Conformal Field Theory: A Bridge Over Troubled Waters

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Abstract
Since 70 years, quantum field theory has been one of the most important research areas of physics. For further progress, it must become a standard domain of mathematical research, too. The article reviews the historical obstacles and shows how they can be overcome. Already now, conformally invariant quantum field theory in two dimensions has become a well defined and beautiful structure. Its emergence is considered in the context of high energy physics, statistical physics, and string theory.

In his 1972 address to the American Mathematical Society, Dyson deplored the 'divorce' between mathematics and physics over the issue of quantum field theory. The present book on the impact of field theory on modern physics, timed in accordance with the International Mathematics Year of 2000 AD, gives hope that the rift will soon be bridged. In conformal quantum field theory in two dimensional spacetime, conditions are particularly favorable for gaining common ground. This area can attract both mathematicians and physicists by its beauty, its transparent mathematical structure and its many applications. Thus its investigation has moved to a rather central position in quantum field theory as a whole. Perhaps one can say that a bridge exists since many years, but has been used much below its capacity.

Analogues in more than two dimensions will be mentioned in the present article, but they have not been developed very far nor found mathematical applications yet. For convenience, the expression 'conformal field theory' will refer to conformally invariant quantum field theories in two dimensions, if nothing else is said.

Its applications are suprisingly diverse. In mathematics, one particular theory has become well known due to its automorphism group, the Fischer-Griess monster, and by the Fields medal for Borcherds. Currently research on the many mirror symmetric cases makes rapid progress, whereas the explicit construction of Kähler-Einstein metrics is just at a planning stage. In physics, conformal field theory became essential for the study of continuous phase transitions in condensed matter physics, but the most important applications concern string theory. Even independently of its status as the best candidate for a theory of quantum gravity, string theory has become an important tool in quantum field theory. In particular, it yields a map from two-dimensional conformal theories to quantum field theories in higher dimensions, even to rather realistic examples in four-dimensional spacetime.

It will be argued that conformal field theory can satisfy all mathematicians who want to understand quantum field theory without giving up their standards of clarity and rigour. In return, many of its important aspects need advanced mathematical techniques. Some still remain out of the reach of physicists, others are handled in a manner which produces lots of undiscovered errors in the literature. A serious involvement of mathematicians will yield much firmer foundations for the things to come.

Apart from conformal field theory, this article will discuss some string theory. The perturbative aspects of the latter are well understood by now and should be easily accessible. Indeed, their bulk just constitutes a direct application of conformal field theory. By themselves, these aspects do not yet transcend the old principles of 20th century physics, but they are incomplete and seem to point in one unique direction of progress. There, non-perturbative string theory (or M-theory, or however one chooses to call it) has started to emerge and should lead to a new insight. Its birth might be a lot easier if mathematicians again get into the habit of acting as good midwives.

There is no lack of good will. Many conferences have been attended by mixed groups of mathematicians and physicists. Most importantly, Princeton has brought together many of the best people of both communities in a dedicated program. Nevertheless, one sometimes gets the feeling that a new generation will be needed to overcome the difficulties.

It may help to recognize what the main obstacles have been, since a discussion of past successes and mistakes can prepare the way for future achievements. In any case, the present article is supposed to cover the early stages of the discoveries. Of course, history is complex, full of turbulence and countercurrents, such that precise statements would need more hedging or much more research. The historical content of the present article should therefore be taken as a signpost, not as a map. Its indications will be approximate, but may be helpful. Moreover, a low density of formulas should make the article accessible to a wider readership. Methodological qualms of the historians will be brazenly ignored. If they deny that it makes sense to ask what might have happened, what about an advanced quantum computer which reruns history, starting from a reasonable subspace of initial conditions.

An invitation for mathematicians to cross the bridge also needs some technical parts, however. Mathematicians still think of quantum field theory as a useful source of ideas (cf. the Seiberg-Witten equations), but otherwise as impenetrable, though some related structures are regarded as good mathematics (topological quantum field theory, probably conformal field theory). ¿From the point of view of a physicist, this is a strange attitude. By nature, conformal field theory was presented in the context of quantum field theory as a whole, so this is the way it should be discussed. Nature has a habit of posing such problems, recall infinitesimals and differential equations. Again, we should have confidence in her guidance.

The article starts with a short introduction to the history of quantum field theory. For readers who want to get a fuller picture and the necessary references, there are good reviews and reprint volumes at different technical levels [Schwinger 1958, Pais 1986, Crease and Mann 1987]. In those reviews the aim is to show how nature was explained. Here, mathematically well-built structures will be regarded as equally important for further progress. We shall see that the tools to build the bridge were available at the end of the 60's, maybe even ten years earlier.

Because of the focus on conformally invariant theories, other issues of mathematically rigorous quantum field theory, like the work of Glimm and Jaffe on superrenormalizable theories will not be discussed, however. Even for the main themes of the article, the selection is partial. Current algebras are one of the major themes of conformal field theory, but here the collaboration between mathematicians and physicists has been very fruitful since twenty years, and there are already nice reviews, e.g. [Goddard and Olive 1988], so the topic will receive less emphasis than would be necessary in a complete survey. Renormalization will be discussed in some detail, since most mathematicians regard it as the major stumbling block which prevents an understanding of quantum field theory. Thus it may help to see that this procedure is rather easy from a mathematical point of view and follows a well known idea of the 19th century. Path integrals will not be mentioned.

In quantum field theory as a whole the corresponding ideas have not yet found a satisfactory form, and even in conformal field theory one needs a discussion in terms of categories and operads, which only seems simple to the very good or the very young. The discussion of string theory is limited to the perspective of conformal field theory. A serious consideration of its non-perturbative aspects would have to start with a description of instantons and solitons. Many connections with conformal invariance could be explained, but not within the scope of the present article. Altogether, it is unavoidable that many readers will have reasons to complain, but at least they should feel encouraged to do better.

After a golden age

A theoretical physicists who looks back to the beginning of the past century has reasons to feel rather humble. A few decades witnessed three revolutionary insights in structures of nature, namely special relativity, quantum mechanics and Einstein's theory of gravity. We still have to embed these in a unified theory. Meanwhile, we covered much territory in the study of the complexities of nature, but further understanding of her basic features proceeds at a snail's pace.

Evidently, the initial rapid progress relied on a close interaction between physicists and mathematicians. We cannot even talk about those three structures without alluding to this fact. Just think of the Poincaré group, Hilbert space and Riemannian geometry.

In the first years of the century, many physicists felt doubtful or uneasy about the importance of contemporary mathematical methods. For example, Mittag-Leffler struggled in vain to get a Nobel prize for Poincaré out of the old establishment. Around 1920, the movement had become irresistible, however. Einstein's Nobel prize document avoids to mention special or general relativity, but this was hardly more than a funny detail. Everyone went to Göttingen, Weyl solved the Schrödinger equation for the hydrogen atom, and Hilbert was an eager competitor when Einstein approached the final form of his equation for gravity. Of longer lasting importance was the clarification of the mathematical structure of quantum mechanics by v. Neumann and Weyl. Von Neumann had to immerse himself in physics because of the bomb and the computer, but Weyl's publication list in physics journals is impressive, too, and his exchange of ideas with Einstein and Pauli was particularly fruitful.

Mathematicians made contributions of three kinds. Sometimes they solved concrete problems which the physicists found too difficult, but this was rare. More importantly, the internal logic of mathematics had led to the discovery of deep structures which found unexpected applications. Finally, the analysis of discoveries made in physics uncovered new mathematical worlds and allowed the physicists to think more clearly and efficiently about their own results.

It is no surprise that the latter task attracts some of the best mathematical minds. Already in 1900, Hilbert saw that a renewed interest in physics would be productive, and he put the development of good axioms for mechanics as the sixth problem on his famous list. Twenty-five years later, classical mechanics had the necessary clear conceptual basis to allow the wonderful emergence of quantum mechanics. In contrast, the infinite dimensional spaces of classical field theory remained less well understood. This contributed a bit to the confusion about quantum field theory, as we shall see.

When a science is restructured during and after a major advance, it is particularly important to put what has been known before in a new context and to provide it with deeper foundations. Conformal invariance in physics emerged in this way, though at first its place seemed to be marginal. Its discovery was triggered by special relativity. This theory had underlined the importance of symmetry groups and stimulated a new mathematical look at Maxwell's equations. Cunningham [1910] and Bateman [1910] determined the maximal group of symmetries of the latter and discovered their conformal invariance. In other words, Maxwell's system of equations is invariant under the maximal group of spacetime transformations which preserve angles, but not necessarily distances. Addressing mathematics and physics audiences, F. Klein repeatedly asked for an explanation, but in vain.

Some progress became possible with the work of E. Noether [1918]. She was a creator of modern abstract algebra and only had a marginal interest in physics, but was motivated by Einstein's gravity theory to explain the general connection between symmetries of Lagrangians and conservation laws. Prompted by Klein to analyse the problem with Noether's method, Bessel-Hagen determined the conserved quantities corresponding to the conformal invariance of Maxwell's equations [1921]. In the following years, several other differential equations were investigated. Most importantly, Pauli proved that the Dirac equation is conformally invariant when the mass vanishes [1940].

By that time, Hodge had found the tools for a deeper analysis [1941], which needed much longer to make headway into the physics community. He had investigated topological questions like Poincaré duality from the point of view of differential forms. In this language, the electromagnetic field can be described by a 2-form F. In the absence of charges, Maxwell's equations take the form

$$dF = 0, \qquad d * F = 0 \ .$$

This simple form even applies to the curved spacetime of Einstein's theory. Indeed, the differential operator d does not depend on any kind of metric structure. The Hodge duality operator $*$, which acts linearly on differential forms, depends on the metric. When applied to k-forms in a spacetime of $2k$ dimensions, however, the dependence on the distance scale cancels out. In particular, in the most important case of four-dimensional spacetime, $*dF$ only depends on the angles, in other words on the conformal structure. Moreover, the Lagrangian density of the electromagnetic field in empty space is given by the integral over $F \wedge *F$. Since the metric again appears only through the Hodge star operation, the Poisson brackets derived from this Lagrangian have the same conformal invariance. This remains true for the corresponding quantum system, but in general conformal invariance is broken when one introduces charges.

Eventually, this approach turned out to be very productive for physics, see e.g. [Atiyah, Hitchin, Singer 1978], but mainstream physicists learned about it only in the 70's, largely through the efforts of Atiyah, who had heard Hodge's lectures as a student. Indeed, in the 30's a fault-line between the communities of the physicists and the mathematicians had started to develop. It must have been hard to spot at the time, since there were greater and more immediate concerns. Within physics, the split between theoreticians and experimentalists now became complete. Einstein still had done moderately respectable experimental work, but Heisenberg's PhD exam was a near disaster, since he had concentrated all his efforts on theory. Some misgivings of the experimentalists were quite natural. Though there was no reason to expect that mathematical physics would regain the prestigeous position it had had in the times of Newton, Euler and Lagrange, the incredible popularity of Einstein and the difficulties of his theories must have suggested to many that something had gone wrong.

Finally, from the 30's to the 50's physics was hopelessly entangled in far more important events (fascism, the war, the bomb, Stalin, McCarthy, ...). This was a worldwide phenomenon, which left little refuge. It had one important positive aspect, however. Due to the international contacts, progress in physics was no longer the prerogative of Europe or North America. Most visibly, Japan and India had started to take part.

All of this left little room for concern about the spreading rift between theoretical physics and mathematics. Around 1970, however, it had become too big to overlook. In his Gibbs Lecture, Dyson put it bluntly: "the marriage between mathematics and physics ... has recently ended in divorce" [Dyson 1972].

The main culprit seemed to be quantum field theory. Here is a fairly typical quotation from a highly regarded textbook: "The mathematically inclined reader undoubtedly by now will have had serious misgivings about the validity and meaningfulness of the renormalization program, since this program has at its point of departure a set of meaningless equations which it then proceeds to manipulate according to rules which are outside the bounds of conventional mathematics to obtain (presumably) finite results (not to mention the fact these prescriptions, as outlined in the present chapter, are applicable only to the power series expansion of the 'meaningless equations,' which power series expansion in all probability does not converge!)" [Schweber 1964, p. 645].

It is clear that something had gone wrong. In a sense, one may put the blame on nature, since she gave ambiguous directions. We considered the discoveries of special relativity, quantum mechanics and Einstein's theory of gravity, but it is somewhat misleading to talk about them on a par, since the three theories do not occupy the same logical level. Einstein's name for his theory of gravity was general relativity, because compatibility with the principles of special relativity was incorporated from its inception. Thus the task of unification would be finished, if one could join gravity and quantum mechanics in one move. ¿From today's point of view, this problem was too difficult and led into a thick fog.

