algebras, Minkowski space, the Poincaré group, the Clifford algebra, the spin group and the different types of spinors. More importantly it also contains our spinor conventions. I learned supersymmetry from Peter van Nieuwenhuizen and these conventions agree mostly with his. I am however solely responsible for any inconsistencies.

Exercise A.1. *Find any inconsistencies and let me know!*

A.1. **Lie algebras.** We now summarise the basic notions of Lie algebras and Lie superalgebras used in the lectures.

A Lie algebra consists of a vector space $\mathfrak{g}$ and an antisymmetric bilinear map

$$[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g} ,$$

called the Lie bracket, which satisfying the Jacobi identity

$$[X,[Y,Z]] = [[X,Y],Z] + [Y,[X,Z]] \qquad \text{for all } X,Y,Z \in \mathfrak{g}.$$

Fixing a basis $\{T_i\}$ for $\mathfrak{g}$, the Lie bracket is specified by the structure constants ${f_{ij}}^k = -{f_{ji}}^k$ defined by

$$[T_i, T_j] = {f_{ij}}^k T_k \ .$$

All Lie algebras considered in these lectures are real; in other words, $\mathfrak{g}$ is a real vector space and the structure constants are real. This means, in particular, that in a unitary representation they are realised as antihermitian matrices.

Most Lie algebras of interest possess an invariant inner product, denoted Tr, since it can often be taken to be the trace in some faithful representation. Relative to a basis, the inner product is specified by a real symmetric matrix $G_{ij} = G_{ji} = \operatorname{Tr} T_i T_j$. Invariance means that

$$\operatorname{Tr}[T_i, T_j] T_k = \operatorname{Tr} T_i [T_j, T_k]$$

which is equivalent to $f_{ijk} := {f_{ij}}^\ell G_{\ell k}$ being totally antisymmetric. For a compact Lie group, one can always choose a basis for the Lie algebra such that $G_{ij} = -\delta_{ij}$. Notice that it is negative-definite.

The exponential of a matrix is defined in terms of the Taylor series of the exponential function:

$$e^{\mathsf{A}} := \mathbb{1} + \mathsf{A} + \tfrac{1}{2}\mathsf{A}^2 + \cdots \ .$$

Suppose we are given a linear representation of a Lie algebra $\mathfrak{g}$. Every element $X \in \mathfrak{g}$ is represented by a matrix X, and hence we can define the exponential $\exp(X)$ in the representation as the exponential of the corresponding matrix $\exp(\mathsf{X})$. Given $X, Y \in \mathfrak{g}$ with corresponding matrices X, Y and consider the product of their exponentials

$\exp(\mathsf{X})\exp(\mathsf{Y})$. It turns out that this is the exponential of a third matrix Z:

$$e^{\mathsf{X}}e^{\mathsf{Y}} = e^{\mathsf{Z}} \quad \text{where } \mathsf{Z} = \mathsf{X} + \mathsf{Y} + \cdots,$$

where the omitted terms consists of nested commutators of X and Y. This implies that there is an element $Z \in \mathfrak{g}$ which is represented by Z. The dependence of Z on X and Y is quite complicated, and is given by the celebrated Baker–Campbell–Hausdorff formula. For our purposes it will be sufficient to notice that

$$Z = X + \left(\frac{-\operatorname{ad} X}{e^{-\operatorname{ad} X} - \mathbb{1}}\right) \cdot Y + \cdots$$

where the omitted terms are at least quadratic in Y. In this formula, $\operatorname{ad} X$ is defined by $\operatorname{ad} X \cdot Y = [X, Y]$ and the expression in parenthesis is defined by its Taylor series. Keeping only those terms at most linear in Y, Z takes the form

$$Z = X + Y + \tfrac{1}{2}[X, Y] + \tfrac{1}{12}[X, [X, Y]] + \sum_{k\geq 2} c_k (\operatorname{ad} X)^{2k} \cdot Y \ , \tag{A-1}$$

where the c_k are rational coefficients. Notice that the sum has only even powers of $\operatorname{ad} X$.

A.2. **Lie superalgebras.** The notion of a Lie superalgebra is a natural extension of the notion of a Lie algebra. By definition, a Lie superalgebra consists of a $\mathbb{Z}_2$-graded vector space $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ and a bilinear operation to be defined presently. In practice we will only consider homogeneous elements; that is, elements in either $\mathfrak{g}_0$ or $\mathfrak{g}_1$. For X a homogeneous element the following are equivalent:

$$|X| = 0 \iff X \in \mathfrak{g}_0 \iff X \text{ is even},$$
$$|X| = 1 \iff X \in \mathfrak{g}_1 \iff X \text{ is odd},$$

which defines what we mean by even and odd. The Lie bracket is now $\mathbb{Z}_2$-graded

$$[-,-] : \mathfrak{g}_i \times \mathfrak{g}_j \to \mathfrak{g}_{i+j}$$

where $i + j$ are added modulo 2. It is again bilinear and obeys

$$[X, Y] = -(-1)^{|X||Y|}[Y, X]$$

and

$$[X, [Y, Z]] = [[X, Y], Z] + (-1)^{|X||Y|}[Y, [X, Z]]$$

for all homogeneous elements $X, Y, Z \in \mathfrak{g}$. We use the *same* notation $[-,-]$ for the bracket of any two elements in a superalgebra. We should remember however that it is symmetric if both elements are odd and antisymmetric otherwise. Furthermore, in a linear representation, the bracket of two odd elements is realised as the anticommutator of the corresponding matrices, whereas it is realised as the commutator in all other cases.

We introduce a useful categorical concept. Given a Lie superalgebra defined by some brackets, by the *opposite superalgebra* we will mean the Lie superalgebra defined by multiplying the brackets by -1. Clearly any Lie superalgebra is isomorphic to its opposite, by sending each generator X to $-X$. We are only introducing this notion for notation: I find it more convenient conceptually to think in terms of representations of the opposite algebra than in terms of antirepresentations of an algebra, and in these lectures we will have to deal with both.

It is a general fact, following trivially from the axioms, that the even subspace of a Lie superalgebra forms a Lie algebra of which the odd subspace is a (real, in the cases of interest) representation. It follows in particular that a Lie algebra is a Lie superalgebra which has no odd elements. Hence the theory of Lie superalgebras contains the theory of Lie algebras, and extends it in a nontrivial way. From a kinematic point of view, supersymmetry is all about finding field theoretical realisations of Lie superalgebras whose even subspace contains a Lie subalgebra isomorphic to either the Poincaré or conformal algebras.

A.3. **Minkowski space and the Poincaré group.** Minkowski space is the four-dimensional real vector space with "mostly plus" metric

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & +1 & & \\ & & +1 & \\ & & & +1 \end{pmatrix} .$$

We fix an orientation $\epsilon_{\mu\nu\rho\sigma}$ by

$$\epsilon^{0123} = -\epsilon_{0123} = +1 .$$

The group of isometries of Minkowski space is called the Poincaré group. The subgroup of isometries which preserve the origin is called the Lorentz group. The Poincaré group is the semidirect product of the Lorentz group and the translation group. Its Lie algebra is therefore also the semidirect product of the Lorentz algebra and the translation algebra. Let $\mathsf{M}_{\mu\nu} = -\mathsf{M}_{\nu\mu}$ be a basis for the Lorentz algebra and let P_μ be a basis for the translation algebra. They satisfy the following brackets:

$$\begin{aligned} [\mathsf{P}_\mu, \mathsf{P}_\nu] &= 0 \\ [\mathsf{M}_{\mu\nu}, \mathsf{P}_\rho] &= \eta_{\nu\rho}\mathsf{P}_\mu - \eta_{\mu\rho}\mathsf{P}_\nu \\ [\mathsf{M}_{\mu\nu}, \mathsf{M}_{\rho\sigma}] &= \eta_{\nu\rho}\mathsf{M}_{\mu\sigma} - \eta_{\mu\rho}\mathsf{M}_{\nu\sigma} - \eta_{\nu\sigma}\mathsf{M}_{\mu\rho} + \eta_{\mu\sigma}\mathsf{M}_{\nu\rho} . \end{aligned} \tag{A-2}$$

The Poincaré group acts transitively on Minkowski space: any point can be reached from the origin by a Poincaré transformation. This transformation is not unique, since there are some transformations which leave the origin fixed: the Lorentz transformations. Therefore Minkowski space (with a choice of origin) can be identified with the space of right cosets of the Lorentz group. Each such coset has a unique representative which is a translation. This allows us to assign a unique element of the Poincaré group to each point in Minkowski space:

$$\begin{array}{ll} x^\mu & \in \text{Minkowski space} \\ \updownarrow & \\ \exp(x^\mu \mathsf{P}_\mu) & \in \text{Poincaré group,} \end{array}$$

which in turn allows us to realise the action of the Poincaré group in Minkowski space as left multiplication in the group.

