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Notes on non-trivial and logarithmic CFTs with c = 0

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Abstract

We examine the properties of two-dimensional conformal field theories (CFTs) with vanishing central charge based on the extended Kac-table for $c_{(9,6)} = 0$ using a general ansatz for the stress energy tensor residing in a Jordan cell of rank two.

Within this setup we will derive the OPEs and two point functions of the stress energy tensor $T(z)$ and its logarithmic partner field $t(z)$ and illustrate this by a bosonic field realization.

We will show why our approach may be more promising than those chosen in the literature so far, including a discussion on properties of the augmented minimal model with vanishing central charge such as full conformal invariance of the vacuum as a state in an irreducible representation.

Furthermore we will present a more general solution of another solution to the $c \to 0$ catastrophe based on a logarithmic CFT tensor model. As an example, we consider a tensor product of the well-known $c = -2$ logarithmic CFT with a fourfold Ising model.

We give an overview of the possible configurations and various consequences on the two point functions and the OPEs of the stress energy tensor $T(z)$ and its logarithmic partner field $t(z)$. We will motivate that due to the full conformal invariance of the vacuum at $c = 0$, we should assume a Jordan cell for the identity since $t(z)$ is presumably a descendant of a new $h = 0$ field.

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Contents

1 Introduction

During the last decade, the interest in $c = 0$ conformal field theories (CFTs) has risen considerably because such theories presumably play an important role in the understanding of percolation and other disorder problems. The problem of vanishing central charge caused a vivid discussion on suitable approaches since the canonical choice of ordinary minimal models does not seem to be sufficient in its field content. There have been several attempts considering the subject before, most notably by Cardy [2], Kogan and Nichols [22, 21] or Gurarie and Ludwig [18, 17]. A further clue comes from the deviation of the partition function from one as observed by Pearce et al. [30] in numerical studies. While the former approaches involve features which are not necessary for $c \neq 0$ theories, this paper will concentrate solely on known techniques and structures to fit $c = 0$ into ordinary (logarithmic) CFTs. Thus, we will assume that the field content of the $c = 0$ theory can entirely be read off from an eventually extended Kac-table with respect to a suitable chiral symmetry algebra. Of course, the simplest case is the Virasoro algebra, on which we mainly focus.

In the second chapter we will start with a sketch of the problems arising at vanishing central charge. We know from numerical simulations, e. g. Pearce et al. [30], that the partition function of their non-trivial $c = 0$ theory is not equal to one. However, this is not what we would expect, if the $c = 0$ theory was just a plain minimal model, since the field content as given by the Kac-table of $c_{(3,2)} = 0$ consists only of the identity. As a direct consequence, the field content has to be modified. For this setup Cardy [2] gave three possible choices as explained in 2.2. From there on we will concentrate on one of them which is based solely on the Kac table. We will see that the Kac-table has to be extended, and the smallest possibility for this seems to be the table for $c_{(9,6)} = 0$ as discussed in [4].

In the third chapter we will derive the three OPEs of the stress energy operator $T(z)$ and its logarithmic partner field $t(z)$, starting with the assumption of two L_0 Jordan cell connected $h = 0$ fields motivated by the Kac table of the augmented minimal model with vanishing central charge $c_{(9,6)} = 0$. Within this ansatz we find that only the vacuum expectation value of $t(z)t(w)$ survives. It is proportional to some arbitrary factor θ which equals the central extension of the algebra between the modes of $T(z)$ and $t(z)$. But contrarily to what one would expect, it is independent of the parameter b introduced by [22] and [17] since it is not proportional to $\langle Tt \rangle$ which itself is proportional to the central charge $c = 0$.

A direct consequence of these results will be given in 3.3 where we discuss the consequences on the OPEs of primary fields within this setup. We will show that the metric is not invertible and thus the OPE is not well defined.

For an introduction to LCFT in general, see [6, 11] and references therein.

In the fourth chapter we will present a bosonic free field construction based on a vertex operator ansatz basically yielding the same results as in our theoretical calculation by setting $\theta = 0$. Thus we suggest that probably a fermionic construction may provide more useful results. In general, fermionic realizations (in contrary to bosonic ones) have the nice feature of a natural truncation due to their nilpotency. Thus only terms up to $\log^2$ may arise in the OPEs of such a theory.

Afterwards, in section 5.1 we will recall the third solution of the $c \to 0$ catastrophe chosen by Gurarie and Ludwig [17] or Kogan and Nichols [22] which introduces fields outside the Kac table explicitly excluding the case of the identity $\mathbb{I}$ residing in a Jordan cell. Additionally we will compare our approach to theirs with respect to the advantages and disadvantages following from the respective assumptions in section 5.2.

The next chapter will elucidate features of the augmented minimal $c_{(9,6)} = 0$ model. This includes an overview on the consequences of the full conformal invariance of the irreducible highest weight representation generated by the identity which can be broken when regarded as a subrepresentation of the indecomposable representation based on the second $h = 0$ field. In 6.2 we will discuss the possibility of null states within our approach which will give us a possibility to subdivide augmented minimal $c = 0$ models with respect to the central extension of the mixed algebra between the modes of $T(z)$ and $t(z)$. This feature will lead us to remarks on percolation as a $c = 0$ model. Additionally, we comment on the field content of the Kac table and the structure of the representations contained therein. Further details will appear in another paper [7].

In the seventh chapter we will present a variation of the fourth loophole added to Cardy's solutions of the $c \to 0$ catastrophe [2] following Kogan and Nichols [22]. This tensor ansatz of an LCFT with central charge c_1 and an ordinary CFT with central charge $c_2 = -c_1$ yields a $c = 0$ theory avoiding the problems arising in the OPEs of primary fields. For this case we present an example of a tensorized CFT of symplectic fermions with a reduced fourfold Ising model in the last section.

All details of the calculations will be given in the appendix, as most of the results of the OPEs in chapters 4 and 7 are given only in parts which are sufficient to see the important features and compare them to previous results as e. g. derived in [17]. Additionally, we will derive the algebra of the modes of $T(z)$ and $t(z)$, $[L_n, l_m]$, to justify why we do not adopt the result of [17]. This will include a general remark on the mode expansion of logarithmic fields and the consequences of the requirement of regularity.

2 General remarks on CFTs with vanishing central charge

2.1 Problems at c = 0

After the introduction of conformal field theories by Belavin, Polyakov and Zamolodchikov [1] twenty years ago and the discovery of logarithmic behavior by Gurarie in 1993 [15] which led to the investigation of so called logarithmic CFT, the understanding of most (L)CFTs, especially the minimal models characterized by the two parameters (p, q) with $q, p \in \mathbb{N}$, $c_{(p,q)} = 1 - 6\frac{(p-q)^2}{pq}$ improved continually. As to CFTs whose field content can not be described solely by the Kac table, i. e. a non-trivial $c_{(3,2)} = 0$ model, this is not the case. There is still a controversial discussion going on about different approaches to (L)CFTs with vanishing central charge which we will try to elucidate in this paper.

For $c = 0$ as an ordinary minimal model, we have $(p, q) = (3, 2)$ and thus a Kac table which consists only of one field, the identity. Keeping the vanishing of the central charge in mind, we know that $L_n|0\rangle = 0$ for all $n \in \mathbb{Z}$ and thus the theory is trivial. But from concrete models, e. g. by Pearce et. al. [30], we know that the partition function differs from one and therefore there are applications where there have to be more fields involved. More concretely, they identify an $h = 1/3$ primary field.

A similar problem occurred in the study of the $c_{(p,1)}$ models whose Kac-tables a priori are empty. Following the procedure which is usually applied to this kind of minimal model, i. e. including the operators on the boundary of the conformal grid into the theory, we get a non trivial $c = 0$ CFT. Additionally it can be shown that they generate indecomposable representations which leads to logarithms in the OPEs of some of these fields [12]. The main advantage of this procedure is that we maintain the properties of all finite Kac-table based CFTs, e. g. the existence of an infinite set of null vectors, thus a rather small field content and the possibility of additional symmetries. It is remarkable that, up to now, in all known logarithmic CFTs, i. e. the $c_{(p,1)}$ models, the identity has a logarithmic partner. Taking the well known formula

$$h_{r,s}(c_{(p,q)}) = \frac{(pr - qs)^2 - (p - q)^2}{4pq} \tag{1}$$

for $q = 1$ and $1 \leq s < 3p$, $1 \leq r < 3q$, i. e. the weights of the operators on the boundary of the conformal grid and those needed for closure under fusion, we always have at least two solutions for $h = 0$: $s_{\pm} = rp \pm (p - 1)$. In a logarithmic theory, these operators cannot be identified with each other.

Thus taking a similar approach to construct a non-trivial $c = 0$ LCFT, we would expect it to contain a degenerate vacuum as well.

There exists a variety of proposals on how to approach $c = 0$. Apart from the suggestions of other LCFTs as discussed above, Cardy [2] tried a general replica ansatz in order to find a loophole to the divergences arising in the OPE of primary fields at $c = 0$.

For any conformal field theory (for the time being we will restrict ourselves to non degenerate vacua and the holomorphic parts), we can write down the OPE of a primary scalar field $\phi(z)$ with conformal weight h,

$$\phi_h(z)\phi_h^\dagger(0) \sim \frac{C_{\Phi\Phi}^{\mathbb{I}}}{z^{2h}}\left(1 + \frac{2h}{c}z^2 T(0) + \ldots\right) + \ldots\,, \tag{2}$$

with $C_{\Phi\Phi}^{\mathbb{I}}$ being the coefficient of the three point function usually normalized to 1 or $\frac{c}{h}$ for $h \neq 0$. For the ordinary minimal model $c_{(3,2)} = 0$ the expression is not problematic since the only possible choice for ϕ is $\mathbb{I}$ and thus $h = 0$. Although, if we seek to describe a model as found by [30], we have to assume additional fields to the identity for which the division by the central charge is not well defined.

2.2 Suggestions how to treat c = 0 properly

According to Cardy [2], there are basically three ways out of the problem as the central charge approaches zero in the OPE of primary fields as given in (2).

(I) $(h, \bar{h}) \to 0$ as $c \to 0$.

(II) $C_{\Phi\Phi}^{\mathbb{I}} \to 0$ as $c \to 0$.

(III) Other additional operators with $h \to 2$ arise in the OPE, canceling the divergences.

Thus the first case can be applied to the ordinary minimal model with the Kac-table

$$c_{(2,3)} = 0 \quad : \quad \boxed{0 \mid 0}\,. \tag{3}$$

The second case has to be taken if we restrict ourselves to the extended Kac-table as for the $c_{(p,1)}$-models. In this case we have to normalize our three-point functions to $\frac{c}{h_\phi}$ and thus the condition $C_{\Phi\Phi}^{\mathbb{I}} \to 0$ as $c \to 0$ is satisfied trivially and the OPE

$$\phi_h(z)\phi_h^\dagger(0) \sim \frac{c}{hz^{2h}} + \frac{2}{z^{2h-2}}T(0) + \ldots \tag{4}$$

stays regular. As discussed above, we expect the identity to have a partner field in these theories, and thus we have to modify the OPE of primary fields. This will be done in the third chapter.

The third case has been chosen by Kogan and Nichols [22] as well as by Gurarie and Ludwig [17]. It includes a new concept of LCFTs which is structurally different from that of $c_{(p,1)}$ models by introducing the logarithmic partner field of the stress energy tensor via the postulate of a non-chiral field X with dimension $h = (2+\alpha, \alpha)$ in the $\alpha \to 0$ limit. A more detailed discussion of this approach will be given in section 5.1. Within this ansatz, fields outside the Kac-table arise in the OPE of primary fields.

However, our paper is based on our objections to this treatment of $c = 0$-LCFTs since we do not have any knowledge about the behavior of such fields outside the Kac table in OPEs among themselves and thus there is no a priori limit on the number of fields available in the emerging CFT. Nothing of what is known for Kac-table based models as null states, symmetries or representation properties can be assumed to be extended to this kind of $c = 0$ theory. Furthermore, this approach introduces fields that have no known direct physical meaning since in all known applications for $c = 0$ the critical exponents of physical quantities are expected to be found in the Kac table. Thus it seems more natural to stay within the known framework of augmented Kac-tables as in the $c_{(p,1)}$-models which simply include the operators on the boundary of the Kac-table rather than introducing new objects. Furthermore, in the work of Pearce et al. [30], the representation belonging to $h_{(1,3)} = 1/3$ which lies on the boundary of the Kac-table is expected to play an important role in addition to the existence of two inequivalent $h = 0$ representations. In our opinion, this justifies a deeper investigation of a Kac-table based $c = 0$ model with at least two $h = 0$ fields which may reside in a Jordan cell as known from ordinary LCFTs (which hopefully will be proven in the near future [7]).

These are the crucial points where we do not agree with the approach of [2] who suggested that the logarithmic partners of Kac-operators always reside outside of the Kac-table. In contrary, we favor the ansatz of sticking to the restrictive structure of an augmented minimal model following e. g. [4] taking the Kac-table of $c_{(9,6)}$ as a basis to describe a $c = 0$ LCFT.

3 OPEs in the augmented minimal model

3.1 The standard assumptions for two-point functions

In the following we will show how our approach differs from the usual constructions. In contrary to our ansatz it is usually assumed that the Jordan cell exists on the $h = 2$ level and *not* on the identity level as well. But in our opinion any theory with arbitrary central charge $c \neq 0$, if extended to a logarithmic CFT, has to possess a global Jordan cell structure. In standard LCFT, i. e. constructed via the same formalism as used for $c_{(p,1)}$-models, primary fields and their logarithmic partners form Jordan cells with respect to L_0. In these models, the identity always resides in such a Jordan cell and, particularly, there can not be a Jordan cell structure at the second level without having this structure in the vacuum sector - at least not in standard LCFT where we do not know of any such behavior.