Instead one could follow the geometrical path indicated by Einstein's gravity theory. This was natural for mathematicians, but not immediately productive for physics. Nevertheless, the search in these directions provided a favorable environment for the development of gauge theories, as we shall see later.

For physicists, a different path was indicated by nature. After quantum mechanics had matured around 1926 (the year of the Schrödinger equation), the next fundamental problem was to put together quantum theory and special relativity. The essential guidance came from the experiments, whereas the mathematical structures remained rather obscure. In favorable circumstances, it still might have been possible to advance together, but many of the links between mathematics and physics were broken by the war. When a deeper study of the weak and strong interactions led to gauge theories, a convergence of the two paths was indicated, but this came too late for a reestablishment of the old contacts. In this sense the lack of mathematical accessability of relativistic quantum field theory is rather a consequence of the separation of mathematics and physics than its cause.

Let us come back to the perspective of 1926. Classical physics deals with rigid bodies and with fields. Now the former were to be regarded as a low velocity approximation, since extended rigid bodies are incompatible with special relativity. When an object is touched, it cannot be affected all at once, since this would surpass the speed of light.

Rigid bodies often had been approximated by point particles. Now they had to be considered in terms of pointlike constituents. In one space dimension, the latter can interact by collisions, but in more dimensions this makes little sense. Thus the only available candidates for the description of interactions in the real world were field theories. Conversely, the discovery of special relativity depended on an analysis of Maxwell's equations for the electromagnetic field. In quantum physics, matter in the form of point particles was easily incorporated in this frame, since the Schrödinger wave function could be regarded as the avatar of a relativistic field. Thus the unification of special relativity and quantum theory demanded the formulation of quantum field theory.

These ideas were well understood in the 1920's. They came very naturally, since already the first steps of quantum mechanics were guided by quantum field theory: The first formula for a quantal phenomenon was Planck's radiation law for the electromagnetic fields emitted by a heated black body. Thus quantum electrodynamics took shape immediately after the birth of modern quantum mechanics, in a 1927 paper by Dirac and, in more appropriate guise, in a paper by Heisenberg and Pauli in 1929. In its initial form, it was sufficient for a calculation of the semiclassical electromagnetic processes observable at that time.

Soon, however, quantum field theory was put in doubt by experimental results and problems of consistency. For experimentalists, further work on the unification of special relativity and quantum theory posed a single basic challenge - study particle interactions at velocities close to the speed of light. Here early researchers were confronted with a bewildering wealth of data, from nuclear physics to cosmic rays. It took a long time until things were sorted out to reveal the underlying structures. In particular, it was far from obvious that the experimental results could be described by any kind of quantum field theory. For a long time, electromagnetism was the only interaction for which it made real sense.

For mathematicians and physicists alike, this greatly diminished the attractiveness of quantum field theory. Most importantly, it contributed to the persistent but unproductive expectation of another revolution in the foundations of physics. In view of the previous decades, this expectation was quite understandable. Quantum mechanics had been developed for atomic physics, particle physics might need something equally revolutionary and exciting. Oppenheimer even gave a number: don't believe the old ideas beyond 100 MeV. For a while, there was a concrete reason for this attitude. Yukawa had predicted a particle of 100 MeV to explain the strong interaction, but when it seemed to be discovered, most of its other properties were wrong. Eventually, muons and pions were distinguished and the paradoxes dissolved away, but the basic attitude surfaced again on many occasions.

Quantum electrodynamics itself set other obstacles against the joint development of quantum field theory in a common effort of physicists and mathematicians. In particular, one immediately had to face one of the old problems of classical physics, namely the infinite energy in the electric field of point charges. In classical physics, spreading out the charge yielded a temporary excuse, but special relativity and quantum mechanics demanded the consideration of point charges, such that a clash was inevitable. There soon came a reason for hope, however. Against his expectations, Weisskopf (with a little help from Furry) showed in 1934 that in quantum electrodynamics the pole divergence of the classical theory is replaced by a mild logarithmic one. In the following years, Kramers explained the basic principles of regularization and renormalization. Weisskopf apparently was slowed down by discontent about his small mistake, but in 1939 he published a clear argument which indicated that any intrinsic inconsistencies of quantum electrodynamics were many orders of magnitude away from the experimentally observable domain.

Altogether, in the 1930's the stage was set for the further development of the theory, and a concerted effort of physicists and mathematicians was not completely out of the question. There was no single overwhelming obstacle. Still, the effort would have demanded an unlikely amount of patience and persistence against many stumbling blocks. Quantum electrodynamics has the typical difficulties of a gauge theory, and mathematics was not quite ready to provide elegant tools for their resolution. For the physicists, experiments had not yet provided a compelling reason to put much effort in the study of the small quantum electrodynamical effects of higher order, and to some extent the bewildering features of the other interactions undermined the faith in quantum field theory as a whole. The mathematicians had no reason to invest much work in something which might not last. They still were busy to consolidate the advances of quantum mechanics and gravity theory. Above all, they saw no compelling internal mathematical reason to develop quantum field theory. In hindsight, v. Neumann's operator algebras came close, but they hardly became part of the mathematical mainstream, and v. Neumann soon had more important things to do.

Since the time for quantum electrodynamics was not yet ripe, joint mathematical and physical progress would have needed another stroke of genius. One possibility would have been the creation of a rigorously solvable but non-free toy model, and from today's point of view conformally invariant theories in two dimensions were by far the best thing to be tried. Indeed, there was a little chance. Einstein had thought about conformal invariance, and Dirac got interested in 1936. At that time, Heisenberg just had started to work on quantum field theories with four-fermion interactions. These can be made conformally invariant in two dimensions, such that a joint effort might have led directly to the Thirring model and perhaps to its solution. After the war, Gürsey searched for a way to make Heisenberg's four-fermion theory conformally invariant, in the line of thought which went back to Cunningham and Bateman. He didn't think about two dimensions, however, and wrote down a four-dimensional version with a cube root, which is impossible to quantize [Gürsey 1956]. For Dirac and Heisenberg, it is unlikely, too, that they considered playing around in two dimensions. Moreover, Dirac became increasingly discontent with quantum field theory as a whole.

Many of Heisenberg's efforts were still creative and successful, but his flirt with mathematics was over. Still, his 1932 concept of a nucleon with two states, put four years later in the language of SU(2) invariance by Cassen and

Condon, had initiated the group theoretic studies which from the 50's onward became one of the major themes of particle physics. Here was perhaps a better chance for joint work with mathematicians, for which such considerations soon became very natural. Some physicists also looked at related structures, even before the experimentalists found convincing reasons to study internal symmetry groups or gauge symmetries.

Einstein had long decided to concentrate on classical gravity and electromagnetism and kept away from quantum theory and the nuclear interactions. He continued to work in the context of classical differential geometry and played around with five dimensions and connections with torsion. These efforts are not highly regarded nowadays, but they familiarized the physics community with the work of E. Cartan and gave much support to the Kaluza-Klein ideas of five-dimensional spacetime. Yang argued that Einstein somehow was looking for the gauge theory found in 1954 by him and Mills [Yang 1982]. Indeed, Einstein repeatedly contemplated parallel transport without the metric constraint of the Levi-Civita connection.

In the five-dimensional line of research, O. Klein himself performed an amazing miracle by writing down the Lagrangian of SU(2) gauge theory during a 1938 conference in Warsaw. He only recognized the U(1) part of the symmetry and saw no clear physical applications, since charged vector mesons had not been found yet [Klein 1939, p. 93].

In 1953, Pauli rediscovered the same SU(2) gauge theory in a conceptually clearer way, when he pushed the Kaluza-Klein ideas one dimension higher and compactified two dimensions on S^2, such that the SO(3) symmetry became manifest. Pauli liked the result and described it in a letter to Pais. He did not publish, however, because he saw no mechanism to give mass to the gauge bosons. Together with Heisenberg he started to work on a fermionic Lagrangian with a four-fermion interaction, but he quickly saw that it made not much sense. Cut down from four to two dimensions it would have been transformed from a wrong unified theory to a fascinating mathematical toy. Altogether, Pauli and Weyl were probably the only ones of the pioneering giants who where both close to the mainstream and imaginative enough to push quantum field theory by inventing, e.g., a mathematically nice conformal field theory. In more fortunate times, Zürich might have witnessed such a step ahead, but it is hard to play in the shadow of war and persecution.

With roots in a dominating wave of mood, the Nazi aversion against Jewish mathematics and physics had pervaded the German universities and the Göttingen environment was destroyed. The focus of research shifted from Europe to the USA.

Progress in the face of mathematics

After the war, the seminal event in the further development of quantum field theory was the Shelter Island conference. The decisive input came from the experimentalists, who made good use of the technology created in the years before. Their results implied that quantum electrodynamics had to be taken very seriously. Mathematicians were absent. Apparently, it had occurred to nobody that they might be of help.

It seems that progress needed a new generation: Very attentive to the experiments, much less to new mathematical structures, conservative in its attachment to the old principles found in the golden age, careful and innovative in calculations. In Princeton, the lonely walks of Einstein and Gödel were parts of a different world, faint reverberations of revolutions in a distant past. The young mathematicians had happy times. They developed fibre bundles, connections, characteristic classes, the deformation theory of complex structures and many other nice things. They laid the groundwork for modern physics, and couldn't care less.

In 1947, work on interacting quantum fields started in earnest. Bethe explained the Lamb shift, and soon after Schwinger calculated the anomalous magnetic moment of the electron. The calculations still were done with the theoretical tools of the prewar period. They started from the quantum theory of free fields and introduced perturbations according to the standard rules of quantum mechanics. Since quantum mechanical perturbation theory makes no use of Lorentz invariance, this procedure compounded the intrinsic difficulties. Soon after, Schwinger and Feynman developed relativistically invariant formalisms, and comparison of quantum electrodynamics with the experiments became very successful. Stueckelberg and Tomonaga had done earlier work in this direction, unfortunately with less impact.

These calculational procedures were correct, but were derived from wrong standard assumptions by dubious mathematical methods. Soon, the standard assumptions were proven to be wrong by a small group of mathematical physicists, whose work was based on an uncontested set of axioms. This caused some uneasiness among the calculators, and kept the mathematical community at a distance. To explain what happened, we have to consider some details of the Heisenberg and Pauli paper of 1929. They were the first to derive the equal time commutators of free fields.

With respect to differentiation by space and time coordinates, quantum fields can satisfy differential equations. In the simplest case, the latter are linear, as for Maxwell's equation. Such fields are called free, since a linear combination of two solutions describes two waves which pass through each other without mutual influence. In the language of today, one starts with the space of classical solutions of the linear differential equation. On this space one needs a symplectic structure, given by a Poisson bracket, or equivalently a Heisenberg Lie algebra. In most cases one has an invariance with respect to a time translation group, the generator of which is the energy. Polarization with respect to the sign of the energy yields the appropriate Hilbert space representation of the Heisenberg Lie algebra (Fock space).

Apparently, in the 20's the symplectic structure of the space of classical solutions had not yet been grasped in depth. Thus the historical procedure was slightly more complicated and involved a special and somewhat formal choice of the classical observables. For free fields, it yielded canonical commutation relations in perfect analogy with Heisenberg's commutation relations

$$[x_i, p_j] = i\delta_{ij} \ ,$$

and the vanishing equal time commutators $[x_i, x_j]$ and $[p_i, p_j]$. Analogously, for a free real scalar field ϕ satisfying the Laplace differential equation, the equal time commutators $[\phi(x), \phi(y)]$ and $[\partial_t\phi(x), \partial_t\phi(y)]$ both are zero. The remaining equal time commutator takes the form

$$[\phi(x), \partial_t\phi(y)] = i\delta(x - y) \ .$$

Note that distribution theory was not yet developed. This caused no physical problems at all, but meant that the use of Dirac's δ had no firm mathematical base yet. One may wonder, if this encouraged the physics community to ignore mathematical niceties. Of course, the mathematical justification was provided in the late 40's by L. Schwartz, in a splendid case of interaction between the two communities. When this was done, one had a good description for free quantum fields as distributions over three-dimensional space. Once a fixed field ϕ is paired with a test function f (physicists write $\int \phi(x)f(x)d^3x$), the result is an element of the Heisenberg Lie algebra and acts on the Hilbert space of the system. For real f the operator is hermitean and describes an observable, as usual in quantum mechanics.

The analogy between particles and free fields given by passing from the Kronecker function δ_{ij} to Dirac's $\delta(x-y)$ was compelling, but proved to be very misleading. Heisenberg's commutation relations for x_i, p_i remain valid when interactions are present. In contrast, Haag showed that an interacting field theory cannot have canonical commutation relations. Indeed, interacting fields cannot even be understood as distributions over three-dimensional space at fixed time. Time averaging is necessary, too [Haag 1955].

