Solutions to the Statistical Physics Exam

1 Series of questions

1.1 Fermions:

a) The correctly normalized and antisymmetrized wave function is

$$\frac{1}{\sqrt{2}}\left(\phi_1(x_1)\phi_2(x_2) - \phi_1(x_2)\phi_2(x_1)\right) .$$

b) The Slater determinant is the completely anti-symmetrized wave function of a system of N identical particles with 1-particle wave functions $\phi_i(x_j)$, $i,j = 1,\ldots,N$. It is given as

$$\mathcal{S}_-\Psi(x_1,\ldots,x_n) = \frac{1}{\sqrt{N!}}\begin{vmatrix} \phi_1(x_1) & \ldots & \ldots & \ldots & \phi_1(x_N) \\ \vdots & \ddots & & & \vdots \\ \vdots & & \phi_i(x_j) & & \vdots \\ \vdots & & & \ddots & \vdots \\ \phi_N(x_1) & \ldots & \ldots & \ldots & \phi_N(x_N) \end{vmatrix} .$$

Alternatively, this can be written as

$$\mathcal{S}_-\Psi(x_1,\ldots,x_n) = \frac{1}{\sqrt{N!}}\sum_{\pi\in S_N} \mathrm{sgn}(\pi)\phi_1(x_{\pi(1)})\ldots\phi_N(x_{\pi(N)}) .$$

c) We have

$$N = \sum_{\vec{p},\sigma}\hat{n}_{\vec{p},\sigma} = 2\sum_{|\vec{p}|\leq k_F} 1 \sim 2V\int_0^{k_F}\frac{\mathrm{d}^3p}{(2\pi)^3} = 2V\int_0^{k_F}\frac{p^2\mathrm{d}p\,\mathrm{d}\Omega}{(2\pi)^3} = \frac{V}{3\pi^2}k_F^3 ,$$

where the factor of 2 takes into account the spin degeneracy. Thus, we have $k_F = (3\pi^2 N/V)^{1/3}$, and with $\epsilon_F = \frac{p_F^2}{2m} = \frac{\hbar^2 k_F^2}{2m}$ we find $\epsilon_F = \frac{\hbar^2}{2m}(3\pi^2 N/V)^{2/3}$.

1.2 Second quantization:

a) Starting with $\hat{n}(\vec{x}) = \sum_\alpha \delta^{(3)}(\vec{x}-\vec{x}_\alpha)$, one has

$$\hat{n}(\vec{x}) = \sum_{i,j} a_i^+ a_j \phi_i^*(\vec{x})\phi_j(\vec{x}) = \Big(\sum_i a_i^+\phi_i^*(\vec{x})\Big)\Big(\sum_j a_j\phi_j(\vec{x})\Big) = \Psi^+(\vec{x})\Psi(\vec{x}) .$$

The total particle number is then simply $N = \int \mathrm{d}^3x\,\hat{n}(\vec{x}) = \int \mathrm{d}^3x\,\Psi^+(\vec{x})\Psi(\vec{x})$.

b) The generic Hamiltonian consists out of three parts, the kinetic energyi T , the potential U, and the inter-particle interaction V. The matrix elements of these three parts, in the momentum representation, read then, respectively:

$$\begin{aligned} \langle\vec{k}'|T|\vec{k}\rangle &= \frac{\hbar^2\vec{k}^2}{2m}\delta_{\vec{k}',\vec{k}} , \\ \langle\vec{k}'|U(\vec{x})|\vec{k}\rangle &= \frac{1}{V}\int \mathrm{d}^3x\,\exp(-\mathrm{i}(\vec{k}'-\vec{k})\cdot\vec{x})U(\vec{x}) \;=\; \frac{1}{V}\tilde{U}_{\vec{k}'-\vec{k}} , \end{aligned}$$