Contrary to the common prerequisites, the limiting procedure (or replica approach) of [17, 22, 29, 28] explicitly uses the absence of such a Jordan cell on the identity level and instead the existence of fields outside the Kac-table as key features. Therefore this non Kac-table based ansatz which has been discussed in the literature so far has to be regarded as a new, non standard approach to $c = 0$ LCFT and thus we also want to draw attention to the treatment of $c = 0$ LCFTs within the standard setting.

Based on our knowledge of the structure of $c_{(p,1)}$-LCFTs, we will concentrate on our proposal how to treat $c_{(3,2)} = 0$ in an analogous manner. Therefore we assume that we have at least a rank two Jordan cell on the identity level and, for the sake of simplicity, only go through detailed calculations for the rank two case since the crucial feature (5) only distinguishes between rank equals one or not.

A consequence of this assumption is the following important property of vacuum expectation values (see e.g. [31] or [5] for an elaborate treatment), where $\tilde{\mathbb{I}}(z)$ denotes the logarithmic partner of the identity $\mathbb{I}$:

$$\begin{aligned} \langle 0|0\rangle = \langle 0|\mathbb{I}|0\rangle &= 0\,, && (5)\\ \langle \tilde{0}|0\rangle = \langle 0|\tilde{0}\rangle = \langle 0|\tilde{\mathbb{I}}(z)|0\rangle &= 1\,, && (6)\\ \langle 0|\tilde{\mathbb{I}}(z)\tilde{\mathbb{I}}(w)|0\rangle &= -2\log(z-w)\,. && (7) \end{aligned}$$

If we keep this in mind, it is clear that any stress tensor within a LCFT will have a vanishing two-point function, where n-point functions are understood as usual, as vacuum expectation values of n field insertions. This must be so, because the central term comes with the identity $\mathbb{I}$ and not its logarithmic partner, i. e.

$$T(z)T(w) = \frac{c/2}{(z-w)^4}\mathbb{I} + \frac{2}{(z-w)^2}T(w) + \frac{1}{(z-w)}\partial T(w)\,. \tag{8}$$

Suppose now that the stress energy tensor $T(z)$ has a logarithmic partner, which we call $t(z)$ to match with the convention in the literature. In fact, there is a rather simple example of how to construct such a partner, $t(z) = {:}\tilde{\mathbb{I}}T{:}(z)$. Standard considerations in rank two LCFT now imply the following behavior in an OPE with the stress energy tensor, revealing the Jordan cell structure:

$$T(z)t(w) = \frac{c/2}{(z-w)^4}\tilde{\mathbb{I}} + \frac{\mu}{(z-w)^4}\mathbb{I} + \frac{2}{(z-w)^2}t(w) + \frac{\lambda}{(z-w)^2}T(w) + \frac{1}{(z-w)^1}\partial t(w)\,. \tag{9}$$

Note that the central charge c now appears together with the logarithmic partner $\tilde{\mathbb{I}}$ of the identity, and that a new central term μ may appear with the identity $\mathbb{I}$. Also, the constant λ with which the original stress energy tensor appears, can be any non-zero number depending on the normalization of the off-diagonal part in the Jordan cell. It is conventional to put $\lambda = 1$. As standard logarithmic pair, $\{T, t\}$ must obey the following two-point functions

$$\langle T(z)T(w)\rangle = 0\,, \tag{10}$$

$$\langle T(z)t(w)\rangle = \frac{b}{(z-w)^4}\,, \tag{11}$$

$$\langle t(z)t(w)\rangle = \frac{1}{(z-w)^4}\left(\theta - 2b\log(z-w)\right)\,. \tag{12}$$

We once more emphasize that this should be true in any rank two LCFT, independent of the value of the central charge c.

3.2 The generic form of the OPE

We will now start with the generic form of the OPE for a pair of two fields of the same weight either being the primary field or its logarithmic partner, respectively, following [5, 9]. In general, it is given by

$$\phi_{h_i}(z)\phi_{h_j}(w) = \sum_k C_{ij}^k (z-w)^{h_k}\left(\phi_{h_k} + \sum_{\{n\}} \beta_{ij}^{k,\{n\}} (z-w)^{|\{n\}|}\phi_{h_k}^{(-\{n\})}(w)\right)\,, \tag{13}$$

where the coefficients $\beta_{ij}^{k,\{n\}}$ of the descendant contributions,

$$\phi_{h_k}^{(-\{n\})} = L_{(-\{n\})}\phi_{h_k} = L_{-n_1}L_{-n_2}\ldots L_{-n_l}\phi_{h_k}\,, \tag{14}$$

are fixed by conformal covariance.

The structure "constants" C_{ij}^k (which in an LCFT can no longer referred to as constants since they merely become functions containing logarithms) can be derived through the two- and three-point functions, i. e. $C_{ij}^k = C_{ijl}D^{lk}$, with

$$\begin{aligned}
D_{ij} &= \langle \phi_{h_i}(\infty)\phi_{h_j}(0)\rangle \propto \delta_{h_i,h_j}\,, &(15)\\
C_{ijk} &= \langle \phi_{h_i}(\infty)\phi_{h_j}(1)\phi_{h_k}(0)\rangle\,. &(16)
\end{aligned}$$

Note that in our case of a rank two LCFT the metric is no longer diagonal but for $h \equiv h_i = h_j$ looks like

$$D_{(i,j)} = \left(\begin{array}{cc} 0 & D^{(0)}_{\Phi\Phi} \\ D^{(0)}_{\Phi\Phi} & D^{(1)}_{\Phi\Phi} - 2D^{(0)}_{\Phi\Phi}\log(z-w) \end{array}\right)(z-w)^{-2h} \tag{17}$$

in the notation following below.

In fact, within the most general ansatz of a rank two Jordan cell, we have for the two-point functions

$$\begin{aligned}
\langle\Phi(z)\Phi(w)\rangle &= 0\,, &(18)\\
\langle\Phi(z)\tilde\Phi(w)\rangle = \langle\tilde\Phi(z)\Phi(w)\rangle &= D^{(0)}_{\Phi\Phi}(z-w)^{-2h}\,, &(19)\\
\langle\tilde\Phi(z)\tilde\Phi(w)\rangle &= \left(D^{(1)}_{\Phi\Phi} - 2\log(z-w)D^{(0)}_{\Phi\Phi}\right)(z-w)^{-2h}\,, &(20)
\end{aligned}$$

where $D^{(0)}_{\Phi\Phi} = D_{\Phi\tilde\Phi} = D_{\tilde\Phi\Phi}$ and $D^{(1)}_{\Phi\Phi} = D_{\tilde\Phi\tilde\Phi}$. For the three-point functions involving the pair $\{\Phi,\tilde\Phi\}$, omitting the explicit dependence on the three points of insertion z_1, z_2, z_3, we have

$$\begin{aligned}
\langle TT\Phi\rangle &= 0\,,\\
\langle TT\tilde\Phi\rangle &= C^{(0)}_{TT\Phi} z_{12}^{h-4} z_{13}^{-h} z_{23}^{-h}\,,\\
&= \langle tT\Phi\rangle = \langle Tt\Phi\rangle\\
\langle tt\Phi\rangle &= \left(C^{(1)}_{TT\Phi} - 2\log(z_{12})C^{(0)}_{TT\Phi}\right) z_{12}^{h-4} z_{13}^{-h} z_{23}^{-h}\,,\\
\langle tT\tilde\Phi\rangle &= \left(C^{(1)}_{TT\Phi} - 2\log(z_{13})C^{(0)}_{TT\Phi}\right) z_{12}^{h-4} z_{13}^{-h} z_{23}^{-h}\,,\\
\langle Tt\tilde\Phi\rangle &= \left(C^{(1)}_{TT\Phi} - 2\log(z_{23})C^{(0)}_{TT\Phi}\right) z_{12}^{h-4} z_{13}^{-h} z_{23}^{-h}\,,\\
\langle tt\tilde\Phi\rangle &= \Big(C^{(2)}_{TT\Phi} - C^{(1)}_{TT\Phi}(\log(z_{12}) + \log(z_{13}) + \log(z_{23}))\\
&\quad - C^{(0)}_{TT\Phi}(\log^2(z_{12}) + \log^2(z_{13}) + \log^2(z_{23}) - 2\log(z_{12})\log(z_{13})\\
&\quad - 2\log(z_{12})\log(z_{23}) - 2\log(z_{13})\log(z_{23}))\Big)\, z_{12}^{h-4} z_{13}^{-h} z_{23}^{-h}\,.
\end{aligned}$$

Note that the constants in front of the logarithms are always given in terms of the constant of previous correlators with less insertions of logarithmic partner fields. Moreover, the constants depend only on the total sum of logarithmic insertions. We are interested in two particular contributions to the OPE, namely the pair $\{\mathbb{I}, \tilde{\mathbb{I}}\}$ and the pair $\{T, t\}$ for $\{\Phi, \tilde{\Phi}\}$, respectively. But let us first write down the generic form of the OPE channel with conformal weight h, expressed in the structure constants of the two- and three-point functions:

$$
\begin{aligned}
T(z)T(0) &= z^{h-4}\frac{C^{(0)}_{TT\Phi}}{D^{(0)}_{\Phi\Phi}}\Phi(0) + \ldots \quad , \\
T(z)t(0) &= z^{h-4}\left(\frac{C^{(0)}_{TT\Phi}}{D^{(0)}_{\Phi\Phi}}\tilde{\Phi}(0) + \frac{C^{(1)}_{TT\Phi}D^{(0)}_{\Phi\Phi} - C^{(0)}_{TT\Phi}D^{(1)}_{\Phi\Phi}}{(D^{(0)}_{\Phi\Phi})^2}\Phi(0)\right) + \ldots \quad , \\
t(z)t(0) &= z^{h-4}\left[\left(\frac{C^{(1)}_{TT\Phi}}{D^{(0)}_{\Phi\Phi}} - 2\log(z)\frac{C^{(0)}_{TT\Phi}}{D^{(0)}_{\Phi\Phi}}\right)\tilde{\Phi}(0) + \left(\frac{C^{(2)}_{TT\Phi}D^{(0)}_{\Phi\Phi} - C^{(1)}_{TT\Phi}D^{(1)}_{\Phi\Phi}}{(D^{(0)}_{\Phi\Phi})^2}\right.\right. \\
&\quad \left.\left. - \log(z)\frac{C^{(1)}_{TT\Phi}D^{(0)}_{\Phi\Phi} - 2C^{(0)}_{TT\Phi}D^{(1)}_{\Phi\Phi}}{(D^{(0)}_{\Phi\Phi})^2} - \log^2(z)\frac{C^{(0)}_{TT\Phi}}{D^{(0)}_{\Phi\Phi}}\right)\Phi(0)\right] + \ldots \quad .
\end{aligned}
$$

We are now in the position to fix most of the constants. The typical normalization for the identity channel is $D^{(0)}_{\mathbb{I}\mathbb{I}} = 1$, whereas $D^{(1)}_{\mathbb{I}\mathbb{I}} = d$ is left undetermined. But from our ansatz (10) - (12), we also know the normalization for the channel of the stress energy tensor, namely $D^{(0)}_{TT} = b$ and $D^{(1)}_{TT} = \theta$. Furthermore, we know how the OPE of the stress energy tensor with itself and with its logarithmic partner must look like, see eqs. (8) and (9). This allows us, by comparing coefficients, to fix further constants, namely $C^{(0)}_{TTT} = 2b$, $C^{(1)}_{TTT} = \lambda b + 2\theta$ and $C^{(0)}_{TT\mathbb{I}} = c/2$, $C^{(1)}_{TT\mathbb{I}} = \mu + cd/2$. These choices are all natural and then exactly reproduce the OPEs TT and Tt as given in eqs. (8) and (9).

Before we continue, we have to address one issue of consistency. So far, we have tried to choose the normalization of the two-point functions of the stress energy tensor and its partner independently of the central charge of the theory. However, these two-point functions do not change, if we insert the identity as third field. Therefore, the structure constants must obey the relations $D_{Tt} = C_{Tt\mathbb{I}} = C_{TT\tilde{\mathbb{I}}}$ and $D_{tt} = C_{tt\mathbb{I}} = C_{Tt\tilde{\mathbb{I}}}$. Hence, $D^{(0)}_{TT} = C^{(0)}_{TT\mathbb{I}}$ and thus $b = c/2$. Furthermore, $D^{(1)}_{TT} = C^{(1)}_{TT\mathbb{I}}$ and thus $\mu = \theta - cd/2$. As a consequence, the only remaining free parameters are the central charge c and the normalizations d and θ in the two-point functions $\langle\mathbb{I}\tilde{\mathbb{I}}\rangle$ and $\langle\tilde{\mathbb{I}}\tilde{\mathbb{I}}\rangle$. Plugging these choices into the remaining OPE tt as given above yields the following structure:

$$
t(z)t(0) = z^{-4}\left(\theta - \log(z)c\right)\tilde{\mathbb{I}}(0)
$$

$$
\begin{aligned}
&+ \quad z^{-4}\left(C^{(2)}_{TT\mathbb{I}} - \theta d + \log(z)(cd-\theta) - \log^2(z)\frac{c}{2}\right)\mathbb{I} \\
&+ \quad z^{-2}\left(1+\frac{4\theta}{c} - 4\log(z)\right) t(0) \\
&+ \quad z^{-2}\left(2\frac{C^{(2)}_{TTT}-\theta}{c} - \frac{4\theta^2}{c^2} - \log(z)(1-\frac{4\theta}{c}) - 2\log^2(z)\right) T(0) \\
&+ \quad \ldots \quad .
\end{aligned} \tag{21}
$$

With the choice $\theta = 0$, (21) looks similar to the result of Gurarie and Ludwig. The difference lies essentially in the terms proportional to z^{-4}, which in our case also involve the logarithmic partner of the identity. Note, however, the close agreement of the numerical coefficients, in particular in the z^{-2} terms. In our approach, we only keep track of the primary fields and their logarithmic partners, but none of the descendants. This is why our formula misses the canonical terms proportional to ∂t, ∂T and $\partial\tilde{\mathbb{I}}$.