In a special relativistic context, this might not have come as a big surprise. There even was a paper by Bohr and Rosenfeld which argued that a careful analysis of measurements implies a spacetime average [1933]. The arguments were clear enough and the paper was never forgotten, but its somewhat obscure style missed its mark on most of the new generation.

Instead, adherence to the canonical commutation relations for quantum fields remained pervasive in the physics literature till recent times, inspite of the fact that everyone knew it was wrong. Most probably, it was much more this attitude than the difficulties of renormalization which made it impossible for mathematicians to digest the intricate and important structures of quantum field theory.

Historians will have to weigh this issue when the dust has settled. Despite of what has been said, they hardly can find a better starting point than the following classic quotation: "In the thirties, under the demoralizing influence of quantum-theoretic perturbation theory, the mathematics required of a theoretical physicist was reduced to a rudimentary knowledge of the Latin and Greek alphabets." (Jost) [Streater and Wightman 1964, p. 31].

The insistence of the physics community on using a wrong basis for successful calculations would be easy to understand, if no alternative formalism had been available. Due to Schwinger and Dyson, this was not the case. Dyson had read much mathematics and brought clarity of thinking to the muddled field. By 1949, Schwinger and Dyson had started to analyse quantum fields in terms of the n-point functions (or rather distributions) $T\langle\phi(x_1,t_1)\ldots\phi(x_n,t_n)\rangle$. Here for an operator A the real or complex number $\langle A\rangle$ is its expectation value in the vacuum state of the Hilbert space, and the analogous notation applies to distributions. The time ordering imposes the condition $t_1 > \ldots > t_n$ on the support of the test functions. Moreover, in 1951 Schwinger published his action principle, which describes how an n-point function varies when one changes the parameters of the interaction.

Thus most of the theoretical tools were ready. On reading the tributes to

Schwinger published after his death [Ng 1996], it seems that some obstacles to progress were personal. Schwinger had been a prodigy and the centre of attention. Apparently, he didn't mind that his calculations remained almost incomprehensible. All that changed after 1948. In Schwinger's own words: "Like the silicon chip of more recent years, the Feynman diagram was bringing computation to the masses" [Schwinger 1983, p. 343]. Dyson had a particularly clear understanding of the issues: "The advantages of the Feynman theory are simplicity and ease of application, while those of Tomonaga-Schwinger are generality and theoretical completeness" [Dyson 1949, p. 486]. Schwinger forbade his students to mention Feynman or Dyson, or to use Feynman graphs. From a European perspective it seems that Einstein and Weyl would have had more reasons for grudges against Hilbert and Schrödinger, but one has to respect a difference of culture.

In 1953, the Wightman axioms [Streater, Wightman 1964] were presented in lectures at Princeton. They were something of a mixed blessing. On one hand, they allowed clear proofs of structural statements, in particular of Haag's insight that the canonical commutation relations are wrong for interacting theories [Haag 1955]. On the other hand, the axioms sacrificed the connection to the concrete quantum field theories which were under development.

One technical detail needs comment. The Wightman axioms concern n-point distributions $\langle \phi(x_1, t_1) \ldots \phi(x_n, t_n) \rangle$, but without time ordering. This seems mathematically convenient, for example when one wants to take Fourier transforms. Nevertheless, for contact with the experiments, the time ordering is natural. This became particularly clear with the LSZ formalism of Lehmann, Symanzik and Zimmermann, which provided a direct calculation of the results of scattering experiments in terms of the time ordered distributions. Different time orderings correspond to different experiments.

The three authors were members of Heisenberg's group, which attracted most of the young people who wanted to work on elementary particles in postwar Germany. Unfortunately, Heisenberg was hardly interested in mathematics and too occupied by his world formula to have much regard for the LSZ achievements. When Lehmann returned from the States, Heisenberg greeted him: "Na, Herr Lehmann, wie geht's der Mathematik?" (how is mathematics?), an episode which Lehmann never forgot. So much for the superiority of European culture.

As an aside, any Third World country which wants to strengthen her scientific basis would be well advised to do a few case studies. The decline of physics in Germany is particularly interesting. One cannot put all of the blame on fascism, since mathematics did not suffer the same fate after the war, largely due to the achievements of Hirzebruch.

The n-point distributions made mathematical sense, but were difficult to deal with. The next big advance was the introduction of the euclidean formalism, as discussed in [Osterwalder 1973]. Early on, Dyson had recognized that some calculations become much easier when one performs an analytic continuation to imaginary values of time (Wick rotation). The gestation of the idea took most of the 1950's, with contributions from Wick, Nakano and, in condensed matter physics, Matsubara [1955]. It first appears in complete form in papers of Schwinger.

In his 1993 lecture in Nottingham [Ng 1996], Schwinger states that it could have been published any time after 1951, but in fact "The Euclidean Structure of Relativistic Field Theory" appeared in 1958. Schwinger made an analytic continuation of the time-ordered n-point distributions to purely imaginary values of time. As Wightman had seen already, the analytic continuation allows to consider the distributions as boundary values of ordinary analytic functions. Thus Schwinger's idea allows to describe physics by functions of some D-dimensional euclidean space instead of distributions with testfunctions over D-dimensional spacetime. By that time, mathematical physicists had mastered the difficulties of distribution theory, such that the due expression of relief was rather muted. Often, the euclidean n-point functions are regarded as distributions, too, but the present article will not follow this habit.

As usual nowadays, Schwinger's euclidean n-point functions will just be written in the form $\langle\phi(x_1)\dots\phi(x_n)\rangle$, where the x_i now denote points in D-dimensional euclidean space. These functions are real analytic and defined everywhere except on the partial diagonals $x_i = x_j$. Since there is no causal structure in euclidean space, the necessity of time ordering disappears. Accordingly, the functions are symmetric under permutation of the x_i. If one considers several fields $\phi_1, \phi_2, \dots$, one has instead

$$\langle A_1\phi_i(x_i)\phi_{i+1}(x_{i+1})A_2\rangle = \langle A_1\phi_{i+1}(x_{i+1})\phi_i(x_i)A_2\rangle \ ,$$

where the A_k stand for products of fields at points different from x_i, x_{i+1}. In spacetime, all possible time orderings can be reached by analytic continuations starting from the same euclidean n-point function, a fact called crossing symmetry.

Since the choice of quantum field theories is quite limited, their n-point functions should be special functions with very interesting properties. Not much is known about them, however. For free theories, they vanish unless n is even, in which case they reduce to sums of products over 2-point functions. The latter are variants of Bessel functions. For conformal field theories, one obtains functions of hypergeometric type. In some other cases in two dimensions, at least the 2-point functions are under good numerical control, but little is known about their analytic properties. It is quite possible that some examples will yield functions of Painlevé type. Unfortunately, interest in special functions was at a low ebb in the past century, but this certainly will change again.

Most quantum field theories have free parameters. The latter take values in some differentiable manifold which is called moduli space. Accordingly, the n-point functions can be differentiated with respect to these parameters. Let ∂_λ be a tangent vector in moduli space. According to Schwinger's action principle, each tangent vector corresponds to some field $t(x)$, such that formally

$$\partial_\lambda \langle \phi(x_1) \ldots \phi(x_n) \rangle = \int \langle t(x) \phi(x_1) \ldots \phi(x_n) \rangle d^D x \ .$$

The expression is formal, since the integral diverges when x approaches one of the x_i and needs to be regularized.

In general, there is no easy way to normalize the field ϕ. Of course, the canonical commutation relations would have provided a natural normalization, but they are wrong. When one changes the normalization by some factor $f(\lambda)$, the derivative of the n-point function changes by a term proportional to $n\langle \phi(x_1) \ldots \phi(x_n) \rangle$. If the divergence of Schwinger's integral is of exactly this type, the freedom of normalization can be used to cancel it. This is the renormalization procedure, which will be discussed in more generality below.

In principle, vector fields can be integrated, such that Schwinger's action principle should allow to recover the moduli space from any of its regular points by higher order derivatives and the summation of the Taylor expansion. In many practical cases, however, the only explicitly known points of the moduli space lie at the boundary, where the space is no longer regular. As a consequence, the Taylor expansion is only asymptotic. This problem can be avoided for conformal field theories, but it will be mentioned again in the context of string theory.

Many moduli spaces do not have a natural metric, such that the integration of a vector field has to follow an arbitrary smooth curve. Equivalently, one can choose local coordinates, also known as renormalization scheme. Indeed, without a metric on moduli space, the perturbing field $t(x)$ does not have a natural normalization. Typically it lives in some finite dimensional vector bundle over moduli space which includes mass perturbations and coupling constant perturbations. When one takes higher order derivatives of the n-point functions, all of these parameters have to be considered together, which requires mass and coupling constant renormalizations of $t(x)$. The finite ambiguities of the latter are fixed by the choice of a renormalization scheme. Changing them leads to a different curve for the integration.

If one wants, one can include the constant field 1 in the vector bundle, but since one wants $\langle 1 \rangle = 1$ it is usually more convenient to require $\langle t(x) \rangle = 0$. This is called the renormalization of the vacuum energy density.

In the 50's, renormalization was well understood on a computational level, but before Wilson's work in the late 60's the concepts were not particularly clear. Nevertheless, the time was ripe for the first quantum field theory which was not free and made complete mathematical sense.

The Thirring model: Conformally invariant quantum field theory is born

In 1958, W. Thirring published a paper with the title 'A Soluble Relativistic Field Theory' (in Mathematical Reviews, it was described by Raychaudhury, Calcutta). The paper kept the promise of its title. Let me quote a few sentences: 'In spite of the great efforts of many people the mathematical structure of relativistic quantum fields is still in the dark. ... In order to study those (features) we propose in the present paper a model of a relativistic field theory... Since the reduction of the number of fields does not simplify the problem sufficiently ... one has to take recourse to a reduction of the dimensionality of the problem... Thus the simplest nontrivial case seems to be a one-dimensional Fermi-field with an interaction $\lambda\bar{\psi}\psi\bar{\psi}\psi$. Although the problem is of considerable complexity it turns out to be soluble. ... (The model) shows explicitly what a relativistic theory can look like. Furthermore it can serve as a testing ground for field theorists."

All of this is true. Perhaps the most remarkable part is the courage to do something simple in two dimensions. Here Thirring was inspired by the investigation of many-body systems in terms of the Bethe ansatz. In two dimensions, one can get interactions by collisions only, without fields. This knowledge led to the correct conjecture that the model would be solvable. Thirring also made some entirely correct remarks about Heisenberg's unified four-fermion interaction theory in four-dimensional spacetime, which may have contributed to some tension between Munich and Vienna. Indeed, despite of the fact that part of Thirring's work had been done at MIT and at the IAS, Princeton, one almost gets the impression that the creation of the model was a provincial non-event. The leading soluble model of the time was due to Lee (1954) and not relativistic. Thirring's remark about the Lee model in his 1958 paper is not particularly deferential, but in his textbook with Henley [1962] he gives it two chapters, whereas his own model does not even seem to be hinted at. Schweber's 1964 textbook doesn't cite it either.

Nevertheless, some of Schwinger's former students had paid attention, and Johnson from MIT devoted a paper to the model [1961]. I quote from the introduction: "Thirring has proposed a two dimensional ... model which is of some interest because its exact solubility enables one to study some of the general conjectures which have been proposed in regard to the behaviour of local relativistic fields. In spite of the model, no general solutions have been proposed which are free from possible criticism because of the rather formal manner in which they have been obtained." In the conclusion, Johnson states: "We have shown how it is possible to solve the two dimensional model of Thirring by making use of the existence of the two vector density conservation laws. ... It was shown how it is possible to define the products of the singular operators $\psi(x)$, in order to determine other covariant operators but that these singular field products do not satisfy the equal time commutation relations with the field operators $\psi(x)$, that one would obtain by means of the canonical commutation relations ...". Again, all of this is correct. Still, some mathematical problems were left, but they were settled in the subsequent years.

Let us describe the model in more detail. It is obtained by perturbing the theory of a massless complex fermion in two dimensions. In the euclidean formulation, the Dirac equation reduces to the Cauchy-Riemann equation and its complex conjugate. Real and imaginary parts of the fermion yield two holomorphic field $\psi_i(z)$ and two anti-holomorphic fields $\bar{\psi}_i(\bar{z})$, $i = 1, 2$. At this point in moduli space, the two conserved vector densities mentioned by Johnson are $j(z) = \psi_1(z)\psi_2(z)$ and $\bar{j}(\bar{z}) = \bar{\psi}_1(\bar{z})\bar{\psi}_2(\bar{z})$. The conservation equations are the Cauchy-Riemann equation for j and its conjugate for $\bar{j}$. The 2-point functions have the form

$$\langle j(z_1)j(z_2)\rangle = (z_1 - z_2)^{-2}$$

and analogously for $\langle \bar{j}\bar{j}\rangle$, whereas $\langle j\bar{j}\rangle = 0$.