Indeed, a translation $\exp(\tau^\mu \mathsf{P}_\mu)$ acts as

$$\exp(\tau^\mu \mathsf{P}_\mu)\exp(x^\mu \mathsf{P}_\mu) = \exp((x^\mu + \tau^\mu)\mathsf{P}_\mu) \ ,$$

whence $x^\mu \mapsto x^\mu + \tau^\mu$. Similarly a Lorentz transformation acts as

$$\exp(\tfrac{1}{2}\lambda^{\mu\nu}\mathsf{M}_{\mu\nu})\exp(x^\mu \mathsf{P}_\mu) = \exp(x^\mu \Lambda_\mu{}^\nu \mathsf{P}_\nu)\exp(\tfrac{1}{2}\lambda^{\mu\nu}\mathsf{M}_{\mu\nu}) \ ,$$

where $\Lambda_\mu{}^\nu$ is the adjoint matrix defined by

$$\Lambda_\mu{}^\nu \mathsf{P}_\nu = \exp(\tfrac{1}{2}\lambda^{\mu\nu}\mathsf{M}_{\mu\nu})\mathsf{P}_\mu \exp(-\tfrac{1}{2}\lambda^{\mu\nu}\mathsf{M}_{\mu\nu}) \ .$$

Therefore the effect of a Poincaré transformation $\exp(\tau \cdot \mathsf{P})\exp(\lambda \cdot \mathsf{M})$ is

$$x^\mu \mapsto x^\nu \Lambda_\nu{}^\mu + \tau^\mu \ .$$

Let us call this transformation $P(\Lambda, \tau)$. Notice that acting on points the order of the transformations is reversed:

$$P(\Lambda_1, \tau_1)P(\Lambda_2, \tau_2) = P(\Lambda_2\Lambda_1, \Lambda_1\tau_2 + \tau_1) \ .$$

Similarly, we can work out the action of the Lie algebra by considering infinitesimal transformations:

$$\delta_\tau x^\mu = \tau^\mu \quad \text{and} \quad \delta_\lambda x^\mu = x^\nu \lambda_\nu{}^\mu \ ,$$

whence we see that P_μ and $\mathsf{M}_{\mu\nu}$ are realised in terms of vector fields

$$\mathsf{P}_\mu \rightsquigarrow \partial_\mu \quad \text{and} \quad \mathsf{M}_{\mu\nu} \rightsquigarrow x_\mu \partial_\nu - x_\nu \partial_\mu \ .$$

Again notice that these vector fields obey the opposite algebra.

A.4. **The Clifford algebra and its spinors.** The Lorentz group has four connected components. The component containing the identity consists of those Lorentz transformations which preserve the space and time orientations, the proper orthochronous Lorentz transformations. This component is not simply connected, but rather admits a simply-connected double cover (the spin cover) which is isomorphic to the group $\mathrm{SL}(2, \mathbb{C})$ of 2×2 complex matrices with unit determinant. The spinorial representations of the Lorentz group are actually representations of $\mathrm{SL}(2, \mathbb{C})$.

A convenient way to study the spinorial representations is via the Clifford algebra of Minkowski space

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = +2\eta_{\mu\nu}\mathbb{1} \ .$$

The reason is that the spin group is actually contained in the Clifford algebra as exponentials of (linear combinations of)

$$\Sigma_{\mu\nu} = \tfrac{1}{4}\left(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu\right) \ .$$

Notice that under the Clifford commutator these elements represent the Lorentz algebra (cf. the last equation in (A-2))

$$[\Sigma_{\mu\nu}, \Sigma_{\rho\sigma}] = \eta_{\nu\rho}\Sigma_{\mu\sigma} - \eta_{\mu\rho}\Sigma_{\nu\sigma} - \eta_{\nu\sigma}\Sigma_{\mu\rho} + \eta_{\mu\sigma}\Sigma_{\nu\rho} \ .$$

As an associative algebra, the Clifford algebra is isomorphic to the algebra of 4×4 real matrices. This means that it has a unique irreducible representation which is real and four-dimensional. These are the *Majorana spinors*.

It is often convenient to work with the complexified Clifford algebra, that is to say, one is allowed to take linear combination of the Dirac γ matrices. The complexified Clifford algebra has a unique irreducible representation which is complex and four-dimensional. These are the *Dirac spinors*.

We can always choose the inner product of spinors in such a way that the Dirac matrices are unitary. The Clifford algebra then implies that γ_0 is antihermitian and γ_i are hermitian. These conditions can be summarised succinctly as

$$\gamma_\mu^\dagger\gamma_0 = -\gamma_0\gamma_\mu \ .$$

One recovers the Majorana spinors as those Dirac spinors for which its Dirac $\bar\psi_D = \psi^\dagger i\gamma^0$ and Majorana $\bar\psi_M = \psi^t C$ conjugates agree:

$$\bar\psi := \bar\psi_D = \bar\psi_M \ , \qquad \text{(A-3)}$$

where C is the charge conjugation matrix. This implies a reality condition on the Dirac spinor:

$$\psi^* = iC\gamma^0\psi \ .$$

I find it easier to work with the Majorana conjugate, since this avoids having to complex conjugate the spinor.

Its historical name notwithstanding, C is *not* a matrix, since under a change of basis it does not transform like a γ matrix. Introducing spinor indices ψ^a, the γ matrices have indices $(\gamma_\mu)^a{}_b$ whereas C has indices C_{ab}. In other words, whereas the γ matrices are linear transformations, the charge conjugation matrix is a bilinear form. We will always use C to raise and lower spinor indices.

The charge conjugation matrix obeys the following properties:

$$C^t = -C \qquad \text{and} \qquad C\gamma_\mu = -\gamma_\mu^t C \ . \qquad \text{(A-4)}$$

Writing the indices explicitly the first of these equations becomes

$$C_{ab} = -C_{ba} \ ,$$

so that C is antisymmetric. This means that care has to be taken to choose a consistent way to raise and lower indices. We will raise and lower indices using the *North-West* and *South-East* conventions, respectively. More precisely,

$$\psi^a = C^{ab}\psi_b \qquad \text{and} \qquad \psi_a = \psi^b C_{ba} \ .$$

This implies that the inner product of Majorana spinors takes the form

$$\bar\varepsilon\psi := \varepsilon_a\psi^a = \varepsilon^b C_{ba}\psi^a = -\varepsilon^b\psi^a C_{ab} = -\varepsilon^b\psi_b \ .$$

The second identity in equation (A-4) can then be written as a symmetry condition:

$$(\gamma_\mu)_{ab} = (\gamma_\mu)_{ba} \ ,$$

where

$$(\gamma^\mu)_{ab} = (\gamma^\mu)^c{}_b C_{ca} = -C_{ac}(\gamma^\mu)^c{}_b \ .$$

We will employ the following useful notation $\gamma_{\mu\nu\ldots\rho}$ for the totally antisymmetrised product of γ matrices. More precisely we define

$$\gamma_{\mu_1\mu_2\ldots\mu_n} := \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n} \operatorname{sign}(\sigma)\gamma_{\mu_{\sigma(1)}}\gamma_{\mu_{\sigma(2)}}\cdots\gamma_{\mu_{\sigma(n)}} \ , \tag{A-5}$$

where the sum is over all the permutations of the set $\{1,2,\ldots,n\}$. Notice the factorial prefactor. For example, for $n=2$ this formula unpacks into

$$\gamma_{\mu\nu} = \tfrac{1}{2}\left(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu\right) \ .$$

The following identity is very convenient for computations

$$\begin{aligned}\gamma_{\mu_1\mu_2\ldots\mu_n}\gamma_\nu = \gamma_{\mu_1\mu_2\ldots\mu_n\nu} + \eta_{\nu\mu_n}\gamma_{\mu_1\mu_2\ldots\mu_{n-1}} - \eta_{\nu\mu_{n-1}}\gamma_{\mu_1\mu_2\ldots\widehat{\mu_{n-1}}\mu_n} \\ + \eta_{\nu\mu_{n-2}}\gamma_{\mu_1\ldots\widehat{\mu_{n-2}}\mu_{n-1}\mu_n} - \cdots + (-1)^{n-1}\eta_{\nu\mu_1}\gamma_{\mu_2\mu_3\ldots\mu_3} \ ,\end{aligned}$$

where a hat over an index indicates its omission. For example,

$$\gamma_{\mu\nu}\gamma_\rho = \gamma_{\mu\nu\rho} + \eta_{\nu\rho}\gamma_\mu - \eta_{\mu\rho}\gamma_\nu \ . \tag{A-6}$$

As an immediate corollary, we have the following useful identities:

$$\gamma^\rho\gamma_\mu\gamma_\rho = -2\gamma_\mu \qquad \text{and} \qquad \gamma^\rho\gamma_{\mu\nu}\gamma_\rho = 0 \ . \tag{A-7}$$

The Clifford algebra is isomorphic *as a vector space* to the exterior algebra of Minkowski space. The above antisymmetrisation provides the isomorphism. This makes it easy to list a basis for the Clifford algebra

$$\mathbb{1} \qquad \gamma_\mu \qquad \gamma_{\mu\nu} \qquad \gamma_{\mu\nu\rho} \qquad \gamma_{\mu\nu\rho\sigma} \ .$$

There are $1+4+6+4+1=16$ elements which are clearly linearly independent.