The result is also very similar to the one derived in [22] (but only for $\theta = 0$, too). Allowing for the terms containing $\tilde{\mathbb{I}}$ to differ, we see that the two characteristic parameters can be described by the central charge and $C^{(2)}_{TTT}$. But $\theta = 0$ implies a vanishing two-point function for $\langle Tt\rangle$ and $\langle tt\rangle$, at least for our ansatz of a Jordan cell structured identity sector with $b = \frac{c}{2} = 0$ which would be consistent with case (II) in 2.2.

There is one very important caveat: The most singular term of the OPE stems from the pair $\{\mathbb{I}, \tilde{\mathbb{I}}\}$, and starts with z^{-4}. The higher level descendants of these fields will influence the terms of the OPE of order z^{-2}. This is particularly important, because these descendants are largely given in terms of T and t. Indeed, we have $L_{-2}\mathbb{I} = T$ and $L_{-2}\tilde{\mathbb{I}} = t + \lambda T$ and thus it is not surprising that (21) differs from concrete calculations.

Most of the results concerning the stress energy tensor have also been derived in [27], especially $b = \frac{c}{2}$ and the normalization of the Jordan cell of the stress energy tensor.

3.3 Consequences on the c → 0 catastrophe

The impact on the OPE of primary fields of the results derived in the previous section is immense.

Therefore let us recall what we know about the general form of the OPE in equations (13ff). Inserting the fact that for our ansatz (and $h = 0$) we have $D^{(0)}_{\Phi\Phi} \propto \langle \frac{c}{2}\tilde{\mathbb{I}}(w) + \mu\mathbb{I}\rangle = 0$, we run into a problem inverting the matrix of two-point functions which is needed to raise indices, since $D^{ij} = (D_{ij})^{-1}$. Hence the OPE of two primary fields in a $c = 0$ theory with a Jordan cell structure on the $h = 0$ level and $T(z), t(z)$ being descendants of the $h = 0$ fields, remains ill defined. The only loophole to this is to define the normalization of the three-point functions to $\frac{c}{h_\phi}$. Thus for $h \neq 0$, the metric is invertible again as Cardy [2] suggested but this way the vacuum expectation values vanish since they are proportional to the central charge.

However, although in this case the vanishing of the vacuum expectation values seems to emerge quite naturally due to the normalization chosen, it may not be the physically correct choice. This particular problem seems to be common in standard LCFT and has already been solved in the case of the fermionic path integral in $c = -2$ LCFT [16]. Due to the presence of zero modes in the path integral, the vev vanishes and thus has to be redefined, e. g. via

$$\langle \cdot \rangle := \langle 0| \cdot |\tilde{0}\rangle + \langle \tilde{0}| \cdot |0\rangle, \tag{22}$$

leaving us with the problem of how expressions like the vev of the OPE of $t(z)t(w)$ may be dealt with.

4 A bosonic free field construction

4.1 Ansatz

To illustrate the results obtained from the most singular term of the OPE by global conformal invariance, we take a free field construction with arbitrary central charge for the stress energy tensor and its logarithmic partner field:

$$\begin{aligned} T(z) &= -\frac{1}{2}{:}\partial\phi(z)\partial\phi(z){:} + \mathrm{i}\sqrt{2}\alpha_0{:}\partial^2\phi(z){:}\,, & (23) \\ t(z) &= {:}\lambda\phi(z)\tfrac{\exp\left(\mathrm{i}\sqrt{2}a\phi(z)\right)}{\mathrm{i}\sqrt{2}\alpha_0}T(z){:}\,. & (24) \end{aligned}$$

For the logarithmic partner of the identity we chose a vertex operator ansatz with conformal weight $h(a) = a^2 - 2a\alpha_0 = 0$ which means that we have two possible weights for the Vertex operator, $a = 0$ for the true identity and $a = 2\alpha_0$ for the second $h = 0$ field. Thus we expect another vertex operator to appear in the OPE behaving like the identity in correlators. This non-trivial ansatz for the logarithmic partner of the identity is necessary to reproduce the standard form of the OPEs typical for LCFT. Thus we define with $a = 2\alpha_0$:

$$\begin{aligned} \tilde{\mathbb{I}}(z) &= \lambda\phi(z)\frac{\exp\left(\mathrm{i}\sqrt{2}a\phi(z)\right)}{\mathrm{i}\sqrt{2}\alpha_0}\,, & (25) \\ \mathbb{I}'(z) &= \exp(\mathrm{i}\sqrt{2}a\phi(z)) \;\equiv \mathbb{I}\,. & (26) \end{aligned}$$

Similar considerations can be found within the Coulomb gas formalism used in [20].

Note that there is a subtlety here. This ansatz can not be directly compared to the general formula, especially not for $t(z)t(w)$ as in (21) since the propagator is not of the standard form. We do not only have the identity $\mathbb{I}$ and its logarithmic partner field $\tilde{\mathbb{I}}$ but in addition also a field that is conjugated to the identity, $\mathbb{I}' = \exp(\mathrm{i}\sqrt{2}a\phi)$ with $a = 2\alpha_0$. Thus contributions by this field have to be taken into account, too. This leads to different prefactors and changes in signs. Additionally, it is not surprising that we are not able to get the coefficients of $T(w)$ on the rhs of $T(z)t(w)$ and those of the most singular terms in the OPE of $t(z)t(w)$ to overlap. This is simply due to the fact that the normalization of the Jordan cell of the stress energy tensor is already fixed by that of the identity. This is the reason why some factors appear twice as often as expected when compared to the OPE derived by Gurarie and Ludwig [17] or Kogan and Nichols [22].

4.2 The non logarithmic OPEs

Of course, the OPE of the stress energy tensor with itself is as usual,

$$T(z)T(w) \sim \frac{\frac{c}{2}}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)}\,, \tag{27}$$

thus, due to the vanishing vev of $\mathbb{I}$, we have

$$\langle T(z)T(w)\rangle = 0. \tag{28}$$

The first different result appears in the correlator of the stress energy tensor with its logarithmic partner:

$$T(z)t(w) \sim \frac{\frac{c}{2}\tilde{\mathbb{I}}(w)}{(z-w)^4} + \frac{2t(w)+\lambda T(w)}{(z-w)^2} + \frac{\partial t(w)}{z-w}\,, \tag{29}$$

where λ depends on the normalization of the off-diagonal entries of the Jordan cell between $T(z)$ and $t(z)$, i. e. $L_0 t(z) = 2t(z) + \lambda T(z)$. These two are exactly what we calculated before. Since the vev of $\tilde{\mathbb{I}}$ does not vanish in an LCFT, we are left with a vacuum expectation value of

$$\langle T(z)t(w)\rangle = \frac{\frac{c}{2}}{(z-w)^4} = 0. \tag{30}$$

However, for $c = 0$ this vanishes, too.

4.3 The logarithmic OPE

The OPE of $t(z)$ with itself is more complicated as one would expect from the derivation from its most singular term due to the appearance of descendants of $\mathbb{I}$ and $\tilde{\mathbb{I}}$ as already mentioned above. Another point to bear in mind are the divergences of lowest order, $\log(z-w)(z-w)^0$, which have been omitted in the literature before.

To keep things as simple as possible, we will just state our result for the case that $4-2a^2>0$ and omit the terms that are dispensable for the comparison with the results of [17] and our general calculation (21) which means that we will restrict ourselves to the contributions of $T, t, \mathbb{I}$ and $\tilde{\mathbb{I}}$ up to first order without logarithmic divergences or composite fields. The full results will be given in the appendix. Choosing $\lambda=\frac{1}{2}$ and for $c=0$, $\alpha_0^2=\frac{1}{24}$ we get:

$$
\begin{aligned}
& t(z)t(w) \\
\sim\ & \left(-2\log(z-w)\tilde{\mathbb{I}}(w)+\frac{1}{2c}+\log^2(z-w)+3\log(z-w)\right)\frac{\frac{c}{2}}{(z-w)^4} \\
& +\frac{(1-4\log(z-w))t(w)+\left(3\log(z-w)+2\log^2(z-w)\right)T(w)}{(z-w)^2} \\
& +\frac{(1-4\log(z-w))\partial t(w)+\left(3\log(z-w)+2\log^2(z-w)\right)\partial T(w)}{2(z-w)}\,.
\end{aligned}
$$

Of course, the vacuum expectation value vanishes for $c=0$ (taking into account that $\langle 0|\mathbb{I}|0\rangle=0$, too) but since these terms may be interesting for the reader, we stated them in spite of that. Additionally we should mention that it is obviously possible to chose an $\tilde{\mathbb{I}}$ which is quasi-primary since its OPE with the stress energy tensor looks like

$$
T(z)\tilde{\mathbb{I}}(w)\sim\frac{\lambda\mathbb{I}}{(z-w)^2}+\frac{\partial_w\tilde{\mathbb{I}}(w)}{(z-w)}\,. \tag{31}
$$

Recapitulating, we have shown that as in [17], we have $\theta=0$, which means that we would have vanishing vacuum expectation values for $c=0$. These results suggest that we can not take a naive free field construction to describe the problem, since all vev's vanish. In order to get the necessary central extensions, more sophisticated constructions should be considered, such as several free fields or deformations of the stress energy tensor similar to the ones introduced in [3]. Contrary to the former calculations [17], we found several more complicated fields than $t(w)$, $T(w)$ or their descendants though there are no other primaries involved.

Therefore we tried to find a way out by searching for $\sum_{\{p,q\}}c_{p,q}=0$ to construct a $c=0$ theory out of tensorized minimal models to get a non trivial CFT with vanishing central charge in chapter seven.

Another possibility would be to find a fermionic theory with vanishing central charge.

5 Discussion of the two LCFT approaches

As already stated in section 2.2, we have a third possible loophole to avoid the $c \to 0$ catastrophe in the OPEs of primary fields. In the following we will explain why the ansatz we chose, i. e. the one based on the augmented minimal model, may be a more natural solution.

Hence we will give a brief overview on the ansatz of Kogan and Nichols [22] followed by our comments on their approach. Additionally we will state some facts about the $c = 0$ case including implications for percolation and a discussion of current research on augmented $c_{(p,q)}$ models with $q > 1$ which have not been treated in the literature so far, focusing on $p = 3, q = 2$.

5.1 The replica approach to vanishing central charge

Following the replica approach of [2], Kogan and Nichols [22] introduced another field $\tilde{T}$ with dimension $h = 2 + \alpha(c)$ which satisfies $\alpha(c) \to 0$ for $c \to 0$ being normalized to

$$\langle \tilde{T}(z)\tilde{T}(0)\rangle = \frac{1}{c}\frac{B(c)}{z^{4+2\alpha(c)}} \tag{32}$$

with $B(c) = -\frac{h^2}{2} + B_1 c + \dots$. (Note that for $c \to 0$ this expression diverges which will be of importance for the existence of null vectors later on.)

Then, after a small c expansion, the OPE of our primary field looks like

$$\begin{aligned}\phi_h(z)\phi_h^\dagger(0) &\sim \frac{1}{z^{2h}}\left(1 + \frac{2h}{c}z^2 T(0) + 2z^{2+\alpha(c)}\tilde{T}(0) + \dots\right) + \dots \\ &\sim \frac{1}{z^{2h}} + \frac{1}{z^{2h}}z^{2+\alpha(c)}\left(\frac{2h}{c}\left(1 - \alpha(c)\log(z)\right)T(0) + 2\tilde{T}(0) + \dots\right) + \dots \quad ,\end{aligned}$$

which is again well-defined, if μ which is given by

$$\mu^{-1} \equiv \lim_{c\to 0} -\frac{2\alpha(c)}{c} = -2\alpha'(c)\,, \tag{33}$$

is not equal to zero.

The logarithmic partner field can now be defined by

$$\frac{h}{\mu}t = \frac{2h}{c}T + 2\tilde{T}\,, \tag{34}$$

satisfying

$$L_0 T = 2T \quad L_0 t = 2t + T. \tag{35}$$

This means that $t(z)$ is a field of the same conformal weight as $T(z)$ living in a Jordan cell due to L_0 being non-diagonalizable. Thus the OPE becomes

$$\phi_h(z)\phi_h^\dagger(0) \sim \frac{1}{z^{2h}}\left(1 + \frac{h}{\mu}\left(t(0) - \log(z)T(0)\right) + \dots\right) + \dots \quad , \tag{36}$$

which yields the following vevs after redefining $t \to t + \gamma T$ with a suitable choice of γ

$$\langle T(z)T(0)\rangle \sim 0\,, \tag{37}$$

$$\langle T(z)t(0)\rangle \sim \frac{b}{z^4}\,, \tag{38}$$

$$\langle t(z)t(0)\rangle \sim \frac{-2b\log z}{z^4}\,. \tag{39}$$

5.2 Comments on the replica approach

The result of [2], Kogan and Nichols [22] or Gurarie and Ludwig [17] can only be obtained by assuming that we are dealing with non- degenerate vacua, which means, that the vacuum expectation value of the identity operator does not vanish and we have only one $h = 0$ field contributing to the OPE. Thus the case that there exists such a Jordan cell is explicitly excluded and the ansatz is not valid for a Jordan cell setup.