In terms of Schwinger's action principle, the perturbation corresponds to the field $t = j\bar{j}$. It turns out that the n-point functions of j and $\bar{j}$ are unaffected by the perturbation. In particular, the two fields and their product t have a natural continuation over Thirring's moduli space and need no renormalization. Moreover, the conservation equations do not change, which accounts for the solvability of the model.

The special properties of $j, \bar{j}$ arise because they are currents, i.e. quantum analogues of the conserved densities which arise by Noether's theorem from continuous symmetries. Because of their close relation to observable quantities, they behave similarly to free fields. This led to the concept of current algebra. In two dimensional theories, the currents of simple Lie groups generate the corresponding affine Kac-Moody algebra, at least when space is compactified to a circle. Unfortunately, the mathematical potential of current algebras was not realized for many years. The work of Kac and Moody in 1967 was independent of physics. In the context of string theory, it was introduced in the physics literature by the mathematicians Lepowsky and Wilson [1978] and again by G. Segal [1981], and became a rare success story of physics and mathematics in cooperation.

The Thirring model fields ψ_i do not remain holomorphic under the perturbation by $j\bar{j}$. Instead, one obtains

$$\langle \psi_i(z_1)\psi_j(z_2)\rangle = (z_1 - z_2)^{-1}|z_1 - z_2|^{-s}\delta_{ij} \ ,$$

where the real number s changes under perturbation. Under the conformal transformation $z \mapsto z' = (az + b)/(cz + d)$ with $ad - bc = 1$, $\psi(z) \mapsto (cz + d)^{-1}|cz + d|^{-s}\psi(z')$, the two-point functions remain invariant. This remains true for all the n-point functions, such that the Thirring model is a conformally invariant theory. Initially, this seems to have been overlooked, and only the special case of invariance under scale transformations $z \mapsto \lambda z$ was commented upon. This is a bit surprising, since in these years Thirring was very much concerned with conformal invariance. In the important 1962 paper where Gell-Mann introduced current algebra to the theory of the strong interactions, he acknowledges that Thirring introduced him to the conformal group. Moreover, conformal invariance had become an issue between Munich and Vienna. There was little internal logic in this local turbulence, but it turned out to be important and may be of interest to historically inclined people.

Heisenberg had developed an interacting spinor theory for all of particle physics and pushed it for many years, though it made no sense. At the time, the new quantum number of strangeness demanded an explanation. Due to Noether, an invariance of the theory had to be found. Heisenberg tried scale invariance, though the theory has a length scale and a non-compact group has a hard time to yield discrete quantum numbers. The contemporary fashion for negative norm states, also present in the Lee model, gave some hope for a cure [Dürr 1959].

In Vienna, Cunningham and Bateman were remembered and Wess used Heisenberg's attempts as justification for the resurrection of the conformal group. In a brief remark, he hinted at a possible use of the conformal group at high energies. Otherwise, he showed in a few pages that Heisenberg had missed the mark [Wess 1960]. Since several of the few good young German theoreticians had flocked around Heisenberg, the paper triggered new interest in the conformal group, and Kastrup started to work on it, though Heisenberg did not pay much attention. Kastrup published papers on the possible importance of conformal invariance at high energies. During a visit to Russia, he explained it to Polyakov, as acknowledged in the first paper of the latter on conformal symmetry [1970]. This paper showed that scale invariance implies full conformal invariance. On the other side of the Atlantic, in his historic paper on the short distance expansion, Wilson ascribes the idea of scale invariance at short distance to Kastrup and his student Mack [Wilson 1969]. The fact that scale invariance implies full conformal invariance was recognized by Callan, Coleman and Jackiw, slightly before Polyakov's work and in a different context [1969]. On the physical relevance of scale and conformal invariance, they cite 1969 papers by Mack and Salam and by Gross and Wess.

Wilson's short distance expansion was the main concept which still was lacking for a rigorous and calculationally efficient description of quantum field theory. It concerns the behaviour of the n-point functions along there singularities. Wilson considered them in Minkowskian spacetime, but the euclidean case is much easier.

It has been mentioned that the euclidean n-point functions $\langle\phi(x_1)\ldots\phi(x_n)\rangle$ are not well defined on the partial diagonal $x_i = x_j$. In general, the functions diverge on these diagonals. For a free field ϕ of dimension h, the leading term at $x_1 = x_2$ is proportional to $|x_1 - x_2|^{-2h}\langle\phi(x_3)\ldots\phi(x_n)\rangle$. The case of several different fields needs a bit more discussion, but is not complicated either.

There had been some speculation on the corresponding behaviour for interacting fields. One idea was that the singularity might be the same as for free fields. In 1964 Wilson conjectured that perturbations just introduce some logarithmic corrections. This was wrong, but one of Wilson's talents was to talk to the right people for correcting mistakes. In particular, he had crucial discussions with Johnson, who familiarized him with the Thirring model. Wilson learned that the latter indeed is scale invariant, but that the dimension h changes with the strength of the interaction. Independently, the same modification to Wilson's original ideas was made by Lowenstein.

Wilson was a mainstream theorist on the way to a Nobel prize, but he did not fear to go against the tide: "The assumption that integrating an operator over space only gives an observable is a basic tenet of canonical field theory... The assumption has been rejected by axiomatic field theory from the beginning" [Wilson 1970, p. 1484]. In the same paper, he discusses a related issue and concludes: "The axiomatic view must in the end replace the popular view" [p. 1483]. It seems that the time was ripe to discuss all of quantum field theory in terms of statements which are at least potentially true.

Before we discuss other aspects of Wilson's work, let us continue the history of the Thirring model. At the end of the 60's, string theory was invented and soon it was recognized that conformal field theory is an essential ingredient [Galli 1970]. Halpern recognized the importance of the Thirring model in this context and informed Virasoro, who gave it publicity [1971]. A comparative investigation of the Thirring model and string physics in the context of conformal field theory was made by Ferrara, Grillo and Gatto [1972].

By 1974, it had become popular to elucidate the properties of quantum field theory by a study of two-dimensional examples. A particularly interesting one was the sine-Gordon model, which describes a bosonic scalar field with trigonometric interaction term. Coleman wrote an elegant and deep paper where he showed that the perturbation by a fermion mass term makes the Thirring model isomorphic to the sine-Gordon model [1975]. This took everyone by surprise, since superficially the two models look entirely different and equally impenetrable in a strict mathematical sense.

On hindsight, people remembered that the equivalence between fermions and bosons in two dimensions had been prefigured by Skyrme [1958,1961], but Skyrme had been too far ahead to have an immediate impact.

After Coleman's paper, at last, one leading mathematician was shocked enough to take things seriously. G. Segal regarded the mass term as an unessential complication and concentrated on the boson-fermion equivalence. This was Coleman's starting point and concerns an isomorphism between two conformally invariant theories. Initially, Segal felt quite sure that boson-fermion equivalence made no sense. When it turned out in the late 70's that the equivalence leads to a combinatorial identity known already to Euler, a dam had been broken. Segal developed a beautiful system of axioms for conformally invariant quantum field theories in two dimensions and transformed the latter into a legitimate field of study for mathematicians [Segal 1988]. But even in their book on loop groups [Pressley, Segal 1986, p. 215] the authors state that a mathematically clear formulation of the isomorphism between the massive Thirring model and the sine-Gordon model still seems not to have been found.

Nature's helping hand

The long delay in the gestation of a correct theory of quantum fields would have been even longer without some direct help from nature. One reason is that the investigation of two-dimensional toy models was not taken very seriously by the particle physicists. Here is a quotation from a paper which reports the discovery of a fundamental property of the Thirring model: "The results are of interest ... because they allow one to see very readily (a) why the Thirring model is solvable and (b) why it has trivial physical consequences. As will be clear from the following, the solvability of this model depends critically on the fact that it is a 2-dimensional model. It is not likely that any of the specific features of this model can be generalized to more realistic cases, or that they will provide a useful guide to the state of affairs in the real world" [Callan, Dashen and Sharp 1968, p. 1883].

Indeed, the highly non-trivial physical consequences of such conformal field theories in the context of string theory could not have been guessed in

1967. No wonder that the authors permitted themselves some sloppiness in the analysis: "At this point, one could introduce the Fock representation for the scalar field, annihilation and creation operators, etc., and verify in detail that the energy and momentum operators have the expected properties, but there is little to be gained by going over these well-known details" [p. 1885]. This was a missed opportunity. For example, the commutation relations for the energy-momentum tensor given in the paper miss the central extension of what is now called the Virasoro algebra. What would have happened, if some interested mathematics student had tried to digest the paper?

Since it seems that no mathematicians were interested, it was very kind of nature to provide her own motivation for the study of such models. In the 50's, physicists were confronted unexpectedly with a rich class of quantum field theory in condensed matter laboratories, which turned out to be conformal field theories in the real world of two-dimensional surface coatings or three dimensional liquids.

After Feynman's breakthrough in 1948, his graph methods soon were transferred to other fields of physics. Their application in condensed matter physics was pioneered by Salam [1953] and Matsubara [1955]. In particular, Matsubara recognized the perfect analogy of imaginary time and temperature, due to the relation between the time translation $\exp(iHt)$ in quantum mechanics and the Boltzmann factor $\exp(-H/T)$ in statistical mechanics.

When continuous phase transitions were studied, it turned out that the analogies went much deeper. At the critical temperatures, the behaviour of the materials is dominated by long range fluctuations of arbitrary scales, and the details of the molecular structure become unimportant. The theory approaches a continuum limit. The correlation functions of the limiting theory behave exactly like the euclidean n-point functions of quantum field theory. In this way, many statistical systems at continuous phase transitions are related to quantum field theories in spacetime by analytic continuation.

Thus nature herself had declared that the Wick rotation introduced by Schwinger makes good sense. Of course, the dimensions of the observed examples are different, since the phase transitions happen in two or three dimensional systems, whereas spacetime has four dimensions. Moreover, the natural constraints on the field theories are not the same. Quantum field theories need a probability interpretation, which is realized by positive scalar products. Under Wick rotation, this becomes Osterwalder-Schrader positivity, which is not a necessary property of phase transitions. On the other hand, statistical observables are given by real numbers. This real structure yields a time reversal invariance of the corresponding quantum field theory, a property not shared by all examples and only approximately true in nature. On a purely mathematical level, these difficulties are not particularly serious, however.

Lab experiments on phase transitions were much cheaper than particle physics with high energy accelerators. Moreover, there were no worries that a breakthrough in the domain of the fundamental laws was necessary. Thus progress was rather steady, both on the experimental and the theoretical side. Soon it became clear that the physics at the critical phase transition point is scale invariant [Kadanoff 1966]. Much of the relevant work on these euclidean quantum field theories was done in the Soviet Union, and Polyakov was one of the most important contributors. He found convincing arguments that scale invariance implies full conformal invariance at the critical point and recognized that this invariance allowed a calculation of the 3-point functions up to a constant factor [Polyakov 1970].

Further developments depended on the analysis of a soluble example in the context of statistical mechanics. This might have been provided by the Thirring model, which had occurred in its bosonic description, and was called the gaussian model. Because of the somewhat misleading simplicity of the bosonic formulation, the subtle features of its fermionic fields were not recognized in this context. Instead, the Ising model played a rôle for the study of continuous phase transitions which was parallel to the one of the Thirring model for particle physicists.

The states of the Ising model put a number 1 or -1 to each site of a rectangular lattice. The latter are called values of the Ising spin. Pairs of nearest neighbours have an interaction energy which depends on the product of their Ising spins. The total energy E is given by a sum over the interaction energies of such pairs. The thermodynamic partition functions at temperature T is given by the average of $\exp(-E/T)$ over all states.

Rectangular lattices can be considered in various dimensions. The thermodynamic functions for the linear or one dimensional model are very easy to calculate. The problem was given by Lenz as part of a PhD thesis to a rather weak student, who did not do any later scientific work. One hardly can imagine an easier way to lasting fame. The two-dimensional model, where the Ising spins sit on a square lattice, was solvable but very hard. The breakthrough calculation was due to Onsager [1944]. There is a critical temperature where the model turns into a rather simple euclidean quantum field theory. In particular, at this point the spin waves of the model satisfy the two-dimensional Dirac equation for free massless fermions, as first noted by Kadanoff [1969]. This equation is conformally invariant, as in the more complicated four-dimensional situation. In contrast to the complex fermion of the Thirring model, the fermion field of the Ising model is real. In this sense, the Ising model at its critical temperature has half as many degrees of freedom as the Thirring model.

