Exercises X

January 6th

E10.1 *Other representations of the Dirac equation*

For a four component spinor ψ_{D}, the Dirac equation takes the form

$$\left(i\gamma^\mu \frac{\partial}{\partial x^\mu} - \frac{mc}{\hbar}\right)\psi_{\mathrm{D}} = 0,$$

where the γ matrices read in standard representation (σ^i are the Pauli matrices):

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}.$$

However, this form of the matrices is only a choice, since they only have to satisfy

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}\mathbb{1}.$$

If γ^μ satisfy this *Clifford algebra* then also all similarity transforms $U\gamma^\mu U^{-1}$ will do.

(1) Construct an $U \in SU(4)$, such that the two component *Weyl spinors* $\psi, \widehat{\psi}$ solve

$$\begin{pmatrix} 0 & W \\ \widehat{W} & 0 \end{pmatrix}\begin{pmatrix} \psi \\ \widehat{\psi} \end{pmatrix} + \frac{mc}{\hbar}\begin{pmatrix} \psi \\ \widehat{\psi} \end{pmatrix} = 0,$$

if $\psi_{\mathrm{D}} = U\begin{pmatrix} \psi \\ \widehat{\psi} \end{pmatrix}$ solves the Dirac equation. $W, \widehat{W}$ are the *Weyl operators*:

$$W = \mathbb{1}\frac{\partial}{\partial x^0} - \vec{\sigma}(\vec{\nabla}), \quad \widehat{W} = -\mathbb{1}\frac{\partial}{\partial x^0} - \vec{\sigma}(\vec{\nabla}).$$

The following form of the Dirac equation was found by E. Majorana in 1937.

(2) Show that the matrix $U = \frac{1}{\sqrt{2}}\gamma^0(\gamma^2 + \mathbb{1})$ is unitary and that $U^2 = \mathbb{1}$.

(3) Show that $D' = UDU$ is real and that $\psi' = U\psi$ solves the equation $D'\psi' = \frac{mc}{\hbar}\psi'$, if ψ solves the Dirac equation $D\psi = \frac{mc}{\hbar}\psi$, with the Dirac operator $D = \mathrm{i}\gamma^\mu\partial_\mu$.

E10.2 *The Weyl equation*

The *Poincaré group* $\mathcal{P}$ consists of pairs (Λ, a) with Lorentz transformations $\Lambda \in \mathcal{L}$ and translations $a \in \mathbb{R}^{1,3}$. It operates on the Minkowski space as $(\Lambda, a)x = \Lambda x + a$.

(1) Show $(\Lambda_1, a_1)(\Lambda_2, a_2) = (\Lambda_1\Lambda_2, \Lambda_1 a_2 + a_1)$ for the composition of two elements of $\mathcal{P}$. Due to that property, $\mathcal{P}$ is called a *semi direct* product $\mathcal{L} \rtimes \mathbb{R}^{1,3}$.

Now, we want to construct the spin $\frac{1}{2}$ representations of $\mathcal{P}$ on the quantum mechanical Hilbert space $\mathcal{L}^2(\mathbb{R}^{1,3}, \mathbb{C}^2)$. Analogously to the spin $\frac{1}{2}$ representation of the rotation group we start with the covering group, i.e. $SL(2,\mathbb{C}) \rtimes \mathbb{R}^{1,3}$. Here, $h : SL(2,\mathbb{C}) \to \mathcal{L}$ is the covering map for $\mathcal{L}$ (cf. H9.1). We define the representations ρ and $\widehat{\rho}$ of $SL(2,\mathbb{C}) \rtimes \mathbb{R}^{1,3}$ with spin $\frac{1}{2}$ for $\psi, \widehat{\psi} : \mathbb{R}^{1,3} \to \mathbb{C}^2$ as

$$(\rho(g,a)\psi)(x) = g\psi(\Lambda^{-1}(x-a)), \quad h(g) = \Lambda \qquad (*)$$
$$(\widehat{\rho}(g,a)\widehat{\psi})(x) = \widehat{g}\,\widehat{\psi}(\Lambda^{-1}(x-a)), \quad h(g) = \Lambda, \quad \widehat{g} = (g^+)^{-1}.$$

ρ is defined in the same way as for rotations. $\widehat{\rho}$ is called *conjugate representation* and is something new: $g = \widehat{g}$ für $g \in SU(2)$, such that $\rho(g,a) = \widehat{\rho}(g,a)$, if $\Lambda = h(g)$ is a rotation. Spinors ψ and $\widehat{\psi}$ transforming according to $(*)$ are called *Weyl spinors*.

(2) Show that ρ as well as $\widehat{\rho}$ are representations of the covering group of $\mathcal{P}$.

(3) Show the *intertwining relations* $W\widehat{\rho}(g,a) = \rho(g,a)W$ und $\widehat{W}\rho(g,a) = \widehat{\rho}(g,a)\widehat{W}$.

(4) Show using (3): if $\widehat{\psi}(x)$ is a solution of the *Weyl equation* $W\widehat{\psi} = 0$, then also $\widehat{\rho}(g,a)\widehat{\psi}$ will be and if $\psi(x)$ is a solution of the *conjugate Weyl equation* $\widehat{W}\psi = 0$, then also $\rho(g,a)\psi$ will be. This means that the solutions of the two Weyl equations both form a Poincaré invariant subspace of $\mathcal{L}^2(\mathbb{R}^{1,3},\mathbb{C}^2)$.

(5) Show $W\widehat{W} = \widehat{W}W = -g^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu} = -\Box$ and deduce that solutions of the Weyl equations satisfy the Klein–Gordon equation with $m = 0$.

(6) Show that $P^{-1}WP = -\widehat{W}$, i.e. the Weyl operators are *not* invariant under parity.

(7) Take the ansatz $\widehat{\psi} = \widehat{u}_0\,\mathrm{e}^{-\mathrm{i}\langle p,x\rangle}$. Show that $\langle p,p\rangle = 0$. Show that for positive energy spin and momentum are antiparallel, but parallel for negative energy.

(8) Analogously, take the ansatz $\psi = u_0\,\mathrm{e}^{-i\langle p,x\rangle}$. What about the directions of spin and momentum? What is the connection between the solution of the two Weyl equations? Which elementary particle do they describe in good approximation?

If one wants to introduce a mass, one will have to consider