It is based on the following algebra between the modes of $T(z)$ and $t(z)$:

$$[L_n, t(z)] = z^n\left\{\left(z\frac{\mathrm{d}}{\mathrm{d}z} + z(n+1)\right)t(z) + (n+1)T(z)\right\} + \frac{\mu\mathbb{I}}{6}n(n^2+1)z^{n-2}\,, \tag{40}$$

where for their ansatz, we have $\mu = b$ meaning that the coefficient of the identity equals the coefficient of the most singular term of the OPE.

Thus, since $\langle\mathbb{I}\rangle \neq 0$, the most singular part of the OPE yields the vacuum expectation value as stated in (38). This is only true if we assume $L_{-2}|0\rangle = T(0)|0\rangle$ not to be zero by construction (since the action of the conformal generators on the vacuum vanish in a $c = 0$ CFT) but to be some kind of generalized null state with respect to which $l_{-2}|0\rangle = t(0)|0\rangle$ is non orthogonal. To keep this assumption it is crucial not to have a logarithmic partner of the identity and thus $\langle\mathbb{I}\rangle \neq 0$.

This assumption is only possible if $b = \mu \neq 0$. However, it seems quite unnatural that for all $c \neq 0$ μ may be set to zero by a redefinition $l_m \to l_m - \frac{2\mu}{c}L_m$. Hence it is not obvious how this limit may equal the value of μ^{-1}

for $c = 0$ due to the discontinuity of being free to choose $\mu = 0$ for $c \neq 0$ but staying with fixed μ for $c = 0$. Thus the replica approach is specially designed for the $c = 0$ case rather than a general extendable ansatz. Moreover, for any other value of the central charge it seems to be unnecessarily complicated whereas a Kac-table based approach can be naturally extended to an LCFT for any generalized rational central charge. Furthermore, there is no physical quantity known to correspond to this arbitrary parameter μ, thus it is rather awkward that it may show up with such a significant role in our (L)CFT. Additionally we will show in section 6.1 that if we want to have null vectors on the second and third level, $\mu = b$ has to vanish. In that case, T decouples again and the replica approach does not solve the divergence problem.

Note that in Kogan and Nichols [22], the term proportional to the identity in the central extension of the algebra between the Laurent modes of $t(z)$ and $T(z)$, μ, is the same as the proportionality factor of $\langle Tt\rangle$. In our calculations, however, these coefficients are different since we assume a Jordan cell on the identity level, yielding

$$[L_n, t(z)] = z^n \left\{ \left(z\frac{\mathrm{d}}{\mathrm{dz}} + z(n+1) \right) t(z) + (n+1)T(z) \right\} + \frac{\mu\mathbb{I} + \frac{c}{2}\tilde{\mathbb{I}}}{6} n(n^2+1) z^{n-2}\,, \tag{41}$$

or, equivalently $T(z)t(0) \sim (\mu\mathbb{I} + b\tilde{\mathbb{I}})z^{-4} + \ldots$. Thus $b = \frac{c}{2}$ in our case and $\langle Tt\rangle$ has to vanish, too. A priori, as already discussed above, there is no constraint on the choice of μ. Following Gurarie and Ludwig [17], we will show how various values of μ affect the theory in the next section.

Additionally we have to state that whithin the replica approach the full conformal invariance of the vacuum is broken whereas in a $c_{(9,6)} = 0$ (L)CFT, there is no level two state which is non orthogonal to $T(0)|0\rangle$, especially not $t(0)|0\rangle$ since the two point function has to vanish. Thus in this setup, we can keep the full (and not only global) conformal invariance of the vacuum.

6 The $\mathbf{c_{(9,6)} = 0}$ augmented minimal model

If we restrict ourselves to the case of not having a Jordan cell structure at the $(h = 0)$-level, we encounter the fact that any two-point function involving T has to vanish. This follows directly from the behavior of the identity sector in a $(c = 0)$-theory unless we introduce a non orthogonal state to $L_{-2}|0\rangle$. We know that by global conformal invariance and the highest weight condition, we have $L_n\mathbb{I} = L_n|0\rangle = 0$ for all $n \geq -1$. In the following, let n be > 0. Starting with a vanishing central charge and $h = 0$, we know

$$0 \;=\; 2nL_0|0\rangle + \frac{c}{12}n(n^2-1)|0\rangle$$

$$
\begin{aligned}
&= [L_n, L_{-n}]|0\rangle \\
&= L_n L_{-n}|0\rangle - L_{-n}L_n|0\rangle \\
&= L_n L_{-n}|0\rangle\,,
\end{aligned}
\tag{42}
$$

and thus we have $L_{-n}|0\rangle = 0$ for all $n \in \mathbb{Z}$. This means that if we expand $T(z)$ in powers of z, i. e.

$$
T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}, \tag{43}
$$

we clearly see that $\langle 0|T(z) = T(z)|0\rangle = 0$ if we impose full conformal invariance. This is possible if all states are orthogonal to $L_n|0\rangle$ which is the case for the minimal model $c_{(3,2)} = 0$. More precisely: the null vector is present in the irreducible vacuum representation but may disappear in the full indecomposable representation based on $|\tilde{0}\rangle$. Note that if we include fields outside the Kac-table without assuming a Jordan cell structure for the identity level with $L_0\tilde{\mathbb{I}} = \mathbb{I}$, non orthogonal states can also be constructed but there are no constraints on their properties as shown in Kogan and Nichols [22] or Gurarie and Ludwig [17].

Nevertheless in our ansatz (the $c_{(9,6)} = 0$ augmented minimal model), the state usually identified with the stress energy tensor seems to decouple completely from the theory since it is even orthogonal to $l_{-2}|0\rangle$ (there may be additional $h = 2$ fields present in the theory which are non orthogonal but we do not know about any of them up to now). However, this also forces any two-point function involving T to vanish (as long as we do not modify the theory as touched in section 3.3). Thus the first two-point function not to vanish is $\langle tt\rangle$.

If we assume $L_{-2}|0\rangle = T(0)|0\rangle$ to be just an ordinary null state, $|\chi^{(2)}_{(h,c)}\rangle$, and not a fundamental property of the vacuum at $c_{(3,2)} = 0$, we can obtain different results.

What we already know is that $L_{-2}|0\rangle$ is a null state with respect to the action of all L_n. Thus we only have to check whether this holds for the action of the l_n, too. Taking a look at

$$
\langle 0|[l_2, L_{-2}]|0\rangle = \langle 0|4l_0|0\rangle + \langle 0|\mu|0\rangle = \mu\,, \tag{44}
$$

we see that it is consistent to assume a non-orthogonal state to $L_{-2}|0\rangle$ if we exclude a Jordan cell for the identity. Even the Jordan cell relation between the usual state associated with the stress energy tensor and its logarithmic partner turns out to be as expected (independent of the assumption about the Jordan cell structure):

$$
L_0 t(z)|0\rangle \equiv L_0 l_{-2}|0\rangle = 2l_{-2}|0\rangle + L_{-2}|0\rangle = 2t(0)|0\rangle + T(0)|0\rangle\,. \tag{45}
$$

Once more we stress that $t(z)$ can only be non-orthogonal in a non Kac based approach to $c = 0$ since otherwise we know that we have a Jordan cell connection between the identity and other states which would cause the two-point function to vanish:

$$\langle 0|[l_2, L_{-2}]|0\rangle = 4\langle 0|l_0|0\rangle + \langle 0|\mu\mathbb{I} + \frac{c}{2}\tilde{\mathbb{I}}|0\rangle = 0\,. \tag{46}$$

6.1 Null vectors in a Kac-table based $\mathbf{c = 0}$ theory

Having agreed upon the proposal that the Kac-table of the augmented $c = 0$ should be taken, we know that under certain circumstances we can have null vectors in our theory. The assumption that no other fields than those of the Kac-table and their descendants may arise is crucial to this calculation since we do not have any knowledge on the properties of non-Kac fields. Thus we point out that there are problems with any arguments based on the assumption of null vectors in a non strictly Kac-based theory.

Assuming $t(z)|0\rangle$ to have a mode expansion like $T(z)|0\rangle$, i. e.

$$t(z)|0\rangle = \sum_{n\in\mathbb{Z}} l_n z^{-n-2}|0\rangle\,, \tag{47}$$

and following the idea of Gurarie and Ludwig [17], we will try to construct universal null vectors that do not only vanish under the action of all L_n for $n > 0$ but also after the application of l_m for $m > 0$. We have to emphasize that this simple expansion of $t(z)$ only holds when acting on a highest weight state, e. g. $|0\rangle$.

6.1.1 The ordinary level two null vector

Now let us have a look at the ordinary null vector on the second level

$$|\chi^{(2)}_{(h,c)}\rangle = \left(L_{-2} - \frac{3}{2(2h+1)}L_{-1}^2\right)|h\rangle\,, \tag{48}$$

$$h = \frac{1}{16}\left(5 - c \pm \sqrt{(c-1)(c-25)}\right)\,. \tag{49}$$

What we already know is that $L_{\{n\}}|\chi^{(2)}_{(h,c)}\rangle = 0$ for all $|\{n\}| > 0$ with $\{n\} = \{n_1, n_2 \ldots, n_k\}$ and $|\{n\}| = \sum_i n_i$. But what about the action of $l_{\{n\}}$ on $|\chi^{(2)}_{(h,c)}(0)\rangle$? For $|\{n\}| \geq 3$ this is obviously trivial since commuting the $l_{\{n\}}$ to the right will leave us with some linear combination of $l_{\{m\}}$ and $L_{\{m'\}}$ with $|\{m\}|, |\{m'\}| > 0$ which vanishes. Thus the interesting cases are the application of l_2 and l_1^2.

Therefore we have to use the algebra of the L_n and l_n. The algebra between the modes of $T(z)$,

$$[L_n, L_m] = (n-m)L_{n+m}\,, \tag{50}$$

is just the same as in any ordinary CFT in spite of lacking the central extension due to the vanishing central charge. The mixed commutator as derived in the appendix A.1 is given by

$$[L_n, l_m]|0\rangle = (n-m)l_{n+m}|0\rangle + (n+1)L_{n+m}|0\rangle + \frac{\mu}{6}n(n^2-1)\delta_{n+m,0}|0\rangle\,. \tag{51}$$

Now we can test the known level two null vector (48) from the ordinary theory by applying the generators of $t(z)$:

$$l_2|\chi^{(2)}_{(h,c)}\rangle = \left[\left(4 - \frac{18}{2(2h+1)}\right)l_0 + h + \mu\right]|h\rangle\,. \tag{52}$$

This result raises the question what the action of l_0 on some state of weight h might be. According to Gurarie and Ludwig [17], it can be chosen to be equal to zero, $l_0|h\rangle = 0$, but in analogy to the generators of $T(z)$ we might also expect it to be $l_0|h\rangle = h|h\rangle$ since for $h = 0$ this does not make any difference.

Unfortunately, there is no criterion to draw the right conclusion which is the correct choice of the action of l_0 on a state. Thus we will state our results of equation (52) for both choices of l_0, i.e. $l_0|h\rangle = h|h\rangle$ or $= 0$, we found $\mu = 0$ for $h = 0$ and $\mu = -\frac{5}{8}$ for $h = \frac{5}{8}$ which give us necessary conditions on the values of μ in order to have a null vector on the second level. Note that these conditions are not sufficient since we are not able to check $l^2_{-1}|\chi^{(2)}_{(h,c)}\rangle = 0$ due to the lack of the algebra $[l_n, l_m]$.

6.1.2 The level three null vector

We could try the same procedure on the level three null state

$$|\chi^{(3)}_{(h,c)}\rangle = \left(L^3_{-1} - 2\,(h+1)\,L_{-2}L_{-1} + h\,(h+1)\,L_{-3}\right)\phi_h\,, \tag{53}$$

$$h = \frac{1}{6}\left(7 - c \pm \sqrt{(c-1)(c-25)}\right)\,. \tag{54}$$

Testing the level three null vector for consistency by applying l_3 to $|\chi^{(3)}_{(h,c)}\rangle$, we find that for both choices of l_0, i.e. $l_0|h\rangle = h|h\rangle$ or $= 0$, we have $\mu = 0$ for $h = 2$ and $\mu = \frac{5}{6}$ for $h = \frac{1}{3}$. However, again we have to bear in mind that these are only necessary conditions, since we did not check the action of $l_2 l_1$ and l_1^3.

6.1.3 Comments on percolation as an augmented $\mathbf{c = 0}$ model

Independently we can conclude that we do not only have different theories for different values of μ but also that any given $c = 0$ theory splits up in certain subsets of primary operators which "cannot give rise to [...] differential equations simultaneously in the same theory" [17].

However, even only from the necessary conditions for the values of μ we see that for $\mu \neq 0$ we can not have a level three and level two null vector differential equation in the augmented $c = 0$ model. Since the results above hold for both approaches to $c = 0$ LCFT, we see that within the replica ansatz, we can not have null states on the second and third level simultaneously since the value $\mu = 0$ is excluded.

Again it is very interesting, that the unphysical parameter μ seems to disappear if we want to have as many null states as possible for the whole theory. This means also that there is no central term in the algebra between the l_n and L_m and all vacuum expectation values of the stress energy tensor and its logarithmic partner vanish since $\mu = \theta$ for $c = 0$ LCFT due to the exclusion of the replica ansatz in that case. However, this is not very surprising since it has already been suggested by Cardy as discussed in section 2.2.