Define γ_5 as

$$\gamma_5 = \tfrac{1}{4!}\epsilon^{\mu\nu\rho\sigma}\gamma_{\mu\nu\rho\sigma} = \gamma_0\gamma_1\gamma_2\gamma_3 \ .$$

It satisfies the following properties:

$$\gamma_\mu\gamma_5 = -\gamma_5\gamma_\mu \qquad \gamma_5^2 = -\mathbb{1} \qquad \gamma_5^\dagger = -\gamma_5 \qquad \gamma_5^t C = C\gamma_5 \ .$$

This last identity can be rewritten as the antisymmetry condition

$$(\gamma_5)_{ab} = -(\gamma_5)_{ba} \ .$$

Using γ_5 we never need to consider antisymmetric products of more than two γ matrices. Indeed, one has the following identities:

$$\begin{aligned} \gamma_{\mu\nu}\gamma_5 &= -\tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}\gamma^{\rho\sigma} \\ \gamma_{\mu\nu\rho} &= \epsilon_{\mu\nu\rho\sigma}\gamma^{\sigma}\gamma_5 \\ \gamma_{\mu\nu\rho\sigma} &= -\epsilon_{\mu\nu\rho\sigma}\gamma_5 \ . \end{aligned}$$

Thus an equally good basis for the Clifford algebra is given by

$$\mathbb{1} \qquad \gamma_5 \qquad \gamma_\mu \qquad \gamma_\mu\gamma_5 \qquad \gamma_{\mu\nu} \ . \tag{A-8}$$

Lowering indices with C we find that $\mathbb{1}$, γ_5 and $\gamma_\mu\gamma_5$ becomes antisymmetric, whereas γ_μ and $\gamma_{\mu\nu}$ become symmetric.

Let ε_1 and ε_2 be anticommuting spinors, and let $\varepsilon_1\bar{\varepsilon}_2$ denote the linear transformation which, acting on a spinor ψ, yields

$$\varepsilon_1\bar{\varepsilon}_2\,\psi = (\bar{\varepsilon}_2\psi)\,\varepsilon_1 \ .$$

Since the Clifford algebra is the algebra of linear transformations in the space of spinors, the basis (A-8) is also a basis of this space and we can expand $\varepsilon_1\bar{\varepsilon}_2$ in terms of it. The resulting identity is the celebrated *Fierz identity*:

$$\begin{aligned} \varepsilon_1\bar{\varepsilon}_2 = -\tfrac{1}{4}(\bar{\varepsilon}_2\varepsilon_1)\,\mathbb{1} + \tfrac{1}{4}(\bar{\varepsilon}_2\gamma_5\varepsilon_1)\,\gamma_5 - \tfrac{1}{4}(\bar{\varepsilon}_2\gamma^\mu\varepsilon_1)\,\gamma_\mu \\ + \tfrac{1}{4}(\bar{\varepsilon}_2\gamma^\mu\gamma_5\varepsilon_1)\,\gamma_\mu\gamma_5 + \tfrac{1}{8}(\bar{\varepsilon}_2\gamma^{\mu\nu}\varepsilon_1)\,\gamma_{\mu\nu} \ , \end{aligned} \tag{A-9}$$

whose importance in supersymmetry calculations can hardly be overemphasised. (For commuting spinors there is an overall minus sign in the right-hand side.) The Fierz identity can be proven by tracing with the elements of the basis (A-8) and noticing that γ_5, γ_μ, $\gamma_\mu\gamma_5$ and $\gamma_{\mu\nu}$ are traceless. An important special case of the Fierz identity is

$$\varepsilon_1\bar{\varepsilon}_2 - \varepsilon_2\bar{\varepsilon}_1 = \tfrac{1}{2}(\bar{\varepsilon}_1\gamma^\mu\varepsilon_2)\,\gamma_\mu - \tfrac{1}{4}(\bar{\varepsilon}_1\gamma^{\mu\nu}\varepsilon_2)\,\gamma_{\mu\nu} \ , \tag{A-10}$$

which comes in handy when computing the commutator of two supersymmetries.

Closely related to the Fierz identity are the following identities involving powers of an anticommuting Majorana spinor θ:

$$\begin{aligned} \theta_a\theta_b &= \tfrac{1}{4}\left(\bar{\theta}\theta\,C_{ab} + \bar{\theta}\gamma_5\theta\,(\gamma_5)_{ab} + \bar{\theta}\gamma^\mu\gamma_5\theta\,(\gamma^\mu\gamma_5)_{ab}\right) \\ \theta_a\theta_b\theta_c &= \tfrac{1}{2}\bar{\theta}\theta\,\left(C_{ab}\theta_c + C_{ca}\theta_b + C_{bc}\theta_a\right) \\ \theta_a\theta_b\theta_c\theta_d &= \tfrac{1}{8}\bar{\theta}\theta\,\bar{\theta}\theta\,\left(C_{ab}C_{cd} - C_{ac}C_{bd} + C_{ad}C_{bc}\right) \ , \end{aligned} \tag{A-11}$$

with all other powers vanishing. These identities are extremely useful in expanding superfields.

A.5. **The spin group.** The spin group is isomorphic to $\mathrm{SL}(2,\mathbb{C})$ and hence has a natural two-dimensional complex representation, which we shall call $\mathbb{W}$. More precisely, $\mathbb{W}$ is the vector space $\mathbb{C}^2$ with the natural action of $\mathrm{SL}(2,\mathbb{C})$. If $w \in \mathbb{W}$ has components $w^\alpha = (w^1, w^2)$ relative to some fixed basis, and $M \in \mathrm{SL}(2,\mathbb{C})$, the action of M on w is defined simply by $(M\,w)^\alpha = M^\alpha{}_\beta w^\beta$.

This is not the only possible action of $\mathrm{SL}(2,\mathbb{C})$ on $\mathbb{C}^2$, though. We could also define an action by using instead of the matrix M, its complex conjugate $\bar{M}$, its inverse transpose $(M^t)^{-1}$ or its inverse hermitian adjoint $(M^\dagger)^{-1}$, since they all obey the same group multiplication law. These choices correspond, respectively to the *conjugate* representation $\overline{\mathbb{W}}$, the *dual* representation $\mathbb{W}^*$, and the *conjugate dual* representation $\overline{\mathbb{W}}^*$.

We will adopt the following notation: if $w^\alpha \in \mathbb{W}$, then $\bar{w}^{\dot\alpha} \in \overline{\mathbb{W}}$, $w_\alpha \in \mathbb{W}^*$ and $\bar{w}_{\dot\alpha} \in \overline{\mathbb{W}}^*$. These representations are not all inequivalent, since we can raise and lower indices in an $\mathrm{SL}(2,\mathbb{C})$-equivariant manner with the antisymmetric invariant tensors $\epsilon_{\alpha\beta}$ and $\bar\epsilon_{\dot\alpha\dot\beta}$. (The $\mathrm{SL}(2,\mathbb{C})$-invariance of these tensors is the statement that matrices in $\mathrm{SL}(2,\mathbb{C})$ have unit determinant.) Notice that we raise and lower also using the North-West and South-East conventions:

$$w_\alpha = w^\beta \epsilon_{\beta\alpha} \qquad \text{and} \qquad w^\alpha = \epsilon^{\alpha\beta} w_\beta \ ,$$

and similarly for the conjugate spinors:

$$\bar{w}_{\dot\alpha} = \bar{w}^{\dot\beta} \bar\epsilon_{\dot\beta\dot\alpha} \qquad \text{and} \qquad \bar{w}^{\dot\alpha} = \bar\epsilon^{\dot\alpha\dot\beta} \bar{w}_{\dot\beta} \ .$$

We choose the perhaps unusual normalisations:

$$\epsilon_{12} = 1 = \epsilon^{12} \quad \text{and} \quad \bar\epsilon_{\dot1\dot2} = -1 = \bar\epsilon^{\dot1\dot2} \ .$$

Because both the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ (when viewed as a real Lie algebra) and $\mathfrak{su}(2) \times \mathfrak{su}(2)$ are real forms of the same complex Lie algebra, one often employs the notation (j, j') for representations of $\mathrm{SL}(2,\mathbb{C})$, where j and j' are the spins of the two $\mathfrak{su}(2)$'s. In this notation the trivial one dimensional representation is denoted $(0,0)$, whereas $\mathbb{W} = (\frac{1}{2}, 0)$. The two $\mathfrak{su}(2)$'s are actually not independent but are related by complex conjugation, hence $\overline{\mathbb{W}} = (0, \frac{1}{2})$. In general, complex conjugation will interchange the labels. If a representation is preserved by complex conjugation, then it makes sense to restrict to the subrepresentation which is fixed by complex conjugation. For example, the Dirac spinors transform like $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$. The subrepresentation fixed by complex conjugation are precisely the Majorana spinors.