$$\begin{aligned}
\langle \vec{k}', \vec{p}' | V(\vec{x} - \vec{x}') | \vec{k}, \vec{p} \rangle &= \frac{1}{V^2} \int \mathrm{d}^3 x \int \mathrm{d}^3 x' \, \exp(-\mathrm{i}(\vec{p}' - \vec{p}) \cdot \vec{x}) \exp(-\mathrm{i}(\vec{k}' - \vec{k}) \cdot \vec{x}') V(\vec{x} - \vec{x}') \\
&= \frac{1}{V^3} \sum_{\vec{q}} \tilde{V}_{\vec{q}} \int \mathrm{d}^3 x \int \mathrm{d}^3 x' \, \exp(-\mathrm{i}(\vec{p}' - \vec{p}) \cdot \vec{x} - \mathrm{i}(\vec{k}' - \vec{k}) \cdot \vec{x}' + \mathrm{i}\vec{q} \cdot (\vec{x} - \vec{x}')) \\
&= \frac{1}{V} \sum_{\vec{q}} \tilde{V}_{\vec{q}} \delta_{\vec{p}', \vec{p} + \vec{q}} \delta_{\vec{k}', \vec{k} - \vec{q}} .
\end{aligned}$$

Collecting this, we find

$$H = \sum_{\vec{k}, \vec{k}'} \left(\frac{\hbar^2 \vec{k}^2}{2m} \delta_{\vec{k}, \vec{k}'} a^+_{\vec{k}'} a_{\vec{k}} + \frac{1}{V} \tilde{U}_{\vec{k}' - \vec{k}} a^+_{\vec{k}'} a_{\vec{k}} + \frac{1}{2V} \sum_{\vec{q}} \tilde{V}_{\vec{q}} a^+_{\vec{k}' + \vec{q}} a^+_{\vec{k} - \vec{q}} a_{\vec{k}} a_{\vec{k}'} \right) ,$$

where, for the last term, we have renamed $\vec{p}$ to $\vec{k}'$.

1.3 Fermions II:

a) From the definition, we have

$$\begin{aligned}
\left(\frac{n}{2}\right)^2 g_{\sigma\sigma'}(\vec{x} - \vec{x}') &= \langle \phi_0 | \Psi^+_\sigma(\vec{x}) \Psi^+_{\sigma'}(\vec{x}') \Psi_{\sigma'}(\vec{x}') \Psi_\sigma(\vec{x}) | \phi_0 \rangle \\
&= \langle \phi_0 | \Psi^+_\sigma(\vec{x}) \Psi_\sigma(\vec{x}) \Psi^+_{\sigma'}(\vec{x}') \Psi_{\sigma'}(\vec{x}') | \phi_0 \rangle - \delta_{\sigma\sigma'} \delta^{(3)}(\vec{x} - \vec{x}') \langle \phi_0 | \Psi^+_\sigma(\vec{x}) \Psi_\sigma(\vec{x}) | \phi_0 \rangle \\
&= \langle \phi_0 | \hat{n}(\vec{x}) \hat{n}(\vec{x}') | \phi_0 \rangle - \delta_{\sigma\sigma'} \delta^{(3)}(\vec{x} - \vec{x}') \langle \phi_0 | \hat{n}(\vec{x}) | \phi_0 \rangle .
\end{aligned}$$

To arrive at this result, one has to make use of the canonical anti-commutation relations for the fermionic field-operators.

b) The case $\sigma \neq \sigma'$ is easier, since then the second term does not contribute. Thus,

$$\begin{aligned}
\left(\frac{n}{2}\right)^2 g_{\sigma\sigma'}(\vec{x} - \vec{x}') &= \frac{1}{V^2} \sum_{\vec{q}, \vec{k}} \langle \phi_0 | \hat{n}_{\vec{k}, \sigma} \hat{n}_{\vec{q}, \sigma'} | \phi_0 \rangle \\
&= \frac{1}{V^2} N_\sigma N_{\sigma'} \;=\; \frac{1}{V^2} \frac{N}{2} \frac{N}{2} \\
&= \left(\frac{n}{2}\right)^2 .
\end{aligned}$$

Thus, $g_{\sigma\sigma'}(\vec{x} - \vec{x}') = 1$ for $\sigma \neq \sigma'$.

