In the lecture course, we encountered the Virasoro algebra \( \mathfrak{Vir} \), given by

\[
[L_n, L_m] = (n - m)L_{n+m} + C \frac{1}{12} (n^3 - n) \delta_{n+m,0}, \\
[L_n, C] = 0.
\]

This is the one important algebra in two-dimensional conformal field theory, since it is the algebra of the generators of arbitrary locally conformal (i.e. locally holomorphic) transformations.

In order to further analyse the nature of a two-dimensional conformally invariant quantum field theory, one has to know how physically relevant representations of the algebra \( \mathfrak{Vir} \) on a space of states look like. We assume the following for a physical sensible space of states:

(1) There exists precisely one state \( |0\rangle \) with the property \( L_n|0\rangle = 0 \forall n \geq -1 \). This state is called the vacuum.

(2) Each representation shall contain precisely one state \( |h\rangle \) with the properties \( L_n|h\rangle = 0 \forall n > 0 \) and \( L_0|h\rangle = h|h\rangle \). Such states are called highest weight states. The \( L_0 \) eigenvalue \( h \) is called the highest weight. As we will see, such representations have an energy spectrum which is bounded from below admitting thus stable ground states. Finally, if \( h = 0 \), the representation is then the vacuum representation built upon the state \( |0\rangle \), since the vacuum is unique.

(3) In the case of unitary representations, we have that \( L_{-n} = L_n^\dagger \). We have seen in the lecture, that unitary representations require \( C, h \geq 0 \).

\[\text{[P1] Field state isomorphism}\]

One important thing is the so-called field-state isomorphism. Suppose we have a primary field \( \Phi_h(z) \) of conformal weight \( h \). This field corresponds one-to-one to a highest weight state \( |h\rangle \) via

\[ |h\rangle = \lim_{z \to 0} \Phi_h(z)|0\rangle. \]

Use the commutator \([L_n, \Phi_h(w)]\) from the lecture to show that the state \( |h\rangle \) defined via the field state isomorphism is indeed a highest weight state, i.e. that it satisfies \( L_n|h\rangle = 0 \forall n > 0 \).

\[\text{[P2] Universal enveloping algebra}\]

Given a highest weight state \( |h\rangle \), we can construct representations with the universal enveloping algebra \( U(\mathfrak{Vir}) \). This is the algebra of all words \( L_{n_1}L_{n_2} \ldots L_{n_k} \) for \( k \in \mathbb{Z}_+ \). Convince yourself that the highest weight representation is formally given by

\[ V|h\rangle = \text{span} \{ L_{-n_1}L_{-n_2} \ldots L_{-n_k}|h\rangle : n_i \geq n_{i+1} > 0 \land k \in \mathbb{Z}_+ \} \]

Such formal representations are called Verma modules.

\[\text{[P3] Gradation}\]

Show that the state \( L_{-n}|h\rangle \) has weight \( h + n \). Convince you that the same is true for a state \( L_{-n_1}L_{-n_2} \ldots L_{-n_k}|h\rangle \) if \( \sum_{i=1}^k n_i = n \). This implements a natural gradation on the Verma modules. Namely, defining

\[ U_n(\mathfrak{Vir}) = \text{span} \left\{ L_{-n_1}L_{-n_2} \ldots L_{-n_k} : n_i \geq n_{i+1} \land k \in \mathbb{Z}_+ \land \sum_{i=1}^k n_i = n \right\}, \]
we may write

\[ V_\langle h \rangle = \bigoplus_{n=0}^{\infty} U_n(\mathfrak{Vir}) | h \rangle . \]

Verify that for any given level \( n \), the vector space \( U_n(\mathfrak{Vir}) \) is finite dimensional. This means that on each level of excitation above the energy of the ground states, only finitely many different excitation states exist. Determine explicitly the dimensions for \( n = 1, 2, \ldots, 5 \). Do you have an idea what the dimension is for generic \( n \)?

\[ \text{[P4] Kac determinant} \]

Let us now concentrate specifically on \( U_1(\mathfrak{Vir}) \) and \( U_2(\mathfrak{Vir}) \). Construct for these vector spaces the (Hermitean) matrices

\[ K_n = \left( \langle h | (L_{-\{n\}_a})^\dagger L_{-\{n\}_b} | h \rangle \right), \]

where \( L_{-\{n\}_a} \) denotes an arbitrary enumeration of the words in \( U_n(\mathfrak{Vir}) \). Then, compute \( \det(K_n) \) and find their zeroes. For \( K_2 \), the set of zeroes will turn out to be functions \( h(c) \). Hint: It is generally true that \( K_n \) factorizes in a certain form such that it contains all zeroes of \( K_{n-1} \). The remaining new zeroes are, in the case of \( K_2 \) then easily found by solving a quadratic equation. It is customary to parametrize the zeroes in the form

\[ h_{r,s}(c) = \frac{(m + 1)r - ms)^2 - 1}{4m(m + 1)}, \]

where

\[ m = \frac{1}{2} \pm \frac{1}{2} \sqrt{\frac{25 - c}{1 - c}}. \]

The choice of sign simply interchanges \( r \leftrightarrow s \) and \( m \leftrightarrow m - 1 \). With these notations, your final result should read \( \det(K_2) = 32(h - h_{1,1})(h - h_{1,2})(h - h_{2,1}) \langle h | h \rangle^2 \). One can show that the general Kac determinant at level \( n \) is of the form

\[ \det(K_n) \propto \det(K_{n-1}) \prod_{r,s=n} (h - h_{r,s}). \]