Another example is the representation $(\frac{1}{2}, \frac{1}{2})$. The real subrepresentation coincides with the defining representation of the Lorentz group—that is, the vector representation. To see this notice that any 4-vector $p_\mu = (p_0, \boldsymbol{p})$ can be turned into a bispinor as follows:

$$\sigma \cdot p \equiv \sigma^\mu p_\mu = \begin{pmatrix} p_0 + p_3 & p_1 - ip_2 \\ p_1 + ip_2 & p_0 - p_3 \end{pmatrix}$$

where $\sigma^\mu = (\mathbb{1}, \boldsymbol{\sigma})$ with $\boldsymbol{\sigma}$ the Pauli matrices:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{A-12}$$

Since the Pauli matrices are hermitian, so will be $\sigma \cdot p$ provided p_μ is real. The Pauli matrices have indices $(\sigma^\mu)^{\alpha\dot{\alpha}}$, which shows how $\mathrm{SL}(2,\mathbb{C})$ acts on this space. If $M \in \mathrm{SL}(2,\mathbb{C})$, then the action of M on such matrices is given by $\sigma \cdot p \mapsto M\, \sigma \cdot p\, M^\dagger$. This action is linear and preserves both the hermiticity of $\sigma \cdot p$ and the determinant $\det(\sigma \cdot p) = -p^2 = p_0^2 - \boldsymbol{p} \cdot \boldsymbol{p}$, whence it is a Lorentz transformation. Notice that both M and $-M$ act the same way on bispinors, which reiterates the fact that the spin group is the double cover of the Lorentz group.

A.6. **Weyl spinors.** Although the Dirac spinors form an irreducible representation of the (complexified) Clifford algebra, they are not an irreducible representation of the spin group. Indeed, since γ_5 anticommutes with γ_μ, it follows that it commutes with $\Sigma_{\mu\nu}$ and is not a multiple of the identity. Schur's lemma implies that the Dirac spinors are reducible under the spin group. In fact, they decompose into two irreducible two-dimensional representations, corresponding to the eigenspaces of γ_5. Since $(\gamma_5)^2 = -\mathbb{1}$, its eigenvalues are $\pm i$ and the eigenspaces form a complex conjugate pair. They are the *Weyl spinors*.

We now relate the Weyl spinors and the two-dimensional representations of $\mathrm{SL}(2,\mathbb{C})$ discussed above. To this effect we will use the following convenient realisation of the Clifford algebra

$$\gamma^\mu = \begin{pmatrix} 0 & -i\sigma^\mu \\ i\bar{\sigma}^\mu & 0 \end{pmatrix} , \qquad \text{where } \bar{\sigma}^\mu = (-\mathbb{1}, \boldsymbol{\sigma}). \tag{A-13}$$

Notice that $\bar{\sigma}^\mu$ is obtained from σ^μ by lowering indices:

$$(\bar{\sigma}^\mu)_{\dot{\alpha}\alpha} = (\sigma^\mu)^{\beta\dot{\beta}} \epsilon_{\beta\alpha} \bar{\epsilon}_{\dot{\beta}\dot{\alpha}} . \tag{A-14}$$

Notice that the indices in γ^μ are such that it acts naturally on objects of the form

$$\psi^a = \begin{pmatrix} \chi^\alpha \\ \bar{\zeta}_{\dot{\alpha}} \end{pmatrix} , \tag{A-15}$$

whence we see that a Dirac spinor indeed breaks up into a pair of two-component spinors. To see that these two-component spinors are precisely the Weyl spinors defined above, notice that in this realisation γ_5 becomes

$$\gamma_5 = \begin{pmatrix} -i\mathbb{1}^\alpha{}_\beta & 0 \\ 0 & i\mathbb{1}_{\dot{\alpha}}{}^{\dot{\beta}} \end{pmatrix} ,$$

so that $\mathbb{W}$ and $\bar{\mathbb{W}}$ are indeed complex conjugate eigenspaces of γ_5.

In this realisation the generators of the spin algebra $\Sigma_{\mu\nu}$ become block diagonal

$$\Sigma_{\mu\nu} = \begin{pmatrix} \frac{1}{2}\sigma_{\mu\nu} & 0 \\ 0 & \frac{1}{2}\bar{\sigma}_{\mu\nu} \end{pmatrix} ,$$

where

$$(\sigma_{\mu\nu})^{\alpha}{}_{\beta} = \tfrac{1}{2}(\sigma_\mu \bar\sigma_\nu - \sigma_\nu \bar\sigma_\mu)^{\alpha}{}_{\beta}$$
$$(\bar\sigma_{\mu\nu})_{\dot\alpha}{}^{\dot\beta} = \tfrac{1}{2}(\bar\sigma_\mu \sigma_\nu - \bar\sigma_\nu \sigma_\mu)_{\dot\alpha}{}^{\dot\beta} \ .$$

Notice that $\sigma_{\mu\nu}$ and $\bar\sigma_{\mu\nu}$ with both (spinor) indices up or down are symmetric matrices.

We collect here some useful identities involving the Pauli matrices:

$$\begin{aligned}
(\sigma^\mu)^{\alpha\dot\beta}(\bar\sigma^\nu)_{\dot\beta\gamma} &= \eta^{\mu\nu}\delta^\alpha_\gamma + (\sigma^{\mu\nu})^\alpha{}_\gamma \\
(\bar\sigma^\mu)_{\dot\beta\alpha}(\sigma^\nu)^{\alpha\dot\gamma} &= \eta^{\mu\nu}\delta^{\dot\gamma}_{\dot\beta} + (\bar\sigma_{\mu\nu})_{\dot\beta}{}^{\dot\gamma} \\
(\sigma^\mu)^{\alpha\dot\beta}(\sigma_\mu)^{\gamma\dot\delta} &= 2\epsilon^{\alpha\gamma}\bar\epsilon^{\dot\beta\dot\delta}.
\end{aligned} \tag{A-16}$$

Using the relation between the γ matrices and the Pauli matrices, it is possible to prove the following set of identities:

$$\begin{aligned}
\sigma^\mu\bar\sigma^\nu\sigma^\rho &= i\epsilon^{\mu\nu\rho\tau}\sigma_\tau + \eta^{\nu\rho}\sigma^\mu - \eta^{\mu\rho}\sigma^\nu + \eta^{\mu\nu}\sigma^\rho \\
\bar\sigma^\mu\sigma^\nu\bar\sigma^\rho &= -i\epsilon^{\mu\nu\rho\tau}\bar\sigma_\tau + \eta^{\nu\rho}\bar\sigma^\mu - \eta^{\mu\rho}\bar\sigma^\nu + \eta^{\mu\nu}\bar\sigma^\rho \\
\sigma^{\mu\nu}\sigma^\rho &= \eta^{\nu\rho}\sigma^\mu - \eta^{\mu\rho}\sigma^\nu + i\epsilon^{\mu\nu\rho\tau}\sigma_\tau \\
\bar\sigma^{\mu\nu}\bar\sigma^\rho &= \eta^{\nu\rho}\bar\sigma^\mu - \eta^{\mu\rho}\bar\sigma^\nu - i\epsilon^{\mu\nu\rho\tau}\bar\sigma_\tau \\
\tfrac{1}{2}\epsilon_{\mu\nu\rho\tau}\sigma^{\rho\tau} &= +i\sigma_{\mu\nu} \\
\tfrac{1}{2}\epsilon_{\mu\nu\rho\tau}\bar\sigma^{\rho\tau} &= -i\bar\sigma_{\mu\nu} \\
\mathrm{Tr}\left(\sigma^{\mu\nu}\sigma^{\rho\tau}\right) &= 2\left(\eta^{\nu\rho}\eta^{\mu\tau} - \eta^{\mu\rho}\eta^{\nu\tau} + i\epsilon^{\mu\nu\rho\tau}\right) \ .
\end{aligned} \tag{A-17}$$

In this realisation, a Majorana spinor takes the form

$$\psi^a = \begin{pmatrix} \psi^\alpha \\ \bar\psi_{\dot\alpha} \end{pmatrix} \ , \tag{A-18}$$

which is the same as saying that the charge conjugation matrix takes the form

$$C_{ab} = \begin{pmatrix} \epsilon_{\alpha\beta} & 0 \\ 0 & \bar\epsilon^{\dot\alpha\dot\beta} \end{pmatrix} \ . \tag{A-19}$$

In particular the (Majorana) conjugate spinor is given by

$$\bar\psi_a = \psi^b C_{ba} = (\psi_\alpha, -\bar\psi^{\dot\alpha}) \ .$$

The passage from Majorana to Weyl spinor inner products is given by:

$$\bar\chi\psi = \chi_a\psi^a = \chi_\alpha\psi^\alpha - \bar\chi^{\dot\alpha}\bar\psi_{\dot\alpha} = -(\chi^\alpha\psi_\alpha + \bar\chi^{\dot\alpha}\bar\psi_{\dot\alpha}) \ . \tag{A-20}$$

where the spinors on the left are four-component Majorana and those on the right are two-component Weyl.

