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REPRESENTATION THEORY OF THE VIRASORO ALGEBRA

In the lecture course, we encountered the Virasoro algebra $\mathfrak{Vir}$, given by

$$\begin{aligned} [L_n, L_m] &= (n-m)L_{n+m} + C\frac{1}{12}(n^3-n)\delta_{n+m,0} \,, \\ [L_n, C] &= 0 \,. \end{aligned}$$

This is the one important algebra in two-dimensional conformal field theory, since it is the algebra of the generators of arbitrary locally conformal (i.e. locally holomorphic) transformations.

In order to further analyse the nature of a two-dimensional conformally invariant quantum field theory, one has to know how physically relevant representations of the algebra $\mathfrak{Vir}$ on a space of states look like. We assume the following for a physical sensible space of states:

(1) There exists precisely one state $|0\rangle$ with the property $L_n|0\rangle = 0 \;\forall\; n \geq -1$. This state is called the *vacuum*.

(2) Each representation shall contain precisely one state $|h\rangle$ with the properties $L_n|h\rangle = 0 \;\forall\; n > 0$ and $L_0|h\rangle = h|h\rangle$. Such states are called *highest weight states*. The L_0 eigenvalue h is called the *highest weight*. As we will see, such representations have an energy spectrum which is bounded from below admitting thus stable ground states. Finally, if $h = 0$, the representation is then the vacuum representation built upon the state $|0\rangle$, since the vacuum is unique.

(3) In the case of unitary representations, we have that $L_{-n} = L_n^\dagger$. We have seen in the lecture, that unitary representations require $C, h \geq 0$.

[P1] *Field state isomorphism*
One important thing is the so-called *field-state isomorphism*. Suppose we have a primary field $\Phi_h(z)$ of conformal weight h. This field corresponds one-to-one to a highest weight state $|h\rangle$ via

$$|h\rangle = \lim_{z\to 0} \Phi_h(z)|0\rangle \,.$$

Use the commutator $[L_n, \Phi_h(w)]$ from the lecture to show that the state $|h\rangle$ defined via the field state isomorphism is indeed a highest weight state, i.e. that it satisfies $L_n|h\rangle = 0 \;\forall\; n > 0$.

[P2] *Universal enveloping algebra*
Given a highest weight state $|h\rangle$, we can construct representations with the universal enveloping algebra $U(\mathfrak{Vir})$. This is the algebra of all words $L_{n_1}L_{n_2}\ldots L_{n_k}$ for $k \in \mathbb{Z}_+$. Convince yourself that the highest weight representation is formally given by

$$V_{|h\rangle} = \mathrm{span}\left\{L_{-n_1}L_{-n_2}\ldots L_{-n_k}|h\rangle : n_i \geq n_{i+1} > 0 \wedge k \in \mathbb{Z}_+\right\}$$

Such formal representations are called *Verma modules*.

[P3] *Gradation*
Show that the state $L_{-n}|h\rangle$ has weight $h+n$. Convince yourself that the same is true for a state $L_{-n_1}L_{-n_2}\ldots L_{-n_k}|h\rangle$ if $\sum_{i=1}^k n_i = n$. This implements a natural *gradation* on the Verma modules. Namely, defining

$$U_n(\mathfrak{Vir}) = \mathrm{span}\left\{L_{-n_1}L_{-n_2}\ldots L_{-n_k} : n_i \geq n_{i+1} \wedge k \in \mathbb{Z}_+ \wedge \sum_{i=1}^k n_i = n\right\} ,$$

we may write

$$V_{|h\rangle} = \bigoplus_{n=0}^{\infty} U_n(\mathfrak{Vir})|h\rangle\,.$$

Verify that for any given *level* n, the vector space $U_n(\mathfrak{Vir})$ is finite dimensional. This means that on each level of excitation above the energy of the ground states, only finitely many different excitation states exist. Determine explicitly the dimensions for $n = 1, 2, \ldots, 5$. Do you have an idea what the dimension is for generic n?

[P4] *Kac determinant*
Let us now concentrate specifically on $U_1(\mathfrak{Vir})$ and $U_2(\mathfrak{Vir})$. Construct for these vector spaces the (Hermitean) matrices

$$K_n = \Big(\langle h|(L_{-\{n\}_a})^\dagger L_{-\{n\}_b}|h\rangle\Big)\,,$$

where $L_{-\{n\}_a}$ denotes an arbitrary enumeration of the words in $U_n(\mathfrak{Vir})$. Then, compute $\det(K_n)$ and find their zeroes. For K_2, the set of zeroes will turn out to be functions $h(c)$. Hint: It is generally true that K_n factorizes in a certain form such that it contains all zeroes of K_{n-1}. The remaining new zeroes are, in the case of K_2 then easily found by solving a quadratic equation. It is customary to parametrize the zeroes in the form

$$h_{r,s}(c) = \frac{((m+1)r - ms)^2 - 1}{4m(m+1)}\,,$$

where

$$m = -\frac{1}{2} \pm \frac{1}{2}\sqrt{\frac{25-c}{1-c}}\,.$$

The choice of sign simply interchanges $r \leftrightarrow s$ and $m \leftrightarrow m-1$. With these notations, your final result should read $\det(K_2) = 32(h-h_{1,1})(h-h_{1,2})(h-h_{2,1})\langle h|h\rangle^2$. One can show that the general *Kac determinant* at level n is of the form

$$\det(K_n) \propto \det(K_{n-1}) \prod_{rs=n} (h - h_{r,s})\,.$$