6.2 The field content of a $\mathbf{c_{(9,6)} = 0}$ augmented minimal model

After having talked so much about the augmented $c_{(9,6)} = 0$ model, we should give at least a brief overview on its features since there has been not much literature published about generalized augmented $c_{(p,q)}$ models with $q > 1$ so far. Following the ideas of [4], we know that the smallest closing set of modular functions larger than the $\frac{1}{2}(p-1)(q-1)$ characters for the minimal $c_{(p,q)}$ model contains $\frac{1}{2}(3p-1)(3q-1)$ individual functions which stay in some suitable linear combination in direct correspondence to the number of highest weight representations or fields in the augmented Kac table. The modular functions can be found by solving the modular differential equation as introduced in [26, 25]. The generalization of this method towards LCFT can be found in [8]. In our example, the $c_{(9,6)}$ model, twenty torus amplitudes can be matched with the twenty representations of the modular group being present in the Kac table of $c_{(9,6)} = 0$ [7]. Closed sets of such functions can only be obtained considering an odd multiple of (p, q) thus usually one tries to get along with the smallest set, i. e. $(3p, 3q)$.

Thus in contrary to the minimal model $c_{(p,q)}$ we technically have to deal with an extended Kac table of $c_{(3p,3q)}$:

$$c_{(9,6)} \quad : \quad \begin{array}{|c|c|c|c|c|c|c|c|} \hline 0 & 0 & \frac{1}{3} & 1 & 2 & \frac{10}{3} & 5 & 7 \\ \hline \frac{5}{8} & \frac{1}{8} & -\frac{1}{24} & \frac{1}{8} & \frac{5}{8} & \frac{35}{24} & \frac{21}{8} & \frac{33}{8} \\ \hline 2 & 1 & \frac{1}{3} & 0 & 0 & \frac{1}{3} & 1 & 2 \\ \hline \frac{33}{8} & \frac{21}{8} & \frac{35}{24} & \frac{5}{8} & \frac{1}{8} & -\frac{1}{24} & \frac{1}{8} & \frac{5}{8} \\ \hline 7 & 5 & \frac{10}{3} & 2 & 1 & \frac{1}{3} & 0 & 0 \\ \hline \end{array} \ . \tag{55}$$

As it is always the case for $c_{(3p,3q)}$ augmented models, we have 3×2 fields in the Kac table which are of weight $h = 0$ and lie within the upper left and lower right corners of the replicated minimal Kac tables on the diagonal. It is conjectured [7] that all fields inside the boundary of the replicated minimal Kac table belong to rank 3 Jordan cells whose detailed structure is not yet known. Thus we may have to alter our calculations in the third and fourth chapter.

Fields on the boundary of the replicated minimal Kac table show up with a multiplicity of 2×2 and belong to rank 2 Jordan cells. The corresponding representation of weight $h_{(r,s)} + rs$ is present 1×2 times as expected, too. Additionally, the fields on the edges of the boundaries show up only 1×2 times as well, with their corresponding representations of weight $h_{(p,q)} + pq/4$ showing up at the anti-diagonal edges.

Thus in the special case of $c = 0$ we have two highest weights which do not form Jordan cells, i. e. $-\frac{1}{24}, \frac{35}{24}$ while the other operators of the boundary of the conformal grid are arranged in triplets of which two states of the same weight form an indecomposable representation and one belongs to an irreducible representation which is differing by an integer in its weight (more precisely rs), i. e.

$$\left(\frac{5}{8}, \frac{5}{8}, \frac{21}{8}\right) = \left(\frac{5}{8}, \frac{5}{8}, 2 + \frac{5}{8}\right) , \tag{56}$$

$$\left(\frac{1}{3}, \frac{1}{3}, \frac{10}{3}\right) = \left(\frac{1}{3}, \frac{1}{3}, 3 + \frac{1}{3}\right) , \tag{57}$$

$$\left(\frac{1}{8}, \frac{1}{8}, \frac{33}{8}\right) = \left(\frac{1}{8}, \frac{1}{8}, 4 + \frac{1}{8}\right) . \tag{58}$$

Due to these indecomposable representations, logarithms arise in the OPEs and especially in the fusion product of the pre-logarithmic field $\phi_{-\frac{1}{24}}$ with itself.

The sector containing the $h = 0$ fields has a more complicated structure. We have three multiple weights $(0, 0, 0), (1, 1)$ and $(2, 2)$ but we do not yet know how they are arranged among the other two fields of weights 5 and 7, respectively. As stated above, it is conjectured [7] that they may form a rank three Jordan cell structure whose details are currently being worked out. Additionally, we can not exclude exotic behavior such as Jordan cells with respect to other generators than L_0, e. g. $\mathcal{W}$-algebra zero modes. Even worse, there might exist indecomposable structures with respect to L_n, $n \neq 0$, as in [24].

As far as we know there has not been any research concerning this issue before. It seems reasonable to assume a structure related to that of the $c_{(p,1)}$ models which has already been discussed in detail [13], [14], [19], but obviously at least for the integer weights it can not be the whole story.

If we accept that the Kac-table of $c = 0$ has to be extended beyond its minimal truncation, we immediately encounter a problem. The field corresponding to the entry $(2, 3)$ in the Kac-table has a negative conformal weight $h_{2,3} = -1/24$. Hence, the theory cannot be unitary. Furthermore, the effective central charge $c_{\text{eff}} = c - 24h_{\min}$ with $h_{\min}$ the minimal eigen value of L_0 is then given by $c_{\text{eff}} = (c = 0) - 24(h = -1/24) = 1$. It follows that such a theory cannot be rational with respect to the Virasoro algebra alone, but only quasi-rational. However, there presumably exists an extended chiral symmetry algebra, $\mathcal{W}(2, 15, 15, 15)$ under which the theory is rational [7]. Fortunately, most of the structures which will interest us in this paper can be studied from the the perspective of the Virasoro algebra.

As a concluding remark, let us note that there seems to be a connection to $c_{(6,1)} = -24$ which is the only rational (L)CFT with equal central charge modulo 24 and thus exhibiting the same modular properties. This theory also has effective central charge one. Unfortunately, the analogies only hold for the boundary of the Kac table and therefore we can only deduce the properties for the representations from the boundary of the Kac-table of the $c = 0$ model and not for the integer weight states.

7 The forgotten loophole

7.1 Tensorized (L)CFTs with $\mathbf{c = 0}$

As already stated by Gurarie [18, 17] and Cardy [2] and further elaborated by Kogan and Nichols [22], there is obviously a fourth way out of the divergence problem at $c = 0$. Taking two non-interacting CFTs with central charges c_1 and $c_2 = -c_1$, respectively, and tensorizing them, we get a CFT with vanishing central charge again but the OPE (2) looks like

$$\phi_h(z)\phi_h^\dagger(0) \sim \frac{C_{\Phi\Phi}^{\mathbb{I}}}{z^{2h}}\left(1 + \frac{2h}{c_1}z^2(T_{c_1}(0) - T_{-c_1}(0)) + \dots\right) + \dots \quad , \qquad (59)$$

which is perfectly well defined for $c = 0$ if $c_1 \neq 0$.

But the result comes with a price, too: we have to introduce a new field $t(z) := T_{c_1}(z) - T_{-c_1}(z)$ which can be shown to satisfy the following OPEs with the stress energy tensor [17]

$$\begin{aligned} T(z)T(0) &\sim \frac{2T(0)}{z^2} + \frac{T'(0)}{z} + \ldots \quad , &(60)\\ T(z)t(0) &\sim \frac{c_1}{z^4} + \frac{2t(0)}{z^2} + \frac{t'(0)}{z} + \ldots \quad , &(61)\\ t(z)t(0) &\sim \frac{2T(0)}{z^2} + \frac{T'(0)}{z} + \ldots \quad . &(62)\end{aligned}$$

Of course, the same procedure can be applied to an LCFT setup as well which we will show in the following. The OPE of the tensorized $c = c_1 + c_2 = 0$ LCFT model consists of an ordinary CFT part from the c_2-sector and a LCFT part from the c_1 sector. Thus we would get a $c = 0$ theory with logarithmic operators without vanishing two-point function.

Operators in the full tensorized theory therefore are just direct products $\phi_h^{(0)} = \phi_{h_1}^{(1)} \otimes \phi_{h_2}^{(2)}$ whose weights are given by the sum of both parts $h = h_1 + h_2$. Thus the OPE of a primary field is given by (see [22])

$$\begin{aligned} \phi_h^{(0)}(z)\phi_h^{(0)}(0) &= \phi_{h_1}^{(1)}(z)\phi_{h_1}^{(1)}(0) \otimes \phi_{h_2}^{(2)}(z)\phi_{h_2}^{(2)}(0) \\ &\sim \frac{1}{z^{2h_1}}\left(\mathbb{I}^{(1)} + z^2\frac{2h_1}{c_1}T^{(1)}(0) + \ldots\right) \\ &\qquad \times \frac{1}{z^{2h_2}}\left(\mathbb{I}^{(2)} + z^2\frac{2h_2}{c_2}T^{(2)}(0) + \ldots\right) + \ldots \\ &\sim \frac{1}{z^{2h}}\left(1 + z^2\left(\frac{2h_1}{c_1}T^{(1)}(0) + \frac{2h_2}{c_2}T^{(2)}(0)\right)\right) ,\end{aligned}$$

which is well defined since the $c_i \neq 0$ and the theories by themselves are regular.

7.2 The general case

In some cases we may not be able to choose a (bosonic) free field construction for the stress-energy-tensor. Thus we have to take a look at the general OPEs for a tensorized theory of an LCFT with central charge c_1 and an ordinary CFT with $c_2 = -c_1$. We start with the known OPEs

$$\begin{aligned} T^{(i)}(z)T^{(i)}(w) &= \frac{\frac{c_i}{2}}{(z-w)^4} + \frac{2T^{(i)}(w)}{(z-w)^2} + \frac{\partial_w T^{(i)}(w)}{(z-w)} , \\ \tilde{\mathbb{I}}^{(1)}(z)\tilde{\mathbb{I}}^{(1)}(w) &= \log^2(z-w)\mathbb{I}^{(1)} + 2\log(z-w)\tilde{\mathbb{I}}^{(1)}(w) , \\ T^{(1)}(z)\tilde{\mathbb{I}}^{(1)}(w) &= \frac{\mathbb{I}^{(1)}}{(z-w)^2} + \frac{\partial_w \tilde{\mathbb{I}}^{(1)}(w)}{(z-w)} , \end{aligned} \qquad (63)$$

and we define

$$\begin{aligned} t^{(1)}(w) &:= \ :T^{(1)}\tilde{\mathbb{I}}^{(1)}:(w)\,, \\ t^{(0)}(w) &:= \ t^{(1)}(w)\otimes\mathbb{I}^{(2)} + (\alpha\mathbb{I}^{(1)}+\beta\tilde{\mathbb{I}}^{(1)}(w))\otimes T^{(2)}(w)\,. \end{aligned}$$

To obtain the two point functions, we make the ansatz:

$$\begin{aligned} T^{(0)}(z) &= T^{(1)}(z)\otimes\mathbb{I}^{(2)}(z)+\mathbb{I}^{(1)}(z)\otimes T^{(2)}(z) \\ t^{(0)}(z) &= t^{(1)}(z)\otimes\mathbb{I}^{(2)}(z)+(\alpha\mathbb{I}^{(1)}(z)+\beta\tilde{\mathbb{I}}^{(1)}(z))\otimes T^{(2)}(z)\,. \end{aligned} \tag{64}$$

This leaves us with the following results for $T^{(0)}(z)T^{(0)}(w)$ and $T^{(0)}(z)t^{(0)}(w)$:

$$\begin{aligned} T^{(0)}(z)T^{(0)}(w) &= T^{(1)}(z)T^{(1)}(w)+T^{(2)}(z)T^{(2)}(w) \\ &\sim \frac{\frac{c_1}{2}+\frac{c_2}{2}}{(z-w)^4}+\frac{2(T^{(1)}+T^{(2)})(w)}{(z-w)^2}+\frac{\partial_w(T^{(1)}+T^{(2)})(w)}{(z-w)} \\ &= \frac{2T^{(0)}(w)}{(z-w)^2}+\frac{\partial_w T^{(0)}(w)}{(z-w)}\,, \end{aligned}$$

whereas the OPE with its logarithmic partner

$$\begin{aligned} & T^{(0)}(z)t^{(0)}(w) \\ = \ & T^{(1)}(z)t^{(1)}(w)\otimes\mathbb{I}^{(2)}+(\alpha\mathbb{I}^{(1)}+\beta\tilde{\mathbb{I}}^{(1)}(w))\otimes T^{(2)}(z)T^{(2)}(w) \\ \sim \ & \frac{\frac{c_1}{2}\left((1-\beta)\tilde{\mathbb{I}}^{(1)}(w)-\alpha\mathbb{I}^{(1)}\right)\otimes\mathbb{I}^{(2)}}{(z-w)^4}+\frac{2t^{(0)}(w)+T^{(1)}(w)\otimes\mathbb{I}^{(2)}}{(z-w)^2}+\frac{\partial_w t^{(0)}(w)}{(z-w)} \end{aligned}$$

yields a non-vanishing vev with a modified b-term:

$$\langle T^{(0)}(z)t^{(0)}(w)\rangle = \frac{\frac{c_1}{2}(1-\beta)}{(z-w)^4}\,.$$