The two-dimensional Ising model is not just a theory of free fermions, however. The average values of the Ising spins turn into a field with scaling dimension 1/8. This result proved to be a highly non-trivial check which uncovered the failures of many calculational methods.

Now two different conformally invariant quantum field theories were available, the Ising model in statistical mechanics and the Thirring model in conventional relativistic quantum field theory. They were used for very much the same theoretical tools, in particular the short distance expansion. Wilson discovered it in 1964 in in the Minkowskian context, Polyakov and Kadanoff in 1969 in the euclidean. Polyakov called it correlation coalescence, Kadanoff reduction hypothesis. Wilson called it operator product expansion, and this terminology has survived, because it clearly has the priority. In the context of statistical mechanics it is not appropriate, however, since there are no operators around. Since it has advantages to have a unique name in both contexts, we use the common synonym short distance expansion.

A scale invariant n-point function of type $\langle\phi(x)\phi(y)A\rangle$ has a leading singularity at $x = y$ proportional to $|x-y|^{-2h}$, where h is the scaling dimension of ϕ. When this leading singularity is subtracted, the next term behaves like $|x-y|^{-2h+h_1}\langle\chi_1(y)A\rangle$. Here χ_1 is some other field of scaling dimension $h_1 > 0$, which can be measured in the way just described. Subtracting this subleading term one finds $|x-y|^{-2h+h_2}\langle\chi_2(y)A\rangle$, where χ_2 now has a larger scaling dimension $h_2 > h_1$. The procedure can be repeated as far as one wants to go. One will find an infinity of fields of ever higher scaling dimension. Note that the χ_i are independent of the fields included in A.

One can apply the same procedure to other n-point functions like $\langle\phi(x_1)$ $\chi_1(x_2)\ldots\rangle$ and so on and produce as many new fields as possible. The short distance expansion now states that for any real number h_0 there is only a finite number of linearly independent fields of scaling dimensions $\leq h_0$. This property can be verified in many concrete examples and may very well be taken as part of the mathematical definition of a quantum field theory.

Lattice systems are scale invariant at the exact temperature of a continuous phase transition. When the temperature is changed a bit, the correlations will show an exponential decay at large distances. When one is sufficiently close to the critical temperature, the corresponding correlation length is still very large compared to the distance between neighbours. With a suitable limiting procedure, one obtains the n-point functions of a quantum field theory which is no longer conformally invariant. In this case, more complicated expressions than $|x-y|^{-2h}$ will occur in the n-point functions. At the very least, one expects logarithmic correction factors. Nevertheless, the basic idea of the short distance expansion applies as before.

Let us consider a euclidean n-point function $\langle\phi(x)\chi(y)A\rangle$, where A is a product of local fields at positions different from x, y. An experimentalist may study the behaviour of this function when x approaches y. Each such measurement can be interpreted as the measurement of some field at y. This is the physical content of the short distance expansion. We can axiomatize it in the following way. Let $\Gamma(y)$ be the vector space of germs of functions which are defined near y, but not at the point y itself. We give a topology to this space by using $o(|x-y|^s)$, $s \in \mathbf{R}$ as a basis of neighborhoods of 0 in $\Gamma(y)$. Let γ be an element of the dual of $\Gamma(y)$. Then for each pair of fields ϕ, χ and each h there must be a field ψ such that $\gamma\langle\phi(x)\chi(y)A\rangle = \langle\psi(y)A\rangle$ for arbitrary A. One just can write $\gamma(\phi(x), \chi(y)) = \psi(y)$.

Consider the vector space F of all fields of a quantum field theory. This vector space is filtered by the scaling dimension. Let $F(h)$ be the subspace of all fields of scaling dimension less or equal to h. We assume that these subspaces are finite dimensional. We also assume that the theory has some degree of asymptotic scale invariance. More precisely, $\psi \in F(h_1+h_2+h_0)$ when $\phi \in F(h_1)$, $\chi \in F(h_2)$ and γ vanishes on $o(|x-y|^h)$ for $h > h_0$. This condition will be important for renormalizability. Finally, $\dim F(h)$ should not increase faster than for free theories. In two dimensions, this yields $\log(\dim F(h)) = O(\sqrt{h})$.

In this way one obtains a nice algebraic structure which is well adapted to calculational purposes. It does not contradict the Wightman axioms, but emphasizes quite different aspects. Whereas those axioms concentrate on one field, or maybe a few, the short distance expansion considers all possible fields at once. For mathematicians, this is certainly the more natural procedure. To some extent, it eliminates the surprise one first feels about the equivalence of the sine-Gordon and the massive Thirring model, since in the latter one immediately has to include its bosonic fields, too.

Regularization and renormalization

With the help of the short distance expansion, it is rather easy to put renormalization in a standard mathematical frame. First we have to generalize the change of normalization of the fields which we considered above. Instead, we will use all the linear transformations of F which conserve the subspaces $F(h)$. The group of these linear transformations will be called $L(F)$.

We want to regard a perturbation of some theory. In accordance with Schwinger's action principle, the deformation is described by a field $t(x)$. We shall see that in a spacetime of D dimensions, the scaling dimension of t must be D or less.

The corresponding derivative of an n-point function $\langle \phi_1(x_1) \dots \phi_n(x_n) \rangle$ is given by $\int d^k x \langle t(x) \phi_1(x_1) \dots \phi_n(x_n) \rangle$. The integral behaves well at infinity, but diverges when x approaches one of the x_i. Thus we regularize it by excluding a small neighborhood of size ϵ around each x_i from the integration domain. Let us denote the resulting integral by $\int_\epsilon$.

The idea of renormalization means that the divergence can be absorbed by a redefinition of the fields. Such a redefinition is given by a linear transformation in $L(F)$ of the fields which maps every subspace $F(h)$ into itself. Using Wilson's short distance expansion, one sees easily that there are transformations $f(\epsilon) \in L(F)$ such that

$$\int_\epsilon d^D x \langle t(x) \phi_1(x_1) \dots \phi_n(x_n) \rangle - \sum_{i=0}^{n} \langle \phi_1(x_1) \dots (f(\epsilon)\phi_i)(x_i) \dots \rangle$$

has a well defined limit when ϵ goes to zero. Indeed, any divergent contribution γ to the integral near x_i vanishes when the n-point function behaves as $o(|x - x_i|^{-D})$, such that $\gamma(t(x)\phi) \in F(h_i)$, when h_i is the scaling dimension of ϕ_i.

The transformation $f(\epsilon)$ is only defined up to addition of a finite linear transformation in $L(F)$. Any choice defines a connection on the filtered vector bundle F over the moduli space. Altogether, we now have well defined first derivatives in the moduli space of a quantum field theory. The calculation gets harder when one looks at higher derivatives, since the perturbing field $t(x)$ will have to be renormalized, too, but this is just a technical difficulty.

As one sees, renormalization is nothing particularly problematic. On the contrary, regularization of divergencies has a long history in mathematics. For example, the Weierstrass product formula for entire function needs the regularization of an infinite product. Let us consider it in more detail. One wants a product formula for an entire holomorphic function $P(z)$ with zeros exactly at given positions z_i, $i = 1, 2, \ldots$, more precisely a function with $\sum(z_i)$ as zero divisor. The sequence z_i must have no accumulation point in the Gauss plane. When the number of zero positions is finite, the product $\prod(z - z_i)$ will do. The most general function with this divisor is $\exp(f(z)) \prod(z - z_i)$, where $f(z)$ is an arbitrary entire function.

Now let us consider the case of an infinite number of positions. Factoring out a power of z if necessary, we may assume that none of the z_i is zero. Let us formulate Weierstrass' solution in terms of the language of quantum field theory. We regularize the problem by restricting the set of zeros to z_i, $i = 1, \ldots, N$. Then we order the z_i in accordance with their absolute value and renormalize the function $\prod_{i=1}^{N}(z - z_i)$ in the form

$$P_N(z) = \exp(f_N(z)) \prod_{i=1}^{N}(z - z_i) \ ,$$

such that the limit $\lim_{N\to\infty} P_N$ is finite.

The situation in quantum field theory is quite analogous. The cut-off by ϵ is analogous to the cut-off by N, the achievement of convergence by the renormalization transformation $f(\epsilon)$ is analogous to the multiplication by $\exp(f_N)$. In renormalizable quantum field theories, fixing a finite number of parameters is sufficient to determine the n-point functions of a given finite set of fields. In the case of the Weierstrass products, this is analogous to the situation where it is sufficient to take for the f_N polynomials of fixed order r. In this case, one can normalize P by demanding that $P(0)$ and the first r derivatives of P at $z = 0$ have prescribed values. This means that the solution P has $r + 1$ free parameters. For $r = 0$, the solution is

$$P(z) = P(0) \lim_{N\to\infty} \prod_{i=1}^{N}(1 - z/z_i) \ .$$

For $r = 1$ one obtains

$$P(z) = P(0) \exp(zP'(0)/P(0)) \lim_{N\to\infty} \prod_{i=1}^{N}(1 - z/z_i) \exp(z/z_i)$$

and so on.

When polynomials do not suffice, the number of free parameters becomes infinite. Quantum field theory is simpler, since the latter case does not seem to have an analogue. Moreover, quantum field theories are far more constrained than entire functions, since they only have a finite number of parameters, in contrast to the infinite set of the z_i.

For conformal field theories, the Weierstrass product formula is more than a far-fetched analogue, since many correlation functions involve Jacobi's theta-functions or Dedekind's η-function. Examples will be given below. Many important properties of these functions are best understood by their product formulas.

As one sees, regularization and renormalization are perfectly standard mathematical procedures. Their unfamiliar context was bound to cause some delay in understanding, but it is hard to comprehend how a delay of many decades could come about.

The structure of conformally invariant theories

One important way to deform a quantum field theory has not been introduced so far. One can change all n-point functions by a simple rescaling of the distances. When this change can be compensated by a transformation in $L(F)$, the theory is called scale invariant. More generally, the change is equivalent to such a transformation in addition to a change of the parameters of the theory. Infinitesimally, this equivalence is expressed by the Callan-Symanzik equation.

When a deformation should respect some symmetry, the corresponding field $t(x)$ must be invariant under the symmetry group. In particular, this is true for Lorentz invariance. Indeed, our formalism does not require Lorentz invariance and can easily be adapted to quantum field theories on general spacetimes. One just has to replace the vector space F of fields by a bundle over spacetime. Let us conserve translational invariance, however, such that fields can be transported in canonical ways between arbitrary points of spacetime. When some component $g_{\mu\nu}$ of the Riemannian metric is changed in a translationally invariant way, the corresponding field t is the component $T^{\mu\nu}$ of the energy momentum tensor. For a rescaling of the distances, this yields $t = T^{\mu}_{\mu}$. For a scale invariant theory this means that the trace of the energy momentum tensor vanishes. Moreover, the integral $\int T^{\mu\nu} d^D x$ must not depend on the distance scale, which means that the scaling dimension of the energy momentum tensor is equal to D.

Scale invariant quantum field theories are conformally invariant, too. This implies that the three point functions are known explicitly. The four-point functions reduce to functions of a single variable. Such theories have a good chance to be solvable in a rather explicit form, but for theories in more than two dimensions, the situation is still rather unclear. Nevertheless, recent developments indicate that these theories are important, too [Maldacena 1998, Witten 1998]. Suppose that you have a quantum field theory in k dimensional Minkowski space which admits a deformation to the corresponding Anti-de-Sitter space. Recall that this is a homogeneous space of negative spatial curvature, with symmetry group $SO(k-1,2)$. Anti-de-Sitter space has a $(k-1)$-dimensional boundary at infinity with a conformal structure, on which $SO(k-1,2)$ acts as the group of conformal transformations. When one takes suitable limits of the n-point functions, the theory in Anti-de-Sitter space reduces to a conformally invariant theory in a space of one lower dimension. In principle, the higher dimensional theory can be recovered from the boundary theory by techniques of algebraic quantum field theory [Rehren 1999].

Perhaps this procedure can be iterated. In this way, the properties of theories in higher dimension would be encoded in conformal field theories in two dimensions. This possibility is due to the typically quantum field theoretical fact that there is more freedom to construct conformal theories than higher dimensional quantum field theories in homogeneous spaces. In other words, the moduli spaces in higher spacetime dimensions have lower dimensions as manifolds, and can be embedded in the moduli spaces of quantum field theories in lower spacetime dimensions. As we shall see, string theory also performs such an encoding. It would be interesting to see if the two encodings are related.