The case $\sigma = \sigma'$ is a bit more complicated. First, check out the matrix element

$$\begin{aligned}
\langle \phi_0 | c^+_{\vec{k}, \sigma} c^+_{\vec{q}, \sigma} c_{\vec{q}', \sigma} c_{\vec{k}', \sigma} | \phi_0 \rangle &= \left(\delta_{\vec{k}, \vec{k}'} \delta_{\vec{q}, \vec{q}'} - \delta_{\vec{k}, \vec{q}'} \delta_{\vec{q}, \vec{k}'} \right) \langle \phi_0 | c^+_{\vec{k}, \sigma} c_{\vec{k}, \sigma} c^+_{\vec{q}, \sigma} c_{\vec{q}, \sigma} | \phi_0 \rangle \\
&= \left(\delta_{\vec{k}, \vec{k}'} \delta_{\vec{q}, \vec{q}'} - \delta_{\vec{k}, \vec{q}'} \delta_{\vec{q}, \vec{k}'} \right) n_{\vec{k}, \sigma} n_{\vec{q}, \sigma} .
\end{aligned}$$

Since the fermionic cretaion- and annihilation-operators are nilpotent, e.g. $(c_{\vec{k}, \sigma})^2 = 0$, we must additionally have $\vec{k} \neq \vec{q}$, and therefore, all the creators and annihilators anti-commute with each other. Thus, we find

$$\left(\frac{n}{2}\right)^2 g_{\sigma\sigma}(\vec{x} - \vec{x}') = \frac{1}{V^2} \sum_{\vec{k}, \vec{q}} \Big(1 - \exp(-\mathrm{i}(\vec{k} - \vec{q}) \cdot (\vec{x} - \vec{x}')) \Big) n_{\vec{k}, \sigma} n_{\vec{q}\sigma} .$$

The second term can be recognized as the single-fermion distribution function $G_\sigma(\vec{x}-\vec{x}')$ squared, so that the final result is, with the abbreviation $r = |\vec{x}-\vec{x}'|$,

$$g_{\sigma\sigma}(\vec{x}-\vec{x}') = 1 - \frac{9}{(k_F r)^6}\Big(\sin(k_F r) - (k_F r)\cos(k_F r)\Big)^2 .$$

1.4 Density matrix and correlators:

a) Of course, $\rho = \frac{1}{Z}\exp(-\beta H)$. The partition function $Z = rmtr\,\exp(-\beta H)$ is the normalization constant such that $\mathrm{tr}\,\rho = 1$.

b) Of course, $\langle\mathcal{O}\rangle \equiv \mathrm{tr}\,(\rho\mathcal{O})$.

c) For this, we need to properties, namely firstly the invariance of the trace under cyclic permutations of products, and secondly that the density matrix commutes with the Hamiltonian, $[\rho, H] = 0$. Then, we have

$$\begin{aligned}
\langle A(t)B(t')\rangle &= \mathrm{tr}(\rho\,\mathrm{e}^{\mathrm{i}Ht/\hbar}A\mathrm{e}^{-\mathrm{i}Ht/\hbar}\mathrm{e}^{\mathrm{i}Ht'/\hbar}B\mathrm{e}^{-\mathrm{i}Ht'/\hbar})\\
&= \mathrm{tr}(\rho\,\mathrm{e}^{\mathrm{i}Ht/\hbar}A\mathrm{e}^{-\mathrm{i}H(t-t')/\hbar}B\mathrm{e}^{-\mathrm{i}Ht'/\hbar})\\
&= \mathrm{tr}(\mathrm{e}^{-\mathrm{i}Ht'/\hbar}\rho\,\mathrm{e}^{\mathrm{i}Ht/\hbar}A\mathrm{e}^{-\mathrm{i}H(t-t')/\hbar}B) \quad \text{by cyclicity}\\
&= \mathrm{tr}(\rho\,\mathrm{e}^{-\mathrm{i}Ht'/\hbar}\mathrm{e}^{\mathrm{i}Ht/\hbar}A\mathrm{e}^{-\mathrm{i}H(t-t')/\hbar}B) \quad \text{because } \rho \text{ commutes with} H\\
&= \mathrm{tr}(\rho\,\mathrm{e}^{\mathrm{i}H(t-t')/\hbar}A\mathrm{e}^{-\mathrm{i}H(t-t')/\hbar}B)\\
&= \langle A(t-t')B(0)\rangle\,.
\end{aligned}$$

2 Exercices

The exercises are more or less identical to some of the tutorial and homeworks. Thus, their solutions can easily be worked out or even simply looked up from the material presented by Guillaume Palacios during the tutorial sessions.