A.7. **Two-component Fierz identities.** One of the advantages of the two-component formalism is that Fierz identities simplify considerably; although there are more of them. For example, suppose that ε and θ are two anticommuting spinors, then we have the following Fierz identities:

$$\begin{aligned} \varepsilon_\alpha\theta_\beta &= -\tfrac{1}{2}\varepsilon\theta\,\epsilon_{\alpha\beta} - \tfrac{1}{8}\varepsilon\sigma^{\mu\nu}\theta\,(\sigma_{\mu\nu})_{\alpha\beta} \\ \bar\varepsilon_{\dot\alpha}\bar\theta_{\dot\beta} &= -\tfrac{1}{2}\bar\varepsilon\bar\theta\,\bar\epsilon_{\dot\alpha\dot\beta} - \tfrac{1}{8}\bar\varepsilon\bar\sigma^{\mu\nu}\bar\theta\,(\bar\sigma_{\mu\nu})_{\dot\alpha\dot\beta} \\ \varepsilon_\alpha\bar\theta_{\dot\beta} &= +\tfrac{1}{2}\varepsilon\sigma^\mu\bar\theta\,(\bar\sigma_\mu)_{\dot\beta\alpha}\ , \end{aligned} \tag{A-21}$$

where we have used the following contractions

$$\begin{aligned} \varepsilon\theta &= \varepsilon^\alpha\theta_\alpha \\ \bar\varepsilon\bar\theta &= \bar\varepsilon^{\dot\alpha}\bar\theta_{\dot\alpha} \\ \varepsilon\sigma^\mu\bar\theta &= \varepsilon_\alpha(\sigma^\mu)^{\alpha\dot\beta}\bar\theta_{\dot\beta} \\ \bar\varepsilon\bar\sigma^\mu\theta &= \bar\varepsilon^{\dot\alpha}(\bar\sigma^\mu)_{\dot\alpha\beta}\theta^\beta \\ \varepsilon\sigma^{\mu\nu}\theta &= \varepsilon_\alpha(\sigma^{\mu\nu})^\alpha{}_\beta\theta^\beta \\ \bar\varepsilon\bar\sigma^{\mu\nu}\bar\theta &= \bar\varepsilon^{\dot\alpha}(\bar\sigma^{\mu\nu})_{\dot\alpha}{}^{\dot\beta}\bar\theta_{\dot\beta}\ . \end{aligned} \tag{A-22}$$

These contractions satisfy the following (anti)symmetry properties:

$$\begin{aligned} \varepsilon\theta &= +\theta\varepsilon \\ \bar\varepsilon\bar\theta &= +\bar\theta\bar\varepsilon \\ \bar\varepsilon\bar\sigma^\mu\theta &= -\theta\sigma^\mu\bar\varepsilon \\ \varepsilon\sigma^{\mu\nu}\theta &= -\theta\sigma^{\mu\nu}\varepsilon \\ \bar\varepsilon\bar\sigma^{\mu\nu}\bar\theta &= -\bar\theta\bar\sigma^{\mu\nu}\bar\varepsilon\ . \end{aligned} \tag{A-23}$$

(For commuting spinors, all the signs change.)

These Fierz identities allow us to prove a variety of useful identities simply by contracting indices and using equations (A-16) and (A-17). For example,

$$\bar\theta\bar\sigma^\mu\theta\,\bar\theta\bar\sigma^\nu\theta = -\tfrac{1}{2}\theta^2\bar\theta^2\,\eta^{\mu\nu} \tag{A-24}$$

and

$$\begin{aligned} \theta\psi\,\theta\sigma^\mu\bar\xi &= -\tfrac{1}{2}\theta^2\,\psi\sigma^\mu\bar\xi \\ \bar\theta\bar\psi\,\bar\theta\bar\sigma^\mu\xi &= -\tfrac{1}{2}\bar\theta^2\,\bar\psi\bar\sigma^\mu\xi\ . \end{aligned} \tag{A-25}$$

These and similar identities come in handy when working out component expansions of superfields.

A.8. **Complex conjugation.** Finally we come to complex conjugation. By definition, complex conjugation is always an involution, so that $(O^*)^* = O$ for any object O. For spinorial objects, we have that

$$(\theta_\alpha)^* = \bar\theta_{\dot\alpha}\ ,$$

which, because of our sign conventions, implies

$$(\theta^\alpha)^* = -\bar\theta^{\dot\alpha}\ .$$

Complex conjugation *always* reverses the order of anticommuting objects. For example,

$$(\theta_\alpha \theta_\beta)^* = \bar\theta_{\dot\beta} \bar\theta_{\dot\alpha} \qquad \text{and} \qquad (\theta^\alpha \theta^\beta \theta^\gamma)^* = -\bar\theta^{\dot\gamma} \bar\theta^{\dot\beta} \bar\theta^{\dot\alpha} \ .$$

In so doing, it does *not* give rise to a sign. This is *not* in conflict with the fact that the objects are anticommuting, since conjugation actually changes the objects being conjugated.

Hermiticity of the Pauli matrices means that

$$\begin{aligned} \left((\sigma^\mu)^{\alpha\dot\alpha}\right)^* &= (\bar\sigma^\mu)^{\dot\alpha\alpha} \\ ((\sigma_{\mu\nu})^\alpha{}_\beta)^* &= -(\bar\sigma_{\mu\nu})_{\dot\beta}{}^{\dot\alpha} \\ \left((\sigma_{\mu\nu})^{\alpha\beta}\right)^* &= +(\bar\sigma_{\mu\nu})^{\dot\alpha\dot\beta} \ . \end{aligned}$$

The last two equations show that complex conjugation indeed exchanges the two kinds of Weyl spinors.

In particular, notice that

$$\begin{aligned} (\varepsilon\theta)^* &= +\bar\theta\bar\varepsilon = +\bar\varepsilon\bar\theta \\ (\varepsilon\sigma^\mu\bar\theta)^* &= -\bar\varepsilon\bar\sigma^\mu\theta = +\theta\sigma^\mu\bar\varepsilon \\ (\varepsilon\sigma^{\mu\nu}\theta)^* &= +\bar\theta\bar\sigma^{\mu\nu}\bar\varepsilon = -\bar\varepsilon\bar\sigma^{\mu\nu}\bar\theta \ . \end{aligned}$$

This rule applies also to conjugating derivatives with respect to anticommuting coordinates. This guarantees that spinorial derivatives of scalars are indeed spinors. For example,

$$(\partial_\alpha)^* = \bar\partial_{\dot\alpha} \qquad \text{and} \qquad (\partial^\alpha)^* = -\bar\partial^{\dot\alpha} \ .$$

More generally, the rule applies to spinorial indices, as in

$$(\epsilon_{\alpha\beta})^* = \bar\epsilon_{\dot\beta\dot\alpha} \ .$$

A useful "reality check" is to make sure that any result involving bar'd objects agrees with the complex conjugate of the corresponding result with unbar'd objects. This simple procedure catches many a wayward sign.