For the OPE of the logarithmic partner fields, we get

$$\begin{aligned} & t^{(0)}(z)t^{(0)}(w) \\ \sim \ & \frac{1}{(z-w)^4}\Bigg(\left(1+\alpha^2\frac{c_2}{2}\right)+\left(\frac{c_1}{2}+\beta^2\frac{c_2}{2}\right)\log^2(z-w) \\ & -2\left(\frac{c_1}{2}+\beta^2\frac{c_2}{2}\right)\log(z-w)\tilde{\mathbb{I}}(w)+\alpha\beta\tilde{\mathbb{I}}c_2+\left(\frac{c_1}{2}+\beta^2\frac{c_2}{2}\right):\tilde{\mathbb{I}}(z)\tilde{\mathbb{I}}_1(w):\Bigg) \\ & +\frac{1}{(z-w)^2}\Bigg(2\left(T^{(0)}(w)-(1-\beta^2)T^{(2)}(w)\right)\log^2(z-w) \\ & -4\left(t^{(0)}(w)-\alpha\beta\tilde{\mathbb{I}}T^{(2)}(w)\right)\log(z-w)+2t^{(0)}(w)+2\alpha\beta\tilde{\mathbb{I}}T^{(2)}(w) \\ & +\left(T^{(0)}(w)-(1-\beta^2)T^{(2)}(w)\right):\tilde{\mathbb{I}}(w)\tilde{\mathbb{I}}(w):\Bigg) \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{(z-w)}\Big(\Big(\partial T^{(0)}(w)-(1-\beta^2)\partial T^{(2)}\Big)\log^2(z-w)\\
&-2\Big(\partial t^{(0)}(w)-\alpha\beta\tilde{\mathbb{I}}\partial T^{(2)}(w)\Big)\log(z-w)\\
&+\partial t^{(0)}(w)+\alpha\beta\tilde{\mathbb{I}}\partial T^{(2)}(w)+\partial\Big(\Big(T^{(0)}(w)-(1-\beta^2)T^{(2)}(w)\Big):\tilde{\mathbb{I}}(w)\tilde{\mathbb{I}}(w):\Big)\Big)\\
&+\Big(\log^2(z-w)-2\log(z-w)\tilde{\mathbb{I}}(w)\Big)\\
&\qquad\times\big(:T^{(0)}(w)T^{(0)}(w):-(1-\beta^2):T^{(2)}(w)T^{(2)}(w):\big)\ ,
\end{aligned}
$$

where we suppressed the labels for the tensor factors as they are clear from the context. Obviously the only possibility to get nothing but "zero charge" quantities on the rhs is to put $\alpha = 0$ and $\beta = 1$ which means, that we are left with vanishing vevs for $\langle TT\rangle$ and $\langle Tt\rangle$. In that case the equations would be of the same form as for the ordinary $c = 0$ LCFT and our construction would be useless. To be exhaustive, we will give the vev of this calculation, too,

$$
\langle t^{(0)}(z)t^{(0)}(w)\rangle \;=\; c_2\frac{\alpha\beta+(1-\beta^2)\log(z-w)}{(z-w)^4}\,.
$$

7.3 An example of a tensor model

One of many possible applications is a tensor product of a $c = -2$ theory and four Ising models. This ansatz has many advantages, e. g. a logarithmic pair in the identity sector of the part with $c = -2$ and the closure under fusion of a small subset of the fields.

As stated in [23] and [22], this corresponds to an $SU(2)_0$ or $OSp(2|2)_{-2}$ model, where the logarithmic structure appears in the $c = -2$ part.

The Ising Model In the Ising Model, we have the following fields

$$
\begin{aligned}
\mathbb{I} &= \phi_{(1,1)}, \phi_{(2,3)} && h = 0\,, && (65)\\
\sigma &= \phi_{(2,2)}, \phi_{(1,2)} && h = \frac{1}{16}\,, && (66)\\
\varepsilon &= \phi_{(2,1)}, \phi_{(1,3)} && h = \frac{1}{2}\,, && (67)
\end{aligned}
$$

with the well-known fusion rules:

$$
\begin{aligned}
\sigma\times\sigma &= \mathbb{I}+\varepsilon\,, && (68)\\
\sigma\times\varepsilon &= \sigma\,, && (69)\\
\varepsilon\times\varepsilon &= \mathbb{I}\,. && (70)
\end{aligned}
$$

For $c = c_{2,1} = -2$, we have an indecomposable representation of the $h = 0$ sector ($\mathcal{R}_{\mathbb{I}}$) consisting of two fields with $h = 0$ whose details are not important for our further discussion and two others, i. e. μ with $h = -\frac{1}{8}$ and ν with $h = \frac{3}{8}$. These fields obey

$$\begin{aligned} \mu \times \mu = \mu \times \nu = \nu \times \nu &= \mathcal{R}_{\mathbb{I}}\,, & (71)\\ \mu \times \mathcal{R}_{\mathbb{I}} = \nu \times \mathcal{R}_{\mathbb{I}} &= \mu + \nu\,, & (72)\\ \mathcal{R}_{\mathbb{I}} \times \mathcal{R}_{\mathbb{I}} &= 2\mathcal{R}_{\mathbb{I}}\,. & (73) \end{aligned}$$

It is easy to check, that the symmetrized fields of the four Ising models $\mathbb{I}, E_1, E_2, E_3, E_4$ and S (where E_i denotes the totally symmetric tensor product of i fields ε and $4 - i$ fields $\mathbb{I}$ and $S = \otimes^4\sigma \equiv (\sigma, \sigma, \sigma, \sigma)$) close under fusion. From these fields tensorized with those of $c = -2$ we can choose a consistent subset $(\mathcal{R}_{\mathbb{I}}, \mathbb{I})$, $(\mathcal{R}_{\mathbb{I}}, E_i)$, (μ, S) and (ν, S). Obviously, (ν, S) has conformal weight $h = \frac{5}{8}$ and (μ, S) has conformal weight $h = \frac{1}{8}$ which are fields assumed to appear in percolation. However, if percolation can be described by a $c = 0$ model such as $(c = 2) \otimes (c = -2)$, the question remains how Watts' differential equation [32] can be derived through a level three null vector condition acting on a four point function of boundary changing operators in this theory [10].

The operator product expansion The OPE of the tensorized $c = 0$ model, consists of an ordinary CFT part from the $c = 2$ sector and and LCFT part from the $c = -2$ sector. To obtain the two point functions, we make the same ansatz as before (64)

$$\begin{aligned} T(z) &= {:}\partial\theta^+(z)\partial\theta^-(z){:}\,, & (74)\\ \tilde{\mathbb{I}}(z) &= {:}\theta^-(z)\theta^+(z){:}\,, & (75)\\ t(z) &= {:}T(z)\tilde{\mathbb{I}}(z){:}\,. & (76) \end{aligned}$$

The results for $\langle Tt\rangle$ and $\langle Tt\rangle$ are exactly the same as for the general case. Since the OPE of $t(z)t(w)$ is relatively short, we will state all terms:

$$\begin{aligned} & t^{(0)}(z)t^{(0)}(w) \\ = \;& t^{(1)}(z)t^{(1)}(w) + \left(\alpha + 2\alpha\beta\tilde{\mathbb{I}} + \beta^2\tilde{\mathbb{I}}(z)\tilde{\mathbb{I}}(w)\right) T^{(2)}(z)T^{(2)}(w) \\ \sim\;& \frac{1}{(z-w)^4}\left(\log^2(z-w) + 2\log(z-w)\tilde{\mathbb{I}}(w) + 1\right) \\ & + \sum_{i=0}^{3} \frac{\log(z-w)\partial^i\tilde{\mathbb{I}}(w)}{i!(z-w)^{4-i}} + \frac{1}{2(z-w)}\partial^3\tilde{\mathbb{I}}(w) \\ & + \frac{1}{(z-w)^2}\left(\left[\log(z-w) - 2\log^2(z-w)\right] T^{(1)}(w) + \left[2 - 4\log(z-w)\right] t^{(1)}(w)\right) \end{aligned}$$

$$+\frac{1}{2(z-w)}\Big(\left[\log(z-w)-2\log^2(z-w)\right]\partial T^{(1)}(w)+\left[2-4\log(z-w)\right]\partial t^{(1)}(w)\Big)$$
$$+\frac{\frac{c_2}{2}(\alpha^2+2\alpha\beta\tilde{\mathbb{I}}-\beta^2(2\log(z-w)\tilde{\mathbb{I}}+\log^2(z-w))}{(z-w)^4}$$
$$+\frac{2(\alpha^2+2\alpha\beta\tilde{\mathbb{I}}-\beta^2(2\log(z-w)\tilde{\mathbb{I}}+\log^2(z-w))T^{(2)}(w)}{(z-w)^2}$$
$$+\frac{\partial_w[(\alpha^2+2\alpha\beta\tilde{\mathbb{I}}-\beta^2(2\log(z-w)\tilde{\mathbb{I}}+\log^2(z-w))T^{(2)}(w)]}{(z-w)}\,.$$

Note that since the θ anti-commute, $:\tilde{\mathbb{I}}(w)\tilde{\mathbb{I}}(w):$ vanishes.

Obviously here, too, it is not possible to reduce the rhs of the equation to terms only consisting of the 'neutral' operators, since it would be necessary to set $\alpha=0$ and $\beta=1$ which means that the OPEs of $T^0(z)t^0(w)$ would vanish. Even the vev of the two-point function of the logarithmic partner vanishes in this case:

$$\langle t^{(0)}(z)t^{(0)}(w)\rangle \;=\; c_2\frac{\alpha\beta+2(1-\beta^2)\log(z-w)}{(z-w)^4}\,. \tag{77}$$

8 Concluding remarks

In our investigation of the structure of (L)CFTs with vanishing central charge we chose a new approach based on the augmented minimal model $c_{(9,6)}=0$, including a Jordan cell structure on the identity level with respect to L_0. From this assumption follows immediately the Jordan cell connection of the level two descendants of $\mathbb{I}$ and its logarithmic partner $\tilde{\mathbb{I}}$, $L_0t(z)=2t(z)+\lambda T(z)$. A special feature of this setup is the vanishing of any two point function involving $T(z)$. Depending on the different resolutions of this puzzle, one is forced to take certain consequences into account. Also, the assumption of a logarithmic partner of the identity naturally leads to a vanishing of all correlation functions which only involve proper primary fields which is consistent with the suitable normalization of the OPEs in this setup. The stress energy tensor is a proper primary field in the case of vanishing central charge.

A possible way to avoid the vanishing of the vacuum expectation values could be the interpretation that this is due to the presence of certain zero modes. Such behavior is well known in fermionic theories such as ghost systems and particularly $c=-2$. Only when the zero modes are canceled due to certain field insertions do we get non-vanishing results. It would be most tempting to try to construct a free field realization of a $c=0$ theory as a Kac-table based theory with anti-commuting fields.

We presented several arguments why the Kac table based ansatz is more promising than the replica approach chosen in the literature so far, especially with respect to its interpretation in physics and determination of the field content. Furthermore we extended the tensor ansatz as a fourth loophole out of the $c \to 0$ catastrophe to an (L)CFT and gave examples for both approaches to $c = 0$.

Since none of the approaches to $c = 0$ currently seems to be able to fulfill all wanted features by now at a time we suggest further investigation. This includes a fermionic realization of the augmented minima model and, above all, general research on the representation theory of the augmented minimal $c_{(9,6)} = 0$ model with extended Kac table. Thus with our ansatz, we therefore discovered an interesting application for augmented minimal models, or, more precisely an extension of the (L)CFT formalism of $c_{(p,1)}$ models to $c_{(p,q)}$ with arbitrary $q \in \mathbb{Z}_+$. Hence, the investigation of the representations inside the boundary of the original replicated Kac table will be an important field of research in the future [7].

We believe that the results of this paper have been a small but important step towards the implementation of percolation models within an LCFT approach. We have shown that we should reconsider widely accepted assumptions such as the postulation of $c = 0$ for percolation and Jordan cell structures on higher level without the same structure for the identity in $c = 0$ theories. More precisely, we proved that there can not be a level three and level two null vector condition in a $c = 0$ augmented minimal model simultaneously for a standard assumption of the action of l_0. This would exclude either Watts' or Cardy's differential equation for the two crossing probabilities in percolation.