In the following, we only will consider conformal field theories in two dimensions. The amount of technical details will just about suffice to put string theory in context. For a history of the crucial years 1984-88 and the relations to statistical mechanics, see [Itzykson, Saleur, Zuber 1988], which contains many references. A recent textbook is [di Francesco, Mathieu, Senechal 1997].

When one starts with a Minkowskian conformal field theory in flat spacetime, Wick rotation yields a euclidean theory on the Gauss plain. By conformal invariance, it is possible to compactify it to a theory on the Riemann sphere. As symmetry group, one obtains the group of linear rational transformations $z \mapsto (az+b)/(cz+d)$ of the Riemann sphere. This will be the symmetry group of the n-point functions.

In two dimensions, the energy momentum tensor is a symmetric 2×2 matrix. Because of scale invariance, its trace vanishes, such that it has only two independent components. By the Noether theorem, they are conserved quantities. More precisely, one linear combination is holomorphic, another anti-holomorphic. These are the famous Virasoro fields, which were first discovered in string theory [Virasoro 1970]. Their short-distance expansions are fixed by conformal invariance.

The symmetry transformations $z \mapsto \lambda z$ introduce a change of the n-point functions which can be compensated by a linear transformation in $L(F)$. In most cases of interest, this transformation can be diagonalized. When a field transforms as $\phi \mapsto \lambda^h \bar{\lambda}^{h'} \phi$, we say that ϕ has conformal dimensions (h, h'). When λ is real, we have a rescaling transformation. Thus $h+h'$ is the scaling dimension of ϕ. When $|\lambda| = 1$, we obtain a rotation, with an action described by the conformal spin $h - h'$. Since a rotation by 2π is trivial, the conformal spin must be integral for bosonic fields. For holomorphic fields, $h' = 0$. Since the scaling dimension of the energy momentum tensor is 2, its holomorphic component has conformal dimensions (2,0) and its anti-holomorphic component has conformal dimensions (0,2).

One could proceed in a purely algebraic way, completely within the framework for quantum field theories which was described above. Instead, let us shorten the path by some geometric intuition. Let us look at some holomorphic transformation $z \mapsto f(z)$ of a neighborhood of $z = 0$. Locally, this is a symmetry, since it does not change the angles. When $f(0) = 0$, it induces a transformation in $L(F)$, since F can be considered as the space of fields at the point 0. The action of the transformations $z \mapsto \lambda z$ on a field ϕ of conformal dimensions (h, h') can be described by stating that the form $\phi(z)(dz)^h(d\bar{z})^{h'}$ is invariant. If this remains true for all f, the field ϕ is called primary. The primary fields span a subspace of F. If this subspace is finite dimensional, the corresponding conformal field theory is called minimal. The Ising model is minimal and has a three dimensional subspace of primary fields, but the Thirring model is not minimal.

The short distance expansion of a holomorphic field ϕ of conformal dimensions $(h, 0)$ on an arbitrary field χ is a Laurent expansion, since it depends holomorphically on z. We write it in the form

$$\phi(z)\chi(w) = \sum_n (z-w)^{n-h}(\phi_n\chi)(w) \ .$$

For all integers n, this defines linear operators ϕ_n on F. They are called the Fourier components of ϕ. When χ has conformal dimensions $(\tilde{h}, \tilde{h}')$, then $\phi_n\chi$ has conformal dimensions $(n+\tilde{h}, \tilde{h}')$. We regard F as graded by the conformal dimensions and see that ϕ_n is an operator of degree $(n, 0)$. The action of local conformal transformations on F is given by the linear operators $L_n, \bar{L}_n$ obtained from the holomorphic and anti-holomorphic Virasoro fields.

For holomorphic fields χ, the fields $\phi_n\chi$ are holomorphic, too, such that one obtains a new algebraic structure [Zamolodchikov 1985, Borcherds 1986, Goddard 1989]. A standard name in the physics literature is W-algebra, but mathematicians prefer to talk about vertex operator algebras. The latter name has the advantage of a clear history in string theory, whereas the W seems to be due to the accidental naming of some field as $W(z)$ by Fateev and Zamolodchikov. Proposed allusions to Weyl, Wigner or Wilson are apocryphal, but may justify the name, which has the advantage of being short.

The field $\phi_h\chi$ is called the normal ordered product of ϕ and χ. It is the first field which occurs in the regular part of the short distance expansion. In the Thirring model, the currents $j, \bar{j}$ have conformal dimensions (1,0) and (0,1). The Virasoro fields are given by the normal ordered products $j_1 j/2$ and $\bar{j}_1\bar{j}/2$ [Callan, Dashen, Sharp 1967]. When $n < h$, the field $\phi_n\chi$ occurs in the singular part. It turns out that it can be described in terms of commutators $[\phi_n, \chi_m]$. Thus one part of the operations of the W-algebra just describes a Lie algebra. For the components L_n of the energy momentum tensor this is the Virasoro algebra. It was discovered by Gelfand and Fuks [1968] and is a central extension of the Lie algebra of vector fields on a circle. The value of the central extension is universally called c for the holomorphic Virasoro field and $\bar{c}$ for the anti-holomorphic one. In many models, they are equal. The values of c for the minimal models lie in a countable set. All of them have $c < 1$, whereas the Thirring model has $c = 1$. When ϕ is holomorphic and χ is anti-holomorphic, then $[\phi_n, \chi_m] = 0$.

The action of the ϕ_n on the space F of all fields yields a representation of the W-algebra. With some effort, the representations of a fixed W-algebra can be given the structure of a tensor category, like the representations of a Lie algebra. The corresponding tensor product is called fusion product.

The representation on the holomorphic fields themselves is called the basic representation and behaves as the neutral element under fusion.

Some W-algebras only have finitely many irreducible representations. These are called rational. In conformally invariant theories with rational W-algebras, all scaling dimensions are rational numbers. Such theories themselves are called rational, too. The minimal theories are characterized by the porperty that already the Virasoro part of the W-algebra has only finitely many irreducible representations. It is sufficient to consider the holomorphic Virasoro field, since for the anti-holomorphic one the situation is analogous. The properties of the Virasoro algebra only depend on the central extension c. The most interesting values occur for those minimal models where all the representations are unitary. This happens for $c = 1 - 6/(p(p+1))$, p an integer greater 2. For $p = 3$ one finds $c = 1/2$ and the Ising model.

The first investigation of these questions was due to Mack and Lüscher. They found that $c = 1/2$ is the lowest possible value and that there is a gap above 1/2. Here is one of the rare cases where progress depended on difficult calculations performed by a mathematician. V. Kac determined the structure of the representations [1979], which later allowed Belavin, Polyakov and Zamolodchikov to determine the values of c for all minimal models [1984]. Soon afterwards, Friedan, Qiu and Shenker determined the unitary cases [1984].

The discovery of the minimal models and their explicit solution by Belavin, Polyakov, Zamolodchikov was the breakthrough event in the history of conformal field theory. It quickly became clear that these models are beautiful and fundamental mathematical structures. For reasons which are hard to understand in depth, very different kinds of such structures, from Platonic solids to singularities, can be classified in terms of the ADE Dynkin diagrams. The same is true for the minimal models [Cappelli, Itzykson, Zuber 1987].

Part of the excitements about these early publications came from the relationship to continuous phase transitions in statistical mechanics. Besides the Ising model, many other well known continuous phase transitions were recognized as minimal models, like the ones for the 3-states Potts model, the tricritical Ising model, and the Lee-Yang edge singularity. Some of them have been realized in the lab, and measurements agree very well with the theoretical calculations.

Many properties of continuous phase transitions now fell into place. For example, some phase transitions are characterized by universal rational numbers, others have free continuous parameters. The former now are described by conformal field theories which have no conformally invariant deformations. In particular, this is true for the minimal models, like the continuum limit of the Ising model. When conformally invariant deformations exist, then they do not change c. The first example was Baxter's eight vertex model, which at the critical point becomes isomorphic to the older but more difficult Ashkin-Teller model. They yield $c = 1$, as for the closely related Thirring model. For a study of all unitary $c = 1$ models, see [Ginsparg 1988].

In some respects even simpler than the minimal models are those for which the Virasoro fields can be described in terms of normal ordered products of fields of conformal dimensions (1,0) and (0,1). Such fields are called currents, and the corresponding conserved integral quantities are called charges. The Thirring model is of this type, with currents $j, \bar{j}$ and single holomorphic and anti-holomorphic charges $j_0, \bar{j}_0$. In more complex models where the two types of charges form simple Lie algebras, the short distance expansion of the currents yields the corresponding affine Kac-Moody algebras [Goddard, Olive 1988].

The Thirring model yields the simplest continuous family of conformal theories and has $c = 1$, too. In its bosonic description, it is given by a the statistical mechanics of maps to a circle. The model is rational when the area of this circle is a rational number. This means that the set of points for which the model is rational is dense in the whole family. At one particular rational point, another continuous deformation is possible, which generates the moduli space of the Ashkin-Teller phase transitions. This is a first example of the rather intricate geometry of such moduli spaces, with many number theoretic aspects. As a first step, It would be important to know the rational points of more complex moduli spaces, since rational theories have very explicit descriptions. So far, there are very few results.

Within a moduli space of conformal theories, consider a perturbations by a field $t(x)$. The integral $\int t(x) dz\, d\bar{z}$ must be invariant under conformal transformations, such that t should be a primary field of conformal dimensions (1,1). In the Thirring model, the field $j\bar{j}$ has these properties. The dimension of the vector space of such fields counts the number of possible infinitesimal deformations. Thus it is an upper bound on the dimension of the moduli space. For generic points of this space, one expects that the two dimensions are equal. For the Thirring model, they are both equal to 1.

The short distance expansion is a local property of the theory. When one wants to calculate the n-point functions, one also has to specify a Riemann surface on which the fields live (in the language of algebraic geometry, an algebraic curve). The simplest case is the Riemann sphere. Here the n-point functions of holomorphic fields are just rational functions. For more general fields, the results are much more complicated. For example, the four-point functions of minimal models already yield hypergeometric functions.

The Riemann sphere is unique, but more complicated Riemann surfaces (or equivalently algebraic curves) have their own continuous parameters. For example, a torus is described by the ratio τ of two independent periods. When these are correctly ordered and varied continuously, τ varies over the upper complex half-plane. The latter is called the Teichmüller space of the curves with torus topology. Points of Teichmüller space describe the same torus when they are related by a different choice of periods. Changes of the periods are described by the modular group. This is the group of linear rational transformations $\tau \to (a\tau+b)/(c\tau+d)$ with integral coefficients. More complicated curves behave in an analogous, but of course more complex way.

The 0-point function on the torus is essentially the partition function of the theory. Since energy and momentum are given by linear combinations of the Virasoro field components L_0 and $\bar{L}_0$, the latter can be defined by

$$Z = \quad tr\ \exp(2\pi i(L_0\tau - \bar{L}_0\bar{\tau}))\ ,$$

where the trace goes over the vector space F of all fields. The 0-point function on a torus with parameter τ has the form

$$\tilde{Z} = \exp(-2\pi i(c\tau - \bar{c}\bar{\tau})/24)Z\ ,$$

where $c, \bar{c}$ are the central extensions of the theory. The prefactor is necessary to get invariance under the modular group. For the Ising model one obtains

$$\tilde{Z} = \frac{1}{2}\sum_{i=2}^{4} |\theta_i(\tau)/\eta(\tau)|\ ,$$

where the θ_i are Jacobi's theta functions and η is Dedekind's function. Note that the scaling dimension 1/8 of the Ising spin can be read off from θ_2.

For a free complex fermion one obtains

$$\tilde{Z} = \frac{1}{2}\sum_{i=2}^{4} |\theta_i(\tau)/\eta(\tau)|^2\ .$$

This function arises at the parameter $R = \sqrt{2}$ of the Thirring model partition function

$$\tilde{Z} = |\eta(\tau)|^{-2} \quad \sum_{m,n} \exp\left(\frac{\pi i}{2}\left((\frac{m}{R} + nR)^2\tau - (\frac{m}{R} - nR)^2\bar{\tau}\right)\right) .$$

Here m, n vary over the integers. The equality of the latter two functions for $R = \sqrt{2}$ is an example of the fermion-boson equivalence mentioned above. In the gaussian description, only the terms with $m = n = 0$ were obvious, which explains why the model was considered to be uninteresting.

String theory

Contrary to the historical developments, we have considered conformal field theory before coming to string theory. The reason is that string theory is more complex. Conformal field theory is just one ingredient, albeit an essential one. For general introductions to string theory and more references, see [Green, Schwarz, Witten 1987] and Polchinski [1998].