Appendix B. Formulas

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & +1 & & \\ & & +1 & \\ & & & +1 \end{pmatrix}$$
$$\epsilon^{0123} = -\epsilon_{0123} = +1$$

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = +2\eta_{\mu\nu}\mathbb{1}$$
$$\gamma_{\mu\nu} := \tfrac{1}{2}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)$$
$$\gamma_{\mu\nu}\gamma_\rho = \gamma_{\mu\nu\rho} + \eta_{\nu\rho}\gamma_\mu - \eta_{\mu\rho}\gamma_\nu$$
$$\gamma^\rho\gamma_\mu\gamma_\rho = -2\gamma_\mu$$
$$\gamma^\rho\gamma_{\mu\nu}\gamma_\rho = 0$$

$$\gamma_5 := \tfrac{1}{4!}\epsilon^{\mu\nu\rho\sigma}\gamma_{\mu\nu\rho\sigma} = \gamma_0\gamma_1\gamma_2\gamma_3$$
$$\gamma_5^2 = -\mathbb{1}$$
$$\gamma_\mu\gamma_5 = -\gamma_5\gamma_\mu$$
$$\gamma_{\mu\nu}\gamma_5 = -\tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}\gamma^{\rho\sigma}$$
$$\gamma_{\mu\nu\rho} = \epsilon_{\mu\nu\rho\sigma}\gamma^\sigma\gamma_5$$
$$\gamma_{\mu\nu\rho\sigma} = -\epsilon_{\mu\nu\rho\sigma}\gamma_5$$

$$C^t = -C$$
$$C\gamma_\mu = -\gamma_\mu^t C$$
$$C\gamma_5 = +\gamma_5^t C$$
$$C\gamma_{\mu\nu} = -\gamma_{\mu\nu}^t C$$
$$\bar\psi_M := \psi^t C$$

$$C_{ab} = -C_{ba}$$
$$(\gamma_\mu)_{ab} := (\gamma_\mu)^c{}_b C_{ca} = (\gamma_\mu)_{ba}$$
$$(\gamma_{\mu\nu})_{ab} = (\gamma_{\mu\nu})_{ba}$$
$$(\gamma_\mu\gamma_5)_{ab} = -(\gamma_\mu\gamma_5)_{ba}$$
$$(\gamma_5)_{ab} = -(\gamma_5)_{ba}$$
$$\psi^a = C^{ab}\psi_b$$
$$\psi_a = \psi^b C_{ba}$$
$$\bar\varepsilon\psi := \varepsilon_a\psi^a = -\varepsilon^b\psi_b$$

$$\gamma_\mu^\dagger = \gamma_0\gamma_\mu\gamma_0$$
$$\gamma_5^\dagger = -\gamma_5$$
$$\gamma_{\mu\nu}^\dagger = \gamma_0\gamma_{\mu\nu}\gamma_0$$
$$\bar\psi_D := \psi^\dagger i\gamma^0$$
$$\bar\psi_D = \bar\psi_M \iff \psi^* = iC\gamma^0\psi$$

$$\begin{aligned}
\varepsilon_1\bar\varepsilon_2 = &-\tfrac{1}{4}(\bar\varepsilon_2\varepsilon_1)\,\mathbb{1}\\
&+\tfrac{1}{4}(\bar\varepsilon_2\gamma_5\varepsilon_1)\,\gamma_5\\
&-\tfrac{1}{4}(\bar\varepsilon_2\gamma^\mu\varepsilon_1)\,\gamma_\mu\\
&+\tfrac{1}{4}(\bar\varepsilon_2\gamma^\mu\gamma_5\varepsilon_1)\,\gamma_\mu\gamma_5\\
&+\tfrac{1}{8}(\bar\varepsilon_2\gamma^{\mu\nu}\varepsilon_1)\,\gamma_{\mu\nu}\\
\varepsilon_1\bar\varepsilon_2 - \varepsilon_2\bar\varepsilon_1 = &+\tfrac{1}{2}(\bar\varepsilon_1\gamma^\mu\varepsilon_2)\,\gamma_\mu\\
&-\tfrac{1}{4}(\bar\varepsilon_1\gamma^{\mu\nu}\varepsilon_2)\,\gamma_{\mu\nu}
\end{aligned}$$

$$\begin{aligned}
\theta_a\theta_b &= \tfrac{1}{4}\left(\bar\theta\theta\, C_{ab} + \bar\theta\gamma_5\theta\,(\gamma_5)_{ab}\right.\\
&\quad\left. + \bar\theta\gamma^\mu\gamma_5\theta\,(\gamma^\mu\gamma_5)_{ab}\right)\\
\theta_a\theta_b\theta_c &= \tfrac{1}{2}\bar\theta\theta\,(C_{ab}\theta_c + C_{ca}\theta_b\\
&\quad + C_{bc}\theta_a)\\
\theta_a\theta_b\theta_c\theta_d &= \tfrac{1}{8}\bar\theta\theta\,\bar\theta\theta\,(C_{ab}C_{cd} - C_{ac}C_{bd}\\
&\quad + C_{ad}C_{bc})\ ,
\end{aligned}$$

$$\epsilon_{12} = 1 = \epsilon^{12}$$
$$\bar\epsilon_{\dot1\dot2} = -1 = \bar\epsilon^{\dot1\dot2}$$
$$(\epsilon_{\alpha\beta})^* = \bar\epsilon_{\dot\beta\dot\alpha}$$
$$w_\alpha = w^\beta\epsilon_{\beta\alpha} \qquad w^\alpha = \epsilon^{\alpha\beta}w_\beta$$
$$\bar w_{\dot\alpha} = \bar w^{\dot\beta}\bar\epsilon_{\dot\beta\dot\alpha} \qquad \bar w^{\dot\alpha} = \bar\epsilon^{\dot\alpha\dot\beta}\bar w_{\dot\beta}$$

$$\sigma^\mu = (\mathbb{1}, \boldsymbol{\sigma}) \qquad \bar\sigma^\mu = (-\mathbb{1}, \boldsymbol{\sigma})$$
$$\boldsymbol{\sigma} : \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$(\bar\sigma^\mu)_{\dot\alpha\alpha} = (\sigma^\mu)^{\beta\dot\beta}\epsilon_{\beta\alpha}\bar\epsilon_{\dot\beta\dot\alpha}$$
$$\left((\sigma^\mu)^{\alpha\dot\alpha}\right)^* = (\bar\sigma^\mu)^{\dot\alpha\alpha}$$

$$(\sigma_{\mu\nu})^{\alpha}{}_{\beta} := \tfrac{1}{2}(\sigma_\mu\bar\sigma_\nu - \sigma_\nu\bar\sigma_\mu)^{\alpha}{}_{\beta}$$
$$(\bar\sigma_{\mu\nu})_{\dot\alpha}{}^{\dot\beta} := \tfrac{1}{2}(\bar\sigma_\mu\sigma_\nu - \bar\sigma_\nu\sigma_\mu)_{\dot\alpha}{}^{\dot\beta}$$
$$((\sigma_{\mu\nu})^{\alpha}{}_{\beta})^* = -(\bar\sigma_{\mu\nu})_{\dot\beta}{}^{\dot\alpha}$$
$$\left((\sigma_{\mu\nu})^{\alpha\beta}\right)^* = +(\bar\sigma_{\mu\nu})^{\dot\alpha\dot\beta}$$
$$(\sigma_{\mu\nu})_{\alpha\beta} = (\sigma_{\mu\nu})_{\beta\alpha}$$
$$(\bar\sigma_{\mu\nu})_{\dot\alpha\dot\beta} = (\bar\sigma_{\mu\nu})_{\dot\beta\dot\alpha}$$

$$(\sigma^\mu)^{\alpha\dot\beta}(\bar\sigma^\nu)_{\dot\beta\gamma} = \eta^{\mu\nu}\delta^\alpha_\gamma + (\sigma^{\mu\nu})^{\alpha}{}_{\gamma}$$
$$(\bar\sigma^\mu)_{\dot\beta\alpha}(\sigma^\nu)^{\alpha\dot\gamma} = \eta^{\mu\nu}\delta^{\dot\gamma}_{\dot\beta} + (\bar\sigma_{\mu\nu})_{\dot\beta}{}^{\dot\gamma}$$
$$(\sigma^\mu)^{\alpha\dot\beta}(\sigma_\mu)^{\gamma\dot\delta} = 2\epsilon^{\alpha\gamma}\bar\epsilon^{\dot\beta\dot\delta}$$

$$\begin{aligned}\sigma^\mu\bar\sigma^\nu\sigma^\rho &= +i\epsilon^{\mu\nu\rho\tau}\sigma_\tau + \eta^{\nu\rho}\sigma^\mu \\ &\quad - \eta^{\mu\rho}\sigma^\nu + \eta^{\mu\nu}\sigma^\rho \\ \bar\sigma^\mu\sigma^\nu\bar\sigma^\rho &= -i\epsilon^{\mu\nu\rho\tau}\bar\sigma_\tau + \eta^{\nu\rho}\bar\sigma^\mu \\ &\quad - \eta^{\mu\rho}\bar\sigma^\nu + \eta^{\mu\nu}\bar\sigma^\rho \\ \sigma^{\mu\nu}\sigma^\rho &= +\eta^{\nu\rho}\sigma^\mu - \eta^{\mu\rho}\sigma^\nu \\ &\quad + i\epsilon^{\mu\nu\rho\tau}\sigma_\tau \\ \bar\sigma^{\mu\nu}\bar\sigma^\rho &= +\eta^{\nu\rho}\bar\sigma^\mu - \eta^{\mu\rho}\bar\sigma^\nu \\ &\quad - i\epsilon^{\mu\nu\rho\tau}\bar\sigma_\tau\end{aligned}$$

