Exercises I

October 21st

E1.1 *Explicit addition of angular momenta*

Considering the angular momenta of two particles, there are two complete sets of commuting angular momentum operators: $(J^{(1)})^2, J_3^{(1)}, (J^{(2)})^2, J_3^{(2)}$ with the basis of eigenfunctions $|j_1m_1j_2m_2\rangle \equiv |j_1m_1\rangle \otimes |j_2m_2\rangle$ and $(J^{(1)})^2, (J^{(2)})^2, J^2, J_3$ with the basis of eigenfunctions $|j_1j_2jm\rangle$, where $J = J^{(1)} \otimes \mathbb{1} + \mathbb{1} \otimes J^{(2)}$ is the total angular momentum. The mutual expansion coefficients or Clebsch–Gordan coefficients of the two bases are scalar products of the form $\langle j_1m_1j_2m_2|j_1j_2jm\rangle$.

(1) Show that the scalar produtcs do not vanish only for $m = m_1 + m_2$.

(2) Why is $j_1 + j_2$ the maximum value of j? By counting the states show that $j \geq |j_1 - j_2|$. What are the possible values for the total angular momentum j?

One can demand the coefficients to be non-negative (in particular real) for the maximal values of m and m_1: $\langle j_1j_1j_2(j-j_1)|j_1j_2jj\rangle \geq 0$.

(3) Show that the states with maximal j and m are just $|j_1j_2\ j = j_1 + j_2\ m = \pm j\rangle = |j_1\ \pm j_1\rangle \otimes |j_2\ \pm j_2\rangle$, i.e. the corresponding coefficient is 1.

In particular, consider the addition of two angular momenta $j_1 = j_2 = 1$.

(4) Start with the state $j = m = 2$ constructed in (3) and calculate the other states with $j = 2$ by applying the lowering operator.

(5) Find the state with $j = m = 1$ by constructing a state with $m = 1$ being orthogonal to the state with $j = 2$, $m = 1$. Verify that this state indeed has the total angular momentum $j = 1$.

(6) Start with the state $j = m = 1$ constructed in (5) and calculate the other states with $j = 1$ by applying the lowering operator.

(7) Find the state with $j = m = 0$ by constructing a state with $m = 0$ being orthogonal to the states with $j = 2$, $m = 0$ and $j = 1$, $m = 0$. Verify that this state indeed has the total angular momentum $j = 0$.

(8) Discuss the symmetry properties of the coefficients by means of this example.

A1.2 *Landau levels*

In assignment A4 of the exam Theoretical Physics II the Hamiltonian for the motion (in the $xy-$plane) of a particle with mass m and charge q in a magnetic field $\vec{B} = B\vec{e}_z$ (constant in time, homogeneous in space) was computed. By introducing complex coordinates $z = x + iy$, $\overline{z} = x - iy$ with derivatives $\frac{\partial}{\partial z} = \frac{1}{2}\frac{\partial}{\partial x} - \frac{i}{2}\frac{\partial}{\partial y}$, $\frac{\partial}{\partial \overline{z}} = \frac{1}{2}\frac{\partial}{\partial x} + \frac{i}{2}\frac{\partial}{\partial y}$ raising and lowering operators

$$a = \frac{1}{\sqrt{2}}\left(\frac{z}{2r_0} + 2r_0\frac{\partial}{\partial \overline{z}}\right), \quad a^+ = \frac{1}{\sqrt{2}}\left(\frac{\overline{z}}{2r_0} - 2r_0\frac{\partial}{\partial z}\right)$$

were defined, satisfying the commutation relation $[a, a^+] = 1$ (known from the harmonic oscillator), where $r_0 = \sqrt{\hbar/m\omega}$ is called magnetic length and $\omega = \frac{qB}{mc}$ is the Larmor frequency. The Hamiltonian $H = \hbar\omega(a^+a + aa^+) = \hbar\omega(a^+a + \frac{1}{2})$ also had an oscillator-like form. The angular momentum was then given by $L_3 = \hbar(z\frac{\partial}{\partial z} - \overline{z}\frac{\partial}{\partial \overline{z}})$.

(1) Define

$$b = \frac{1}{\sqrt{2}}\left(\frac{\overline{z}}{2r_0} + 2r_0\frac{\partial}{\partial z}\right), \quad b^+ = \frac{1}{\sqrt{2}}\left(\frac{z}{2r_0} - 2r_0\frac{\partial}{\partial \overline{z}}\right)$$

and show that they satisfy the oscillator-like commutation relation $[b, b^+] = 1$.

(2) Show that a, a^+ and b, b^+ mutually commute, i.e. $[a, b] = 0$ etc.