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A Appendix

A.1 The algebra between the modes of $\mathbf{T(z)}$ and $\mathbf{t(z)}$

What remains to be shown is that the algebra (51) is correct although it differs from the one given in [17] which contains a misprint in eq. (45) although eq. (39) of the same paper is correct. In general, although the two approaches are different, the algebra $[L_n, l_m]$ remains the same since $c = 0$ and thus the only possible term (the central extension) which could be different stays the same. We use the general ansatz for a logarithmic field,

$$\tilde{\phi}_h(z) = \sum_{m\in\mathbb{Z},q\in\mathbb{N}_0} l_{m,q} \log^q(z) z^{-m-h}\,, \tag{78}$$

to calculate the commutator by comparison of the powers of $\log(w)$ and w on both sides of the equation. Note that this method is only applicable to the mixed algebra and the ordinary between the L_n alone since otherwise the residue theorem would have to be applied to a non analytic function, i. e. $\log(w)$.

$$\begin{aligned}
& [L_n, t(w)] \\
= & \sum_{\substack{m\in\mathbb{Z}\\ q\in\mathbb{N}_0}} [L_n, l_{m,q}] \log^q(w) w^{-m-2} \\
= & \oint_0 \mathrm{d}z\, z^{n+1} T(z) t(w) \\
= & \oint_0 \mathrm{d}z\, z^{n+1} \left(\frac{c/2\tilde{\mathbb{I}} + \mu\mathbb{I}}{(z-w)^4} + \frac{2t(w) + \lambda T(w)}{(z-w)^2} + \frac{\partial t(w)}{(z-w)^1} \right) \\
= & w^{n-2} \frac{n(n^2-1)}{6} \left(c/2\tilde{\mathbb{I}} + \mu\mathbb{I} \right) + (n+1) w^n \left(2t(w) + T(w)\right) + w^{n+1} \partial_w t(w) \\
= & \sum_{\substack{m\in\mathbb{Z}\\ q\in\mathbb{N}_0}} \log^q(w) w^{-m-2} \Bigg[\left(\frac{n(n^2-1)}{6} \left(c/2\tilde{\mathbb{I}} + \mu\mathbb{I} \right) \delta_{n+m,0} + (n+1) L_{n+m} \right) \delta_{q,0} \\
& \qquad (n-m) l_{n+m,q} + (q+1) l_{n+m,q+1} \Bigg]\,.
\end{aligned}$$

from which we may extract the commutator

$$\begin{aligned}
[L_n, l_m] = & \left(\frac{n(n^2-1)}{6} \left(c/2\tilde{\mathbb{I}} + \mu\mathbb{I} \right) \delta_{n+m,0} + (n+1) L_{n+m} \right) \delta_{q,0} \\
& + (n-m) l_{n+m,q} + (q+1) l_{n+m,q+1}\,.
\end{aligned} \tag{79}$$

A rather practical than elegant way out of the problem of complicated commutators as (79) is the application of the whole thing to the vacuum (or any other highest weight state). Imposing regularity at $w \to 0$ we can conclude, that all modes with

$q \neq 0$ have to vanish in that case. Thus we are left with an analytic expression for $t(w)$, i. e.

$$t(w)|0\rangle = \sum_{m\in\mathbb{Z}} l_m w^{-m-2}|0\rangle\,, \tag{80}$$

and we could even calculate the OPE as in the usual way for non logarithmic fields,

$$\begin{aligned}
& [L_n, l_m]|0\rangle \\
= & \frac{1}{(2\pi\mathrm{i})^2}\oint_0 \mathrm{d}w\, w^{m+1}\oint_0 \mathrm{d}z\, z^{n+1}\left(\frac{c/2\tilde{\mathbb{I}}+\mu\mathbb{I}}{(z-w)^4}+\frac{2t(w)+\lambda T(w)}{(z-w)^2}+\frac{\partial t(w)}{(z-w)^1}\right)|0\rangle \\
= & \frac{1}{2\pi\mathrm{i}}\oint_0 \mathrm{d}w\, w^{m+n-1}\frac{n(n^2-1)}{6}(\frac{c}{2}\tilde{\mathbb{I}}+\mu\mathbb{I})|0\rangle \\
& +\frac{1}{2\pi\mathrm{i}}\oint_0 \mathrm{d}w\, w^{m+n+1}(n+1)(2t(w)+\lambda T(w))|0\rangle+\frac{1}{2\pi\mathrm{i}}\oint_0 \mathrm{d}w\, w^{m+n+2}\partial t(w)|0\rangle \\
= & \left(\frac{n(n^2-1)}{6}(\frac{c}{2}\tilde{\mathbb{I}}+\mu\mathbb{I})\delta_{n,-m}+(n+1)(2l_{n+m}+\lambda L_{n+m})-(n+m+2)l_{n+m}\right)|0\rangle\,,
\end{aligned}$$

and therefore

$$[L_n, l_m]|0\rangle = (n-m)l_{n+m}|0\rangle + (n+1)\lambda L_{n+m}|0\rangle \;+\frac{n(n^2-1)}{6}\mu\delta_{n+m,0}|0\rangle \tag{81}$$

with $T(z)t(w)$ as given in (9).

A.2 Mode expansion of $\mathbf{t(z)}$ in $\mathbf{c=-2}$

As a concrete example for a non trivial mode expansion containing logarithms, we chose the $c=-2$ CFT which is known to have a special realization containing

$$\theta^{\pm} = \theta_0^{\pm}\log(z) + \xi^{\pm} + \sum_{n\neq 0}\theta_n^{\pm} z^{-n}\,, \tag{82}$$

with the modes of $\theta^{\pm}$ obeying the canonical anti-commutation relations:

$$\begin{aligned}
\{\theta_n^{\pm},\theta_m^{\mp}\} &= \frac{1}{n}\delta_{m+n,0}\,, & (83)\\
\{\xi^{\pm},\theta_0^{\mp}\} &= \pm 1\,. & (84)
\end{aligned}$$

The logarithmic partner of the identity is given by

$$\tilde{\mathbb{I}}(z) \;:=\; :\theta^-\theta^+:(z) = \sum_{n\in\mathbb{Z}}\left(\imath_n + \log(z)\tilde{\imath}_n + \log^2(z)\hat{\imath}_n\right)z^{-n}\,. \tag{85}$$

Inserting (82), we observe

$$
\begin{aligned}
\tilde{\mathbb{I}}(z) &:= \,:\theta^-\theta^+:(z) \\
&= \,:\Big(\theta_0^- \log(z) + \xi^- + \textstyle\sum_{n\neq 0}\theta_n^- z^{-n}\Big)\Big(\theta_0^+ \log(z) + \xi^+ + \textstyle\sum_{m\neq 0}\theta_m^+ z^{-m}\Big): \\
&= \log^2(z):\theta_0^-\theta_0^+: \\
&\quad + \log(z)\left[\left(:\theta_0^-\xi^+: + :\xi^-\theta_0^+:\right) + \left(\sum_{n\neq 0}\left(:\theta_n^-\theta_0^+: + :\theta_0^-\theta_n^+:\right)z^{-n}\right)\right] \\
&\quad +:\xi^-\xi^+: + \sum_{n\neq 0}\left[:\theta_n^-\theta_{-n}^+: + \left(:\xi^-\theta_n^+: + :\theta_n^-\xi^+:\right)z^{-n}\right] \\
&\qquad + \sum_{\substack{m,n\neq 0\\ n\neq m}} :\theta_n^-\theta_{-m}^+:z^{m-n}\,.
\end{aligned}
$$

Now we can identify the terms. For $n \neq 0$ the modes of $\tilde{\mathbb{I}}(z)$ are

$$
\begin{aligned}
\imath_0 &= :\xi^-\xi^+: + \sum_{n\neq 0} :\theta_n^-\theta_{-n}^+:\,, \\
\imath_n &= \sum_{n\neq 0}\left(:\xi^-\theta_n^+: + :\theta_n^-\xi^+:\right)z^{-n} + \sum_{n,m\neq 0, n\neq m} :\theta_n^-\theta_{-m}^+:z^{m-n}\,, \\
\tilde{\imath}_0 &= :\theta_0^-\xi^+: + :\xi^-\theta_0^+:\,, \\
\tilde{\imath}_n &= \sum_{n\neq 0}\left(:\theta_n^-\theta_0^+: + :\theta_0^-\theta_n^+:\right)\,, \\
\hat{\imath}_0 &= :\theta_0^-\theta_0^+:\,, \\
\hat{\imath}_n &= 0\,.
\end{aligned}
$$

Since $t(z) = :T(z)\tilde{\mathbb{I}}(z):$ with $T(z) = :\partial\theta^+(z)\partial\theta^-(z):$, we have to check the mode expansion of $T(z)$. Taking the derivative of (82) with respect to z, we see that the logarithm and the ξ modes vanish. Thus, taking the normal ordered product $:T(z)\tilde{\mathbb{I}}(z):$ and expanding it by modes yields the same structure as in (85). Eventually, some of the modes which vanished for $\tilde{\mathbb{I}}$ may not vanish for $t(z)$, i. e. in general the $\hat{l}_n$ may differ from zero, where $\hat{l}_n = l_{n,2}$ in the notation of (78).

A.3 Calculations for the bosonic free field construction

The computation simplifies greatly if we take advantage of known OPEs such as $\langle TT\rangle$ or $\langle\tilde{\mathbb{I}}\tilde{\mathbb{I}}\rangle$, taking the ansatz

$$
\begin{aligned}
t(z)t(w) &= :\tilde{\mathbb{I}}(z)T(z)::\tilde{\mathbb{I}}(w)T(w): \\
&\sim (\tilde{\mathbb{I}}(z)\tilde{\mathbb{I}}(w))(T(z)T(w)) + (\tilde{\mathbb{I}}(z)T(w))(T(z)\tilde{\mathbb{I}}(w)) \\
&\quad +(\tilde{\mathbb{I}}(z)\tilde{\mathbb{I}}(w)):T(z)T(w): + (T(z)T(w)):\tilde{\mathbb{I}}(z)\tilde{\mathbb{I}}(w): \\
&\quad +(\tilde{\mathbb{I}}(z)T(w)):T(z)\tilde{\mathbb{I}}(w): + (T(z)\tilde{\mathbb{I}}(w)):\tilde{\mathbb{I}}(z)T(w):\,.
\end{aligned}
$$

In this case we have to pay attention carefully to the normal ordering of terms. In general, we get a very lengthy expression:

$$
\begin{aligned}
&t(z)t(w) \sim \\
&\left(4\lambda^2\log^2(z-w)+\frac{\lambda^2}{2\alpha_0^2}\log(z-w)-4\lambda\log(z-w)\tilde{\mathbb{I}}(w)\right)\frac{\frac{c}{2}}{(z-w)^4} \\
&+\frac{\lambda^2}{(z-w)^4}+\frac{\frac{c}{2}}{(z-w)^4}{:}\tilde{\mathbb{I}}(w)\tilde{\mathbb{I}}(w){:} \\
&-2\lambda\log(z-w)\partial\tilde{\mathbb{I}}(w)\frac{\frac{c}{2}}{(z-w)^3}+\frac{\frac{c}{2}}{(z-w)^3}{:}\partial\tilde{\mathbb{I}}(w)\tilde{\mathbb{I}}(w){:} \\
&-\lambda\log(z-w)\partial^2\tilde{\mathbb{I}}(w)\frac{\frac{1-24\alpha_0^2}{2}}{(z-w)^2}+\frac{\frac{c}{2}}{2(z-w)^2}{:}\partial^2\tilde{\mathbb{I}}(w)\tilde{\mathbb{I}}(w){:}+\frac{2{:}T(w)\tilde{\mathbb{I}}(w)\tilde{\mathbb{I}}(w){:}}{(z-w)^2} \\
&+\frac{(\frac{\lambda^2}{\alpha_0^2}\log(z-w)+8\lambda^2\log^2(z-w))T(w)}{(z-w)^2}-\frac{8\lambda\log(z-w)t(w)}{(z-w)^2}+\frac{2\lambda t(w)}{(z-w)^2} \\
&+\frac{1}{(z-w)^2}\left(-\partial\tilde{\mathbb{I}}(w)\partial\tilde{\mathbb{I}}(w)-2\lambda^2\partial\phi(w)\partial\phi(w)+\left(\frac{\lambda^2}{\mathrm{i}\sqrt{2}\alpha_0}+\lambda\mathrm{i}a\sqrt{2}\tilde{\mathbb{I}}(w)\right)\partial^2\phi(w)\right) \\
&-\lambda\log(z-w)\partial^3\tilde{\mathbb{I}}(w)\frac{\frac{c}{2}}{3(z-w)}+\frac{\frac{c}{2}}{6(z-w)}{:}\partial^3\tilde{\mathbb{I}}(w)\tilde{\mathbb{I}}(w){:}+\frac{\partial{:}T(w)\tilde{\mathbb{I}}(w)\tilde{\mathbb{I}}(w){:}}{(z-w)} \\
&+\frac{(\frac{\lambda^2}{2\alpha_0^2}\log(z-w)+4\lambda^2\log^2(z-w))\partial T(w)}{(z-w)}-\frac{4\lambda\log(z-w)\partial t(w)}{(z-w)}+\frac{\lambda\partial t(w)}{(z-w)} \\
&+\frac{1}{(z-w)}\left(\partial\tilde{\mathbb{I}}(w)\left(-2\lambda\partial\phi(w)\partial\phi(w)-\left(\frac{\lambda}{\mathrm{i}\sqrt{2}\alpha_0}+\mathrm{i}a\sqrt{2}\tilde{\mathbb{I}}(w)\right)\partial^2\phi(w)\right)\right. \\
&\left.+\left(\frac{\lambda^2}{2\mathrm{i}\sqrt{2}\alpha_0}+\frac{\lambda\mathrm{i}a\sqrt{2}}{2}\tilde{\mathbb{I}}(w)\right)\partial^3\phi(w)-\lambda^2\partial\phi(w)\partial^2\phi(w)\right) \\
&-2\lambda\log(z-w)\partial^4\tilde{\mathbb{I}}(w)\frac{\frac{c}{2}}{24} \\
&-2\lambda\log(z-w)\partial\tilde{\mathbb{I}}(w)\partial T(w)-\frac{4\lambda}{3}\log(z-w)T(w)\partial^2\tilde{\mathbb{I}}(w) \\
&+\left(4\lambda^2\log^2(z-w)+\frac{\lambda^2}{2\alpha_0^2}\log(z-w)-4\lambda\log(z-w)\tilde{\mathbb{I}}(w)\right) \\
&\cdot\left(\frac{1}{2}{:}\partial^3\phi(w)\partial\phi(w){:}+\frac{\mathrm{i}\sqrt{2}\alpha_0}{6}\partial^4\phi(w)+{:}T(z)T(w){:}\right) \\
&+\left({:}\tilde{\mathbb{I}}(w)\tilde{\mathbb{I}}(w){:}+\frac{\lambda^2}{2\alpha_0^2}\log(z-w)\right)\frac{\frac{c}{2}}{(z-w)^{4-2a^2}} \\
&+\left({:}\partial\tilde{\mathbb{I}}(w)\tilde{\mathbb{I}}(w){:}\right)\frac{\frac{c}{2}}{(z-w)^{3-2a^2}} \\
&+\left({:}\partial^2\tilde{\mathbb{I}}(w)\tilde{\mathbb{I}}(w){:}\right)\frac{\frac{c}{2}}{2(z-w)^{2-2a^2}}+\frac{{:}T(w)\tilde{\mathbb{I}}(w)\tilde{\mathbb{I}}(w){:}}{(z-w)^{2-2a^2}}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\frac{\lambda^2}{\alpha_0^2}\log(z-w){:}T(w)\exp(2\mathrm{i}a\sqrt{2}\phi(w)){:}}{(z-w)^{2-2a^2}}\\
&+\left({:}\partial^3\tilde{\mathbb{I}}(w)\tilde{\mathbb{I}}(w){:}\right)\frac{\frac{c}{2}}{6(z-w)^{1-2a^2}}+\frac{\partial{:}T(w)\tilde{\mathbb{I}}(w)\tilde{\mathbb{I}}(w){:}\exp(2\mathrm{i}a\sqrt{2})}{(z-w)^{1-2a^2}}\\
&+\frac{\frac{\lambda^2}{2\alpha_0^2}\log(z-w){:}\partial T(w)\exp(2\mathrm{i}a\sqrt{2}\phi(w)){:}}{(z-w)^{1-2a^2}}\\
&+\left(\frac{\lambda^2}{2\alpha_0^2}\log(z-w)(z-w)^{-2a^2}\exp(2\mathrm{i}a\sqrt{2}\phi(w))+(z-w)^{-2a^2}{:}\tilde{\mathbb{I}}(z)\tilde{\mathbb{I}}(w){:}\right)\\
&\cdot\left(\frac{1}{2}{:}\partial^3\phi(w)\partial\phi(w){:}+\frac{\mathrm{i}\sqrt{2}\alpha_0}{6}\partial^4\phi(w)+{:}T(z)T(w){:}\right).
\end{aligned}
$$