$$\tfrac{1}{2}\epsilon_{\mu\nu\rho\tau}\sigma^{\rho\tau} = +i\sigma_{\mu\nu}$$
$$\tfrac{1}{2}\epsilon_{\mu\nu\rho\tau}\bar\sigma^{\rho\tau} = -i\bar\sigma_{\mu\nu}$$
$$\operatorname{Tr}(\sigma^{\mu\nu}\sigma^{\rho\tau}) = 2(\eta^{\nu\rho}\eta^{\mu\tau} - \eta^{\mu\rho}\eta^{\nu\tau} + i\epsilon^{\mu\nu\rho\tau})$$

$$(\text{Majorana})\quad \psi^a = \begin{pmatrix}\psi^\alpha \\ \bar\psi_{\dot\alpha}\end{pmatrix}$$
$$C = \begin{pmatrix}\epsilon_{\alpha\beta} & 0 \\ 0 & \bar\epsilon^{\dot\alpha\dot\beta}\end{pmatrix}$$
$$\bar\psi_a = \psi^b C_{ba} = (\psi_\alpha, -\bar\psi^{\dot\alpha})$$
$$\bar\chi\psi = \chi_a\psi^a = -(\chi\psi + \bar\chi\bar\psi)$$
$$(\psi_\alpha)^* = \bar\psi_{\dot\alpha}$$
$$(\psi^\alpha)^* = -\bar\psi^{\dot\alpha}$$

$$\gamma^\mu = \begin{pmatrix}0 & -i\sigma^\mu \\ i\bar\sigma^\mu & 0\end{pmatrix}$$
$$\gamma_5 = \begin{pmatrix}-i\mathbb{1}^{\alpha}{}_{\beta} & 0 \\ 0 & i\mathbb{1}_{\dot\alpha}{}^{\dot\beta}\end{pmatrix}$$
$$\Sigma_{\mu\nu} = \begin{pmatrix}\tfrac{1}{2}\sigma_{\mu\nu} & 0 \\ 0 & \tfrac{1}{2}\bar\sigma_{\mu\nu}\end{pmatrix}$$

$$\varepsilon\theta := \varepsilon^\alpha\theta_\alpha$$
$$\bar\varepsilon\bar\theta := \bar\varepsilon^{\dot\alpha}\bar\theta_{\dot\alpha}$$
$$\varepsilon\sigma^\mu\bar\theta := \varepsilon_\alpha(\sigma^\mu)^{\alpha\dot\beta}\bar\theta_{\dot\beta}$$
$$\bar\varepsilon\bar\sigma^\mu\theta := \bar\varepsilon^{\dot\alpha}(\bar\sigma^\mu)_{\dot\alpha\beta}\theta^\beta$$
$$\varepsilon\sigma^{\mu\nu}\theta := \varepsilon_\alpha(\sigma^{\mu\nu})^{\alpha}{}_{\beta}\theta^\beta$$
$$\bar\varepsilon\bar\sigma^{\mu\nu}\bar\theta := \bar\varepsilon^{\dot\alpha}(\bar\sigma^{\mu\nu})_{\dot\alpha}{}^{\dot\beta}\bar\theta_{\dot\beta}$$
$$\theta\sigma^\mu\bar\sigma^\nu\varepsilon := \theta_\alpha(\sigma^\mu)^{\alpha\dot\alpha}(\bar\sigma^\nu)_{\dot\alpha\beta}\varepsilon^\beta$$
$$\bar\theta\bar\sigma^\mu\sigma^\nu\bar\varepsilon := \bar\theta^{\dot\alpha}(\bar\sigma^\mu)_{\dot\alpha\alpha}(\sigma^\nu)^{\alpha\dot\beta}\bar\varepsilon_{\dot\beta}$$

$$\varepsilon\theta = +\theta\varepsilon$$
$$\bar\varepsilon\bar\theta = +\bar\theta\bar\varepsilon$$
$$\bar\varepsilon\bar\sigma^\mu\theta = -\theta\sigma^\mu\bar\varepsilon$$
$$\varepsilon\sigma^{\mu\nu}\theta = -\theta\sigma^{\mu\nu}\varepsilon$$
$$\bar\varepsilon\bar\sigma^{\mu\nu}\bar\theta = -\bar\theta\bar\sigma^{\mu\nu}\bar\varepsilon$$
$$\theta\sigma^\mu\bar\sigma^\nu\varepsilon = -\eta^{\mu\nu}\theta\varepsilon + \theta\sigma^{\mu\nu}\varepsilon$$
$$\bar\theta\bar\sigma^\mu\sigma^\nu\bar\varepsilon = +\eta^{\mu\nu}\bar\theta\bar\varepsilon + \bar\theta\bar\sigma^{\mu\nu}\bar\varepsilon$$
$$(\varepsilon\theta)^* = +\bar\theta\bar\varepsilon = +\bar\varepsilon\bar\theta$$
$$(\varepsilon\sigma^\mu\bar\theta)^* = -\bar\varepsilon\bar\sigma^\mu\theta = +\theta\sigma^\mu\bar\varepsilon$$
$$(\varepsilon\sigma^{\mu\nu}\theta)^* = +\bar\theta\bar\sigma^{\mu\nu}\bar\varepsilon = -\bar\varepsilon\bar\sigma^{\mu\nu}\bar\theta$$

$$\varepsilon_\alpha\theta_\beta = -\tfrac{1}{2}\varepsilon\theta\,\epsilon_{\alpha\beta} - \tfrac{1}{8}\varepsilon\sigma^{\mu\nu}\theta\,(\sigma_{\mu\nu})_{\alpha\beta}$$
$$\bar\varepsilon_{\dot\alpha}\bar\theta_{\dot\beta} = -\tfrac{1}{2}\bar\varepsilon\bar\theta\,\bar\epsilon_{\dot\alpha\dot\beta} - \tfrac{1}{8}\bar\varepsilon\bar\sigma^{\mu\nu}\bar\theta\,(\bar\sigma_{\mu\nu})_{\dot\alpha\dot\beta}$$
$$\varepsilon_\alpha\bar\theta_{\dot\beta} = +\tfrac{1}{2}\varepsilon\sigma^\mu\bar\theta\,(\bar\sigma_\mu)_{\dot\beta\alpha}$$

$$\theta_\alpha\theta_\beta = -\tfrac{1}{2}\theta^2\,\epsilon_{\alpha\beta}$$
$$\bar\theta_{\dot\alpha}\bar\theta_{\dot\beta} = -\tfrac{1}{2}\bar\theta^2\,\bar\epsilon_{\dot\alpha\dot\beta}$$
$$\theta_\alpha\bar\theta_{\dot\beta} = -\tfrac{1}{2}\bar\theta\bar\sigma^\mu\theta\,(\bar\sigma_\mu)_{\dot\beta\alpha}$$

$$\begin{aligned}
\theta\psi\,\theta\varepsilon &= -\tfrac{1}{2}\theta^2\psi\varepsilon\\
\bar\theta\bar\psi\,\bar\theta\bar\varepsilon &= -\tfrac{1}{2}\bar\theta^2\bar\psi\bar\varepsilon\\
\theta\psi\,\bar\theta\bar\varepsilon &= +\tfrac{1}{2}\theta\sigma_\mu\bar\theta\,\bar\varepsilon\bar\sigma^\mu\psi\\
\theta\psi\,\theta\sigma^\mu\bar\xi &= -\tfrac{1}{2}\theta^2\,\psi\sigma^\mu\bar\xi\\
\bar\theta\bar\psi\,\bar\theta\bar\sigma^\mu\xi &= -\tfrac{1}{2}\bar\theta^2\,\bar\psi\bar\sigma^\mu\xi\\
\bar\theta\bar\sigma^\mu\theta\,\theta\sigma^\nu\bar\varepsilon &= +\tfrac{1}{2}\theta^2\bar\theta\bar\varepsilon\eta^{\mu\nu} + \tfrac{1}{2}\theta^2\bar\theta\bar\sigma^{\mu\nu}\bar\varepsilon\\
\bar\theta\bar\sigma^\mu\theta\,\bar\theta\bar\sigma^\nu\varepsilon &= -\tfrac{1}{2}\bar\theta^2\theta\varepsilon\eta^{\mu\nu} + \tfrac{1}{2}\bar\theta^2\theta\sigma^{\mu\nu}\varepsilon
\end{aligned}$$