In 1968, Veneziano invented an amplitude for a scattering process with two incoming and two outgoing particles which shared several features with strong interaction processes. When a natural generalization to arbitrary particle numbers was found, Nambu, Nielsen and Susskind recognized that these amplitudes describe a one-dimensional object moving in space. The surface described by its motion is called a worldsheet. Its embedding into spacetime is described by functions $X^\mu(\sigma, \tau)$, where σ, τ are coordinates on the worldsheet and X^μ yields the corresponding spacetime positions.

Calculational problems arise, because there is no canonical parametrization of the worldsheet. Some natural choice can be made, however. The causal structure of the ambient spacetime induces a causal structure on the worldsheet, with two lightlike tangent directions at each point. These directions can be integrated to lightlike curves. One chooses the coordinates such that their equations are given by $d\tau = d\sigma$ and $d\tau = -d\sigma$. This introduces a Minkowskian conformal structure on the σ, τ parameter space. One chooses $d\tau$ to be timelike and $d\sigma$ to be spacelike.

Strings have finite spatial extent, such that the range of σ is compact. For open strings, the standard choice is an interval of length π, for closed strings a circle of circumference π. No further natural choices can be made, which means that the worldsheet dynamics is conformally invariant. In other words, the possible states of a single string are described by a conformal field theory. When one continues to a euclidean conformal field theory, one must make a Wick rotation in τ, not in the time coordinate of X. The euclidean coordinate is called z.

By the analytic continuation, the worldsheet becomes a Riemann surface. Let us consider the case of closed strings only. Both in Minkowskian and in euclidean space, the worldsheet has the topology of a cylinder. By conformal invariance, it can be compactified to a Riemann sphere with two special points, one for the incoming and one for the outgoing state. Such special points are called punctures. String interactions are introduced by considering arbitrary Riemann surfaces with different numbers of punctures. Calculating the scattering of n strings involves three steps. First, the string states have to be identified with fields on the worldsheet. Secondly, one has to calculate the corresponding n-point functions for all Riemann surfaces with n punctures. The surfaces are not necessarily connected, since some groups of strings can interact independently of the others. Thirdly, one has to integrate over all of these configurations, in particular over the position of the punctures. In addition, one has to integrate over the finite dimensional moduli space of complex structures on Riemann surfaces with a given genus (number of handles). This integral is not needed when one applies conformal field theory to statistical systems, since there the Riemann surface is fixed.

Finally, one has to sum over the genera. Each term is multiplied by a power of the coupling constant. The exponent is proportional to an integral over the curvature, and can be normalized to $g-1$. The leading contribution is given by $g = 0$ and as many connected components as possible. For vanishing coupling, this leads to a free theory, exactly as for a quantum field theory. Indeed, a string theory can be regarded as a quantum field theory which includes graviton fields. In some limit, gravity decouples and one obtains a field theory of conventional type. For the latter, the perturbation series is a sum over Feynman diagrams. A tubular neighborhood of such a graph yields a Riemann surface of some genus g. This allows to identify one of the field theory couplings with the string coupling. We shall see that the others correspond to parameters of a conformal field theory on the string worldsheet.

The sum over g is certainly not convergent, which provides a technical reason to develop non-perturbative string theory. A deeper reason is the following. As for quantum field theory, free string theory can be considered as a boundary stratum on some moduli space. This stratum is characterized by the vanishing of a coupling constant, but in many cases its codimension is larger than one. Thus an expansion in the coupling constant cannot recover the full theory. In particular, it has no reason to be convergent. One example is given by quantum electrodynamics, where the following picture can be conjectured. To get a well defined quantum field theory, one has to introduce magnetic monopoles. These become infinitely heavy when the interaction goes to zero and their effects are not included in the perturbation expansion. Since monopoles can have an electric charge, one has an additional dimension of the moduli space which cannot be captured by perturbation theory.

In string theory, the rôle of the magnetic monopoles is taken over by branes of various dimensions. One can approach the full picture by a description of all possible boundary strata, but this goes much beyond the scope of the present article. Nevertheless, the reader should keep in mind that the following description of conformal worldsheet physics is perturbative and thus incomplete.

When a string state is described by a field ϕ on the worldsheet, the integration over the corresponding puncture position takes the form $\int \phi(z)\, dz\, d\bar{z}$. This must make sense independently of the choice of the coordinate z. In other words, string states are described by primary fields of conformal dimensions (1,1). There is another way to get the same result. When the string is considered in the background of some particle wave in spacetime, this yields a conformally invariant deformation of the theory, at least infinitesimally. Since deformations are described by the primary fields of conformal dimensions (1,1), the same must be true for the particle states arising from the string. With reference to spontaneous symmetry breaking in quantum field theory, the existence of particle states may be described as a Goldstone phenomenon.

When one considers strings in flat spacetime, the coordinates of the latter can be regarded separately. For a space coordinate X^i appropriate fields are given by $\exp(ip_i X^i(z))$, with arbitrary p_i. With a conventional choice of the length scale, the scaling dimension of this field is $p_i^2/4$.

A new situation appears for the time coordinate X^0. Due to Lorentz invariance, the field $\exp(ip_0 X^0(z))$ has scaling dimension $-p_0^2/4$. Fields with negative scaling dimensions of arbitrary size do not occur in statistical mechanics, but they can be made to fit in the framework of conformal field theory. Indeed, without such negative contributions to the scaling dimension, one never would get an infinite number of particle states. Here we can take an arbitrary field with $h = h'$ and adjust the value of p_0^2 such that the scaling dimension becomes 2. This produces at least a (1,1) field, though in general it will not be primary.

When we disregard the latter problem, we can consider the fields $\partial X^\mu \bar{\partial} X^\nu \exp(ipX)$. They have conformal dimensions (1,1) when $p^2 = 0$, such that they describe massless particles. When one considers their behaviour under spacetime rotations, one sees that they include spin 2 particles, i.e. states which behave like gravitons. For general reasons, the coupling of such states must be described by Einstein's theory. Thus any consistent string theory is a theory of quantum gravity.

Later, this fact was recognized as the best feature of string theory, but it was a nuisance as long as the theory was supposed to work for the strong interaction. Other problems of the original string theory had to be solved quite apart of this deeper issue, namely the existence of tachyons and the wrong dimension of spacetime.

A tachyon appears when one considers the simple field $\exp(ipX)$. This is a primary (1,1) field, if $p^2 = -8$. To get rid of this unwanted particle with negative squared mass, the conformal symmetry had to be extended to a superconformal one. The fields of such theories can have integral or half-integral conformal spin. Those with a half-integral difference $h - h'$ are fermionic. In addition to the Virasoro fields one has fields G and $\bar{G}$ of conformal dimensions (3/2,0) and (0,3/2). There are two different fermion numbers associated to the holomorphic and anti-holomorphic variables. The short distance expansion with G changes the first one by one unit, that with $\bar{G}$ the second one. The Fourier components L_n of the Virasoro field and those of G together yield a superalgebra, in which the Virasoro algebra is embedded. The model has two sectors (discovered separately by Ramond and by Neveu and Schwarz), but we shall consider just the latter. In this sector, the fermionic fields have half-integral coefficients. Apart from such modifications, superconformal field theory can be regarded as a special case of conformal field theory, so most of the preceding description remains valid.

Fields related by the action of $G_{1/2}, \bar{G}_{1/2}$ are called superpartners. For superconformal deformations, the corresponding (1,1) fields must be superpartners of (1/2,1/2) fields. The physically relevant deformations are described by bosonic fields, such that the (1/2,1/2) fields must be fermionic with respect to both fermion numbers. The superstring still has fields $\exp(ipX)$, which have conformal dimensions (1/2,1/2) for $p^2 = -4$, but these fields are of bosonic nature and do not correspond to physical particles. This elimination of the tachyonic fields is due to Gliozzi, Olive and Scherk.

The issue of the spacetime dimension arose in a different way. When one calculates the norm of a field of type ∂X^μ, Lorentz invariance yields a result proportional to $g_{\mu\mu}$. In particular, one can find negative norms which are incompatible with a probability interpretation. In the 50's and 60's much ink had flown in unsuccessful attempts to make sense out of negative norms and no one was motivated to try again. Fortunately, Virasoro recognized that not all fields yield physical states [1970]. The concepts of primary fields and conformal dimensions did not exist yet, but he only found the correct constraints and described them by the Fourier modes of the Virasoro fields. One year later, Galli obtained the interpretation in terms of conformal invariance [1970].

Numerical investigations showed up to a certain degree of complexity that the physically allowed fields all have positive norm, but a general proof was difficult to obtain. Then it turned out that allowed negative norm fields do exist when the spacetime dimension is greater than 26, or 10 for the superstring. This made sense of an observation of Lovelace [1971], which had not been taken very seriously because it was too outlandish. Looking at Riemann surfaces of torus topology, Lovelace had shown that the bosonic string theory was found to require a spacetime of 26 dimensions. Now it became clear that this number was a deep structural property of the bosonic string theory and would not go away. Indeed, Brower [1972] and Goddard and Thorn [1972] used the 26 dimensions to prove that the norms make physical sense (the no-ghost theorem). Later it turned out that the value of this critical dimension has deep relations to the conformal invariance of the world sheet physics and the corresponding modular invariance [Brink, Nielsen 1973]. Moreover, Beilinson and Manin found out that the strange 26 was closely related to analytic torsion results of Mumford, which allowed them to write the measure for the integration over the moduli space of Riemann surfaces in a very elegant form [1986].

The critical dimension translates into the value $c = \bar{c} = 26$ of the central extensions. For the superstring one needs 10 dimensions and $c = \bar{c} = 15$. The latter value is due to the superpartners of the 10 coordinates X^μ, which contribute half as much to the central extension. The simplest way to obtain a model in four dimensions is the old Kaluza-Klein idea. One just wraps up all superfluous dimensions in a small circle. For the bosonic string this yields 22 copies of the Thirring model. The corresponding 44 currents of type j and $\bar{j}$ yield 44 photons, all with separate interactions of electromagnetic type. The values of $c, \bar{c}$ do not change. Obviously, this model is not particularly realistic. It exemplifies, however, that the spacetime dimension of the model can be changed at will, as long as one keeps conformal invariance and the correct central extensions. For the superstring, similar remarks apply.

To write down a general bosonic string model in four dimensions, one just needs to replace the 22 copies of the Thirring model by an arbitrary conformal field theory with $c = \bar{c} = 22$. The latter is called the internal conformal theory. In analogy to the Kaluza-Klein case, one still says that it describes 22 compactified dimensions, even if this is not always a geometrically correct interpretation. To compactify the superstring to fourdimensional spacetime, one needs six compactified dimensions and $c = \bar{c} = 9$ instead. Every possible compactification corresponds to a theory in a space of less than 10 spacetime dimensions. In this way one gets, e.g., an encoding of four-dimensional quantum field theories by conformal or superconformal field theories in two dimensions.

In particular, consider a field $\phi(z)\exp(iXp)$, where ϕ belongs to the internal theory. When one adjusts p^2 to get overall conformal dimensions (1,1), one sees that for a particle state of mass m the corresponding field must have contributions $h = h' = 1 + m^2/8$ from the internal conformal theory. Of particular interest are the internal (1,1) fields, which correspond to massless Higgs bosons.

When the conformal theory includes an affine Kac-Moody algebra with holomorphic currents j_a of conformal dimensions (1,0), the fields $j_a\bar{\partial}X^\mu$ $\exp(iXp)$ with $p^2 = 0$ describe the quanta of a vector potential A_a^μ belonging to the corresponding finite dimensional gauge group. Thus the states of the string now include non-abelian gauge fields, and the theory starts to look a bit more like the standard model. In the superstring theory, one also gets fermions. Their interactions with the Higgs bosons and the gauge fields are of standard type, though one has not yet managed to obtain precisely the standard model.

Of course, the bosonic model always will have the tachyonic $\exp(ipX)$ fields and cannot be used by itself. Nevertheless, the bosonic string can be used for either the holomorphic or the anti-holomorphic coordinates. To get rid of the tachyon, it is indeed sufficient to use a field G but no $\bar{G}$. This yields models with $c = 26$ but $\bar{c} = 15$, called heterotic string models. They were found by Gross, Harvey, Martinec and Rohm. Heterotic strings do not have a pure spacetime version, since the spacetime contributions to c and $\bar{c}$ have to match. The archetypal heterotic string lives in 10 dimensions, where the compactified part is purely holomorphic, with $c = 16$. There are many constraints on purely holomorphic conformal theories which exist on arbitrary Riemann surfaces. In particular, c must be a multiple of 8. For $c = 8$, the only example is the affine Kac-Moody algebra based on E_8. For $c = 16$, one either can take the tensor product of two $c = 8$ models or use the affine Kac-Moody algebra based on $SO(32)$. For $c = 24$, there are 71 possibilities [Schellekens 1993]. One of them has a remarkable symmetry group of about 10^{54} elements, the Fischer-Griess monster [Borcherds 1986]. This closeness of string theory to beautiful exotic structures is still a deep mystery. To get to four dimensions, one needs a less exotic internal conformal field theory with $c = 22$ and $\bar{c} = 9$.

