Exercise VI for the lecture course CONFORMAL FIELD THEORY

Michael Flohr
Derek Harland

Free field realization

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FREE FIELD REALIZATION AND VERTEX OPERATORS

We learned in the lecture that in conformal field theory, fields are divided into two classes, the primary fields $\Phi_h(z,\bar{z})$, and their descendant fields. If we know how to compute correlation functions of primary fields, we can compute any correlation function. What we miss so far is an explicit realization of a primary field. We know one exception so far: Considering a theory of a free massless scalar Boson $\phi(z,\bar{z})$, we saw that the current $J(z) = \mathrm{i}\partial_z\phi(z,\bar{z})$ is a chiral primary field of weight $h = 1$. But how does one construct primary fields of other scaling dimensions?

[P1] *Vertex operators*
Consider the normal ordered expression

$$V_k(z,\bar{z}) = :\!\exp(ik\phi(z,\bar{z}))\!:\,,$$

and compute its OPE with the energy momentum tensor. Remember that $T(z) = -\frac{1}{2}:\!\partial\phi(z)\partial\phi(z)\!:$, and that $\langle\phi(z,\bar{z})\phi(w,\bar{w})\rangle = -\log|z-w|^2$. By symmetry, the analogous result holds for the OPE with $\bar{T}(\bar{z})$. Read off from the form of these OPEs that the vertex operators $V_k(z,\bar{z})$ are primary fields ith conformal weights$h = \bar{h} = \frac{k^2}{2}$. Note that V_k has the same conformal weight as V_{-k}.

[P2] *Operator algebra*
Compute the leading term of the OPE of two vertex operators with Wick's theorem or by inserting the mode expansion of the free field and using the Baker-Hausdorff formula. It should yield

$$V_{k'}(z,\bar{z})V_k(w,\bar{w}) = |z-w|^{2kk'}V_{k+k'}(w,\bar{w}) + \ldots$$

Thus, the vertex operators form an operator algebra. The number k is called the charge of the vertex operator. Verify that this is indeed the $U(1)$ charge with respect to the current $J(z) = \mathrm{i}\partial\phi(z,\bar{z})$. The charge of a product of operators is therefore additive.

[P3] *Correlators*
The two-point function of two such vertex operators can be found in many ways, e.g. by exploiting $SL(2,\mathbb{C})$ invariance or the OPE computed above, and turns out as

$$\langle V_k(z,\bar{z})V_{k'}(w,\bar{w})\rangle = (z-w)^{-2\frac{k^2}{2}}(\bar{z}-\bar{w})^{-2\frac{k^2}{2}}\delta_{k+k',0} = |z-w|^{-k^2}\delta_{k+k',0}\,.$$

Argue why the two-point function can only be non-vanishing, if the charge balance condition $k + k' = 0$ is satisfied. *Hint*: The identity operator not only has scaling dimension $h = \bar{h} = 0$, but also must have vanishing charge with respect to $J(z)$.

Argue, using Wick's theorem extensively, that n-point functions of arbitrary vertex operators of the free bosonic CFT can all be computed yielding the quite simple result

$$\langle\prod_i V_{k_i}(z_i,\bar{z}_i)\rangle = \prod_{i>j}|z_j - z_i|^{k_ik_j}\delta_{\sum_i k_i,0}\,,$$

provided $|z_i| > |z_j|$ for $i < j$. Thus, these n-point functions are trivially zero unless the charge balance $\sum_i k_i = 0$ is kept, i.e. total momentum is conserved.

[P4] *Conservation of momentum*
Show explicitly that the charge balance condition arises from momentum conservation. Note that the operator $J(z) = \mathrm{i}\partial\phi(z,\bar{z})$ is a conserved current with zero mode $a_0 = p$, as can be infered from its mode expansion $J(z) = pz^{-1} + \sum_{n\neq 0}a_nz^{-n-1} \equiv \sum_n a_nz^{-n-1}$. Remember that the vacuum was defined in such a way that $\langle 0|p = p|0\rangle = 0$.