$$\begin{aligned}
[\mathsf{M}_{\mu\nu},\mathsf{P}_\rho] &= +\eta_{\nu\rho}\mathsf{P}_\mu - \eta_{\mu\rho}\mathsf{P}_\nu\\
[\mathsf{M}_{\mu\nu},\mathsf{M}_{\rho\sigma}] &= +\eta_{\nu\rho}\mathsf{M}_{\mu\sigma} - \eta_{\mu\rho}\mathsf{M}_{\nu\sigma}\\
&\quad -\eta_{\nu\sigma}\mathsf{M}_{\mu\rho} + \eta_{\mu\sigma}\mathsf{M}_{\nu\rho}\\
[\mathsf{P}_\mu,\mathsf{D}] &= +\mathsf{P}_\mu\\
[\mathsf{K}_\mu,\mathsf{D}] &= -\mathsf{K}_\mu\\
[\mathsf{P}_\mu,\mathsf{K}_\nu] &= +2\eta_{\mu\nu}\mathsf{D} - 2\mathsf{M}_{\mu\nu}\\
[\mathsf{M}_{\mu\nu},\mathsf{K}_\rho] &= +\eta_{\nu\rho}\mathsf{K}_\mu - \eta_{\mu\rho}\mathsf{K}_\nu\\
[\mathsf{M}_{\mu\nu},\mathsf{Q}_a] &= -(\Sigma_{\mu\nu})_a{}^b\mathsf{Q}_b\\
[\mathsf{Q}_a,\mathsf{Q}_b] &= +2(\gamma^\mu)_{ab}\mathsf{P}_\mu\\
[\mathsf{K}_\mu,\mathsf{Q}_a] &= +(\gamma_\mu)_a{}^b\mathsf{S}_b\\
[\mathsf{M}_{\mu\nu},\mathsf{S}_a] &= -(\Sigma_{\mu\nu})_a{}^b\mathsf{S}_b\\
[\mathsf{P}_\mu,\mathsf{S}_a] &= -(\gamma_\mu)_a{}^b\mathsf{Q}_b\\
[\mathsf{S}_a,\mathsf{S}_b] &= -2(\gamma^\mu)_{ab}\mathsf{K}_\mu\\
[\mathsf{Q}_a,\mathsf{S}_b] &= +2C_{ab}\mathsf{D} - 2(\gamma_5)_{ab}\mathsf{R}\\
&\quad +(\gamma^{\mu\nu})_{ab}\mathsf{M}_{\mu\nu}\\
[\mathsf{R},\mathsf{Q}_a] &= +\tfrac{1}{2}(\gamma_5)_a{}^b\mathsf{Q}_b\\
[\mathsf{R},\mathsf{S}_a] &= -\tfrac{1}{2}(\gamma_5)_a{}^b\mathsf{S}_b\\
[\mathsf{D},\mathsf{Q}_a] &= -\tfrac{1}{2}\mathsf{Q}_a\\
[\mathsf{D},\mathsf{S}_a] &= +\tfrac{1}{2}\mathsf{S}_a
\end{aligned}$$

$$\begin{aligned}
[\mathsf{M}_{\mu\nu},\mathsf{Q}_\alpha] &= -\tfrac{1}{2}(\sigma_{\mu\nu})_\alpha{}^\beta\mathsf{Q}_\beta\\
[\mathsf{M}_{\mu\nu},\bar{\mathsf{Q}}_{\dot\alpha}] &= +\tfrac{1}{2}(\sigma_{\mu\nu})_{\dot\alpha}{}^{\dot\beta}\bar{\mathsf{Q}}_{\dot\beta}\\
[\mathsf{Q}_\alpha,\bar{\mathsf{Q}}_{\dot\beta}] &= +2i(\bar\sigma^\mu)_{\dot\beta\alpha}\mathsf{P}_\mu
\end{aligned}$$

$$\begin{aligned}
e^X e^Y &= e^Z\\
Z = X + Y &+ \tfrac{1}{2}[X,Y]\\
&+ \tfrac{1}{12}[X,[X,Y]] + \cdots
\end{aligned}$$

$$\begin{aligned}
\partial_\alpha\theta^\beta &= \delta_\alpha{}^\beta & \bar\partial_{\dot\alpha}\bar\theta^{\dot\beta} &= \delta_{\dot\alpha}{}^{\dot\beta}\\
\partial_\alpha\theta_\beta &= \epsilon_{\alpha\beta} & \bar\partial_{\dot\alpha}\bar\theta_{\dot\beta} &= \bar\epsilon_{\dot\alpha\dot\beta}\\
\partial_\alpha\theta^2 &= 2\theta_\alpha & \bar\partial_{\dot\alpha}\bar\theta^2 &= 2\bar\theta_{\dot\alpha}\\
\partial^2\theta^2 &= -4 & \bar\partial^2\bar\theta^2 &= -4
\end{aligned}$$

$$\begin{aligned}
Q_\alpha &:= \partial_\alpha + i(\sigma^\mu)_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_\mu\\
\bar Q_{\dot\alpha} &:= \bar\partial_{\dot\alpha} + i(\bar\sigma^\mu)_{\dot\alpha\alpha}\theta^\alpha\partial_\mu\\
[Q_\alpha,\bar Q_{\dot\alpha}] &= +2i(\bar\sigma^\mu)_{\dot\alpha\alpha}\partial_\mu\\
D_\alpha &:= \partial_\alpha - i(\sigma^\mu)_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_\mu\\
\bar D_{\dot\alpha} &:= \bar\partial_{\dot\alpha} - i(\bar\sigma^\mu)_{\dot\alpha\alpha}\theta^\alpha\partial_\mu\\
[D_\alpha,\bar D_{\dot\alpha}] &= -2i(\bar\sigma^\mu)_{\dot\alpha\alpha}\partial_\mu
\end{aligned}$$

$$\begin{aligned}
U := \theta\sigma^\mu\bar\theta\partial_\mu &= -\bar\theta\bar\sigma^\mu\theta\partial_\mu\\
D_\alpha &= e^{iU}\partial_\alpha e^{-iU}\\
\bar D_{\dot\alpha} &= e^{-iU}\bar\partial_{\dot\alpha}e^{iU}\\
Q_\alpha &= e^{-iU}\partial_\alpha e^{iU}\\
\bar Q_{\dot\alpha} &= e^{iU}\bar\partial_{\dot\alpha}e^{-iU}
\end{aligned}$$

$$\bar{D}_{\dot{\alpha}}\Phi = 0$$
$$\Phi = e^{-iU}\left[\phi + \theta\chi + \theta^2 F\right]$$
$$\Phi = \phi + \theta\chi + \theta^2 F + i\bar{\theta}\bar{\sigma}^\mu\theta\partial_\mu\phi - \tfrac{i}{2}\theta^2\bar{\theta}\bar{\sigma}^\mu\partial_\mu\chi + \tfrac{1}{4}\theta^2\bar{\theta}^2\,\square\,\phi$$

$$\bar{V} = V$$
$$e^V \mapsto e^{-\bar{\Lambda}}e^V e^{-\Lambda}$$
$$\bar{D}_{\dot{\alpha}}\Lambda = 0 \quad D_\alpha\bar{\Lambda} = 0$$

In WZ gauge:

$$V = \bar{\theta}\bar{\sigma}^\mu\theta v_\mu + \bar{\theta}^2\theta\lambda + \theta^2\bar{\theta}\bar{\lambda} + \theta^2\bar{\theta}^2 D$$
$$W_\alpha := -\frac{1}{8g}\bar{D}^2 e^{-2g\,V}D_\alpha e^{2g\,V}$$
$$\overline{W}_{\dot{\alpha}} := -\frac{1}{8g}D^2 e^{-2g\,V}\bar{D}_{\dot{\alpha}} e^{2g\,V}$$
$$\bar{D}_{\dot{\alpha}}W_\alpha = 0 \qquad D_\alpha\overline{W}_{\dot{\alpha}} = 0$$
$$D^\alpha W_\alpha = \bar{D}^{\dot{\alpha}}\overline{W}_{\dot{\alpha}}$$

$$\mathcal{L} = \int d^2\theta d^2\bar{\theta}\,\left(\bar{\mathbf{\Phi}}e^{2g\,V}\mathbf{\Phi} + \operatorname{Tr}\mu V\right) + \left[\int d^2\theta\ \operatorname{Tr}\tfrac{1}{4}W^\alpha W_\alpha + \text{c.c.}\right] + \left[\int d^2\theta\, W(\mathbf{\Phi}) + \text{c.c.}\right]$$
$$W(\mathbf{\Phi}) = a_I\Phi^I + \tfrac{1}{2}m_{IJ}\Phi^I\Phi^J + \tfrac{1}{3}\lambda_{IJK}\Phi^I\Phi^J\Phi^K$$