The late discovery of the heterotic string was due to the fact that string research slowed down a lot after 1974. At that time it had become clear that QCD is a better theory for the strong interaction. Though Scherk and Schwarz had shown that one could reinterpret string theory as a theory of quantum gravity [1974], there was not very much support for such an arcane research line. One of the ideas which appeared shortly before the theory entered a long hibernation period was the classification of the possible rational theories by modular forms. In particular, this yielded a candidate which later would be interpreted as the partition function of the compactified part of the heterotic string [Nahm 1977]. Unfortunately, present mathematical techniques only allow to apply this procedure to rational conformal fields theories, for which the partition function can be written as a finite sum $Z = \sum_i Z_i \bar{Z}_i$ with holomorphic functions Z_i and anti-holomorphic functions $\bar{Z}_i$. Nevertheless, it was striking that the method indicated an incredibly large number of possible theories, in stark contrast to the initial hopes that one was heading for something unique. At present, the situation has not changed very much. There are vague hopes that non-perturbative string theory will select particular models, but it also is possible that one will end up on a moduli space with more parameters than in the standard model. For every choice of parameters, one will have a quantum version of Einstein's gravity theory, however.

The bold switch from the interpretation of string theory as a theory of the strong interaction to a theory of quantum gravity by Scherk and Schwarz must have been one of the strangest events in the history of physics. In particular, the basic distance scale had to be changed by twenty orders of magnitude, from the proton diameter to the Planck length. For mathematicians this should be easier to digest than for physicists, since no change in the mathematical structure is involved. For physicists, however, the emergence of string theory now appears as an accident. It would not have happened if the discovery of the SU(3) gauge interaction of the quarks had come a few years earlier. Even in hindsight, one sees no way how a direct study of quantum gravity could have led to this theory. Indeed, a direct attack has been tried from several points of view, but with very limited success. It seems that one cannot unify quantum theory and gravitation without incorporating much knowledge about other interactions.

In any case, present research on quantum gravity cannot follow the traditional pattern of physics. One hundred years ago, Planck himself estimated its characteristic length scale by combining Newton's constant, the speed of light, and his new quantum of action. He found $4,13 \cdot 10^{-33} cm$ [1900]. At that time, physicists and chemists were getting the first precise ideas about what happens at $10^{-8} cm$, so Planck must have felt like looking into an abyss. Of the 25 orders of magnitude to be covered we now have explored not quite ten, thus a naive extrapolation predicts another 150 years before we really understand what is going on at the basic scale. Planck's report about his discovery is brief and sober. Nevertheless, he states that the units he found would keep their meaning for all times and all cultures, including extraterrestrian and non-human ones.

Without the ability to do experiments in quantum gravity, it is hard to know if theoretical investigations are on the right track. Everyone who keeps trying is inspired by Einstein's success with general relativity. He had little experimental input, but relied on his keen sense for structure and mathematical beauty. His belief in the harmony of the spheres was as deep as Kepler's, and when he had found an indication of congruence between nature and a mathematical structure he did everything to uncover it fully. Quantum mechanics remained as a jarring note like the irrational numbers to the early Pythagoreans. Thus string theory would have disappointed Einstein as far as quantum physics is concerned. But if it is correct, it justifies some of his attempts in the search for a unified theory. On one hand, he wanted to generalize the metric tensor $g_{\mu\nu}$ to an object with an antisymmetric part.

String theory has such an object, called the B field. Together with the metric tensor it is obtained from the fields $\partial X^\mu \bar{\partial} X^\nu \exp(ipX)$ which have been considered above. Einstein also was right in his high regard for the Kaluza-Klein approach.

Einstein's example can be used as an encouragement and as a warning. His successful gravity theory was based on at least one elementary fact which no one else could explain - the equality of inertial and gravitational mass. When he let loose of such guidance, he still did important research, but went astray. String theory does not do too badly on this account. On one hand, one can at least come close to the standard model. Moreover, superstring theory at least suggests that the experimentalists will find supersymmetry in the near future. The theory develops in a search for deep and beautiful structures, but it has the advantage of holding on to the tenuous guide offered by low energy experiments. Currently, no other theory of quantum gravity can make such claims. Despite its unbelievable origin, string theory is by far the most promising approach to unify all of the known interactions. The one possible exception of the latter claim is the cosmological constant, since it is separated by another abyss of many orders of magnitude from the rest of physics.

Missed and open opportunities

Let us come back to Dyson's 1972 address to the American Mathematical Society. It was titled 'Missed Opportunities' and concerned problems in the communication between mathematicians and physicists. In particular, he considered quantum field theory and the unification with gravity, but his first example concerned a communication problem between Dyson the physicist and Dyson the mathematician. As a mathematician, he had played around with powers of Dedekind's η-function and obtained nice identities for the exponents 3,8,10,14,15,21,24,26,28,35,36,... In this combinatorial context he did not recognize the dimensions of the simple Lie groups, plus 26, which would have been evident to him in a physics context. From today's point of view, the regret about this little failure may have caused him to miss a greater opportunity. He must have heard about the incredible 26 dimensions of the bosonic string, which Lovelace had found the year before, but apparently thought little about this coincidence. Otherwise he would have stumbled on the importance of Dedekind's η-function in string theory. Two years later Scherk and Schwarz established the importance of string theory for the unification of quantum field theory and gravity.

Many of the present author's missed and taken opportunities also concern the interaction with mathematics. A very rapid course by D. Zagier led to a classification of string theories by modular functions. On the other hand, searching the CERN library for books discussing infinite dimensional Lie algebras was a frustrating enterprise. Even worse was the inability to find the dimensions of the next representations of E_8, when the classification yielded $q^{-1/3} + 248q^{2/3} + \ldots$. Unfortunately, the visit of Kac to CERN came far too late, but to the author it proved the value of an environment where physicists and mathematicians could make the effort to learn about their respective discoveries. Princeton and some other places made a good start, but it would be nice to have a few more. Here are some problems which may be tackled in such an environment.

In the Kaluza-Klein formalism, a fifth dimension is hiding because it is compactified to a circle. When one considers experiments at fixed energy and makes the period of the compactified dimension very large, the five dimensional geometry emerges again. This process can be generalized – taking suitable limits of quantum field theories one can obtain classical geometries. In this context, the latter are called target spaces.

Let us look at the Kaluza-Klein situation in the context of string theory. We have considered fields $\exp(iXp)$, where for simplicity we consider a single position component X. When it is compactified with period l, the choice of p is constrained by $\exp(ilp) = 1$, or $p = 2\pi n/l$ with integral n. The scaling dimension of such a field is proportional to $(n/l)^2$, thus small for large l and small n. In particular, the integer $h - h'$ has to vanish. The short distance expansion of such fields involves weak singularities only. In the limit where l becomes large, it reduces to the ordinary product $\exp(iXp_1)\exp(iXp_2) = \exp(iX(p_1 + p_2))$. Because of scale invariance, the large l limit produces a unique commutative algebra of all fields whose scaling dimension approaches zero.

In the classical limit, every smooth function on the Kaluza-Klein circle can be Fourier expanded with basis $\exp(iXp)$. Thus one obtains the algebra of all smooth functions on the circle. Moreover, the space of these functions is graded by the scaling dimension $2h$, which is the eigenvalue of the Laplace operator. The geometry of the circle can easily be reconstructed from this information.

This example can be generalized to all kinds of manifolds. Particularly attractive are Calabi-Yau manifolds, where one can work with the highly constrained superconformal theories. Much less is known about these manifolds than about the circle, for example about their Einstein metrics. In these cases, the conformal theories may be easier to control than their classical limits. One certainly may hope to obtain the algebra of smooth functions and the corresponding eigenvalues of the Laplace operator from the conformal data.

When the parameters of a conformal field theories are varied, one may obtain quite different classical limits. In particular, a connected moduli space may have several different boundary components. In this way, it is possible to relate different classical geometries by non-classical paths. As a simple example, consider again a string on the Kaluza-Klein circle. There are more fields than we have considered so far, since one can wind the string around the circle. When the circle is large, this yields particle states of large mass. When the period l becomes very small, winding costs hardly any energy, whereas the $\exp(iXp)$ fields have large scaling dimensions and describe particles of large mass. When l goes to zero, the short distance expansion of the basic winding fields is just given by the additive group of the winding numbers. In this way, a Kaluza-Klein theory with period l becomes isomorphic to one with period l^{-1}. This isomorphism is known as T-duality. It is one of many dualities which arise in string theory, so the name 'dual model' used around 1970 was quite prescient. Perhaps the most famous of the dualities is mirror symmetry, which is a specific property of conformal field theories with a high degree of supersymmetry.

The winding fields are examples of solitonic objects, since the winding number is time-independent. In the euclidean conformal theory, one also has instanton contributions given by a map of the string Riemann surface to the target space. The most studied case is the one of embeddings of Riemann spheres in algebraic target manifolds, since mathematicians have been much interested in counting the number of different embeddings. As shown by Candelas' group, mirror symmetry yields the correct numbers for the quintic in four dimensional projective space. This started the huge interest of mathematicians in this topic. Usually, mathematicians try to replace the quantum field theoretic approach by more classical methods, but in the end this may well turn out to be the more arduous approach, quite comparable to a proof of the prime number theorem without using analysis.

The moduli space of superconformal theories of fixed central extensions seems to be connected. For $c = \bar{c} = 9$ each Calabi-Yau manifold of three complex dimensions is a possible target space and yields one boundary component of the moduli space. By following all ramifications of the moduli space it should be possible to classify all such Calabi-Yau manifolds, for a start.

In modern algebraic geometry, geometric and number theoretic problems occur side by side. The same is true of conformal field theory, though physicists so far are ill equipped to handle these issues. For example, what is the meaning of Dysons formula for η^{26}? For the moment, this is a mystery without much of a clue, but one can start with simpler problems. Above, we have discussed rational points in the moduli space of string theories. At these points, coupling constants like the Yukawa couplings can be calculated explicitly. Usually, the rational models are among the few which such calculations are possible at present. For example, let us take models with $c = \bar{c} = 6$ and sufficiently large supersymmetry. In this case one obtains K3 surfaces as target space. The moduli space turns out to have 80 dimensions, but the largest submanifold under explicit control has just 16 dimensions. In addition, however, there are many rational points sprinkeled around which are perfectly well understood. Will the rational points turn out to be densely distributed? More specifically, the same moduli space occurs for torus compactifications of the heterotic string to six-dimensional flat spacetime. In the latter case the rational points are well known, and they correspond precisely to the complex multiplication points of the K3 moduli space. Are those the rational points of the latter moduli space?

In simple examples, the conformal dimensions of rational models are obtained by applying the dilogarithm function to algebraic numbers. The corresponding sums are described by torsion elements in the Bloch group [Nahm, Recknagel, Terhoeven 1993]. Thus there is a link between conformal field theory and one of the most active areas of present mathematical research. In particular, there seem to be relations to Grothedieck's program for a description of the Galois group of all algebraic numbers and to the theory of motives. Kontsevich recently conjectured that the motivic Galois group acts on the moduli space of conformal field theories [1999].

Obviously, mathematicians have much to gain from physics. In view of the higher reliability of the answer (and regarding costs as irrelevant) physicists were more inclined to ask nature than to ask a mathematician. Quite typically, Schweber's textbook concludes with the following sentences:

"In the final analysis, however, it will probably be the new information that will be obtained from the high energy machines and colliding beam machines to go into operation in the next few years which will help unravel the puzzle of the elementary particles and their interactions. In particular, we may discover whether the notions of space and time upon which present-day field theories are based are in fact valid." But meanwhile we have learned more respect for the sixteen orders of magnitude which separate us from the Planck scale. If it gets too expensive to ask the direct questions, we just have to push the mathematical analysis of what little clues there are. There is hope, since sometimes it did work. Kepler managed to extract the secrets of the planetary motions from pretelescopic data, but it would have been much harder without some knowledge about ellipses.

As a final encouragement for those willing to use the bridge, let me quote a German poet: "Nur Beharrung führt zum Ziele, nur die Fülle führt zur Klarheit und im Abgrund wohnt die Wahrheit" (to reach the goal you must be persistent, to see clearly you have to understand a wealth of phenomena, and truth lives in the abyss). Schiller's poem talks about causal time and three-dimensional space, but two euclidean dimensions make a good start.

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