(3) Show that $L_3 = \hbar(b^+b - a^+a)$.

(4) Show that $a\psi_{0,0} = b\psi_{0,0} = 0$ for $\psi_{0,0} = \frac{1}{\sqrt{\pi}}\,\mathrm{e}^{-|z|^2/4r_0^2}$.

(5) Consider the states $\psi_{n,m} = \frac{1}{\sqrt{n!m!}}(b^+)^m(a^+)^n\psi_{0,0}$. Which energy and which angular momentum do they have? What is the degeneracy of the energy levels?

(6) Which functional form have the states $\psi_{n,m}$, in particular for $n = 0$?

Homework I

Return: October 28th

H1.1 *Clebsch–Gordan coefficients, general case* $(j_1) \otimes (j_2)$

(1) Let $A(j, m) = \sqrt{(j + m)(j - m + 1)}$. Show by applying $J_\pm$ to $|j_1 j_2 j m\rangle$:

$$A(j,m)\langle j_1 m_1 j_2 m_2 | j_1 j_2 j (m-1)\rangle = A(j_1, m_1+1)\langle j_1 (m_1+1) j_2 m_2 | j_1 j_2 j m\rangle + A(j_2, m_2+1)\langle j_1 m_1 j_2 (m_2+1) | j_1 j_2 j m\rangle,$$

$$A(j,m+1)\langle j_1 m_1 j_2 m_2 | j_1 j_2 j (m+1)\rangle = A(j_1, m_1)\langle j_1 (m_1-1) j_2 m_2 | j_1 j_2 j m\rangle + A(j_2, m_2)\langle j_1 m_1 j_2 (m_2-1) | j_1 j_2 j m\rangle.$$

(2) Using the second recursion formula of (1), show that

$$\langle j_1 m_1 j_2 (j-m_1) | j_1 j_2 j j\rangle = (-1)^{j_1 - m_1}\langle j_1 j_1 j_2 (j - j_1) | j_1 j_2 j j\rangle \times \sqrt{\frac{(j_1+m_1)!(j_1+j_2-j)!(j_2+j-m_1)!}{(2j_1)!(j_2-j+m_1)!(j_2-j_1+j)!(j_1-m_1)!}}.$$

(3) Why is the transformation matrix between the two bases unitary? Deduce

$$\sum_{m_1=-j_1}^{j_1}\langle j_1 j_2 j' m | j_1 m_1 j_2 (m-m_1)\rangle\langle j_1 m_1 j_2 (m-m_1) | j_1 j_2 j m\rangle = \delta_{j'j}.$$

(4) With the help of (2), (3) and the convention mentioned in E1.1, show that

$$\langle j_1 j_1 j_2 (j-j_1) | j_1 j_2 j j\rangle = \sqrt{\frac{(2j_1)!(2j+1)!}{(j_1+j_2+j+1)!(j_1-j_2+j)!}}.$$

Use the following identity, arising from addition theorems for binomial coefficients:

$$\sum_{n\in\mathbb{Z},\, n\geq -a,\, n\geq -c,\, n\leq b,\, n\leq d}\frac{(a+n)!(b-n)!}{(c+n)!(d-n)!} = \frac{(a+b+1)!(a-c)!(b-d)!}{(c+d)!(a+b-c-d+1)!}.$$

(5) Using the first recursion formula of (1) as well as (2) and (4), show that

$$\langle j_1 m_1 j_2 m_2 | j_1 j_2 j m\rangle = \sqrt{\frac{(2j+1)(j_1+j_2-j)!(j_1-m_1)!(j_2-m_2)!(j+m)!}{(j_1+j_2+j+1)!(j_1-j_2+j)!(j_2-j_1+j)!(j_1+m_1)!(j_2+m_2)!(j-m)!}} \times\delta_{m,m_1+m_2}\sum_{n=0}^{j-m}(-1)^{j_1-m_1-n}\binom{j-m}{n}\frac{(j_1+m_1+n)!(j_2+j-m_1-n)!}{(j_1-m_1-n)!(j_2-j+m_1+n)!}.$$

(6) Compare the coefficients with the ones for $(\ell)\otimes\left(\frac{1}{2}\right)$ from Theoretical Physics II, H9.1 (Spin-orbit coupling), $\left(\frac{1}{2}\right)\otimes\left(\frac{1}{2}\right)$ from Theoretical Physics II, E10.1 (Helium atom) and the case $(1)\otimes(1)$ computed in E1.1.

(30 points)

Website for this lecture: http://www.th.physik.uni–bonn.de/th/People/wisskirc/tp3