Now we can make the assumption $2a^2 > 4$ ($a > \sqrt{2}$), in order to get rid of the terms proportional to $(z-w)$ to some powers of a meaning that no additional fields appear in the singular part of the OPE. In this case $\alpha_0 > \frac{1}{\sqrt{2}}$, which means that $c = 0$ would be excluded. But for our $c = -24$ proposal for percolation, it is justified. For $c = 0$ ($a = \frac{1}{\sqrt{6}}$) we would get fractional exponents. Hence the OPE would no longer be valid for a local chiral field.

A rank three Jordan cell realization Based on the augmented $c_{(9,6)} = 0$ LCFT, we know that we should have three $h = 0$ fields in the theory, probably belonging to a rank three Jordan cell structure generated by L_0. Such a rank three Jordan cell can indeed be constructed in our free field realization as follows: We take the simplest possible ansatz for a third field with vanishing central charge:

$$\hat{\mathbb{I}}(z) = {:}\phi\phi{:}(z). \tag{86}$$

Now we can check for its properties, computing the OPE with all $h = 0$ fields and the stress energy tensor. Obviously, the OPE with the identity $\mathbb{I}$ has to be trivial.

Furthermore we have to keep in mind that the identification of the two vertex operators corresponding to the solutions of $h(a) = a(a - 2\alpha_0) = 0$ has to hold in both directions. Hence we get the following OPEs:

$$
\begin{aligned}
T(z)\hat{\mathbb{I}}(w) &= \frac{\mathbb{I}-4\alpha_0^2\tilde{\mathbb{I}}(w)}{(z-w)^2}-\frac{\partial\hat{\mathbb{I}}(w)}{(z-w)}\,,\\
\tilde{\mathbb{I}}(z)\hat{\mathbb{I}}(w) &= -2\hat{\mathbb{I}}(w)\log(z-w)-2\log^2(z-w)\\
&\quad -a^2\tilde{\mathbb{I}}(w)\log(z-w)-\tilde{\mathbb{I}}(w)\log(z-w)\,,\\
\hat{\mathbb{I}}(z)\hat{\mathbb{I}}(w) &= 2\log^2(z-w)+4\log(z-w)\hat{\mathbb{I}}(w)\,.
\end{aligned}
$$

Obviously, $\hat{\mathbb{I}}$ is a standard logarithmic field.

We would like to remark that we have assumed a rank two Jordan cell for the vacuum sector throughout the paper for the sake of simplicity. Our general results are not affected by a higher rank Jordan cell structure (up to some trivial relabeling and keeping track of more than one logarithmic partner of the identity). The point which makes the difference is whether there is an indecomposable Jordan structure for the identity or not, and this is what we have elaborated on in this paper.

A.4 c = −2 and the fourfold Ising model

The OPE of the tensorized $c = 0$ model, consists of an ordinary CFT part from the sector with $c = +2$ and and LCFT part from the $c = -2$ sector. The basic features of the special representation of the $c = -2$ theory are stated in (82) and (83). The contraction rules follow from

$$\theta^+(z)\theta^-(w) = -\log(z-w). \tag{87}$$

To obtain the two-point functions, we make the same ansatz as before. With the help of the anticommutation relations we find that

$$-\frac{1}{2}{:}\partial^2\tilde{\mathbb{I}}(w)\tilde{\mathbb{I}}(w){:} - {:}\partial\tilde{\mathbb{I}}(w)\partial\tilde{\mathbb{I}}(w){:} = -\frac{1}{2}\partial^2{:}\tilde{\mathbb{I}}(w)\tilde{\mathbb{I}}(w){:} = 0$$

and

$$-\frac{1}{3}{:}\partial^3\tilde{\mathbb{I}}(w)\tilde{\mathbb{I}}(w){:} - \partial{:}\partial\tilde{\mathbb{I}}(w)\partial\tilde{\mathbb{I}}(w){:} = -\frac{1}{3}\partial^3{:}\tilde{\mathbb{I}}(w)\tilde{\mathbb{I}}(w){:} = 0$$

as well as

$${:}\tilde{\mathbb{I}}(w)\tilde{\mathbb{I}}(w){:} \quad \text{and} \quad \partial{:}\tilde{\mathbb{I}}(w)\tilde{\mathbb{I}}(w){:} = 0\,.$$

Thus the OPE reduces to

$$\begin{aligned}
& t(z)t(w) \\
\sim\; & \frac{1}{(z-w)^4}\Big(\log^2(z-w) - 2\log(z-w)\tilde{\mathbb{I}}(w) + 1\Big) \\
& -\frac{1}{(z-w)^3}\log(z-w)\partial\tilde{\mathbb{I}}(w) \\
& +\frac{1}{(z-w)^2}\left(\left[\log(z-w) - 2\log^2(z-w)\right]T(w) + \left[2 - 4\log(z-w)\right]t(w)\right) \\
& -\frac{1}{(z-w)^2}\left(\frac{\log(z-w)}{2}\partial^2\tilde{\mathbb{I}}(w)\right) \\
& +\frac{1}{2(z-w)}\left(\left[\log(z-w) - 2\log^2(z-w)\right]\partial T(w) + \left[2 - 4\log(z-w)\right]\partial t(w)\right) \\
& +\frac{1}{2(z-w)}\left(\partial^3\tilde{\mathbb{I}}(w) - \frac{\log(z-w)}{3}\partial^3\tilde{\mathbb{I}}(w)\right) \\
& +\Big(\log^2(z-w) - 2\log(z-w)\tilde{\mathbb{I}}(w)\Big){:}T(z)T(w){:} \\
& +\frac{\log(z-w)}{24}\left({:}\partial^4\theta^+(w)\theta^-(w){:} + {:}\theta^+(w)\partial^4\theta^-(w){:}\right) \\
& -\log^2(z-w)\left({:}\partial\theta^+(w)\partial^3\theta^-(w){:} + {:}\partial^3\theta^+(w)\partial\theta^-(w){:}\right) \\
& +2\log(z-w)\left({:}\partial\theta^+(w)\partial^3\theta^-(w){:} + {:}\partial^3\theta^+(w)\partial\theta^-(w){:}\right)\tilde{\mathbb{I}}(w)\,.
\end{aligned}$$

A.5 Calculations for the general tensorized model

We take the ansatz

$$\begin{aligned}
t^{(0)}(z)t^{(0)}(w) &= t^{(1)}(z)t^{(1)}(w) + \left(\alpha + \beta\tilde{\mathbb{I}}(z)\right)\left(\alpha + \beta\tilde{\mathbb{I}}(w)\right)T^{(2)}(z)T^{(2)}(w) \\
&\sim \left(T^{(1)}(z)T^{(1)}(w) + :T^{(1)}(z)T^{(1)}(w):\right)\left(\tilde{\mathbb{I}}^{(1)}(z)\tilde{\mathbb{I}}^{(1)}(w) + :\tilde{\mathbb{I}}^{(1)}(z)\tilde{\mathbb{I}}^{(1)}(w):\right) \\
&\quad + \left(T^{(1)}(z)\tilde{\mathbb{I}}(w) + :T^{(1)}(z)\tilde{\mathbb{I}}(w):\right)\left(\tilde{\mathbb{I}}^{(1)}(z)T^{(1)}(w) + :\tilde{\mathbb{I}}^{(1)}(z)T^{(1)}(w):\right) \\
&\quad + \left(\alpha^2 + 2\alpha\beta\tilde{\mathbb{I}} + \beta^2\tilde{\mathbb{I}}(z)\tilde{\mathbb{I}}(w)\right)T^{(2)}(z)T^{(2)}(w)\,.
\end{aligned}$$

Note that there are no contractions between the two parts; the tensorized fields factorize into their respective OPEs.

$$\begin{aligned}
&t^0(z)t^0(w) \\
\sim\; &\left(\frac{\frac{c_1}{2}}{(z-w)^4} + \frac{2T^{(1)}(w)}{(z-w)^2} + \frac{\partial_w T^{(1)}(w)}{(z-w)} + :T^{(1)}(z)T^{(1)}(w):\right) \\
&\qquad \times \left(\log^2(z-w) - 2\log(z-w)\tilde{\mathbb{I}}(w) + :\tilde{\mathbb{I}}^{(1)}(z)\tilde{\mathbb{I}}^{(1)}(w):\right) \\
&+ \left(\frac{\mathbb{I}}{(z-w)^2} + \frac{\partial_w\tilde{\mathbb{I}}(w)}{(z-w)} + :T^{(1)}(z)\tilde{\mathbb{I}}(w):\right) \\
&\qquad \times \left(\frac{\mathbb{I}}{(z-w)^2} - \frac{\partial_w\tilde{\mathbb{I}}(w)}{(z-w)} + :\tilde{\mathbb{I}}^{(1)}(z)T^{(1)}(w):\right) \\
&+ \left(\alpha^2 + 2\alpha\beta\tilde{\mathbb{I}}(w) + \beta^2\tilde{\mathbb{I}}(z)\tilde{\mathbb{I}}(w)\right) \\
&\qquad \times \left(\frac{\frac{c_2}{2}}{(z-w)^4} + \frac{2T^{(2)}(w)}{(z-w)^2} + \frac{\partial_w T^{(2)}(w)}{(z-w)} + :T^{(2)}(z)T^{(2)}(w):\right)\,.
\end{aligned}$$

After inserting the OPEs (63) and sorting the terms by order of $(z-w)$ we get

$$\begin{aligned}
\sim\; &\frac{1}{(z-w)^4}\Big[\left(1 + \alpha^2\frac{c_2}{2}\right) - \left(\frac{c_1}{2} + \beta^2\frac{c_2}{2}\right)\log^2(z-w) \\
&+ 2\left(\frac{c_1}{2} + \beta^2\frac{c_2}{2}\right)\log(z-w)\tilde{\mathbb{I}}(w) + \alpha\beta\tilde{\mathbb{I}}^{(1)}c_2 + \left(\frac{c_1}{2} + \beta^2\frac{c_2}{2}\right):\tilde{\mathbb{I}}^{(1)}(z)\tilde{\mathbb{I}}^{(1)}(w):\Big] \\
&+ \frac{1}{(z-w)^2}\Big[+2t_0 + \alpha\beta\tilde{\mathbb{I}}^{(1)}T^{(2)}(w) + \left(T_0(w) - (1-\beta^2)T^{(2)}(w)\right):\tilde{\mathbb{I}}^{(1)}(w)\tilde{\mathbb{I}}^{(1)}(w): \\
&+2\left(T_0(w) - (1-\beta^2)T^{(2)}(w)\right)\log^2(z-w) - 4\left(t_0 - \alpha\beta\tilde{\mathbb{I}}T^{(2)}(w)\right)\log(z-w)\Big] \\
&+ \frac{1}{(z-w)}\Big[\partial t_0 + \frac{1}{2}\alpha\beta\tilde{\mathbb{I}}^{(1)}\partial T^{(2)}(w) + \partial\left(\left(T_0(w) - (1-\beta^2)T^{(2)}(w)\right):\tilde{\mathbb{I}}^{(1)}(w)\tilde{\mathbb{I}}^{(1)}(w):\right) \\
&+ \left(\partial T_0(w) - (1-\beta^2)\partial T^{(2)}\right)\log^2(z-w) - 2\left(\partial t_0 - \alpha\beta\tilde{\mathbb{I}}\partial T^{(2)}(w)\right)\log(z-w)\tilde{\mathbb{I}}^{(1)}(w)\Big] \\
&+ \left(\log^2(z-w) + 2\log(z-w)\tilde{\mathbb{I}}^{(1)}(w)\right)\left(:T_0(w)T_0(w): - (1-\beta^2):T^{(2)}(w)T^{(2)}(w):\right)\,.
\end{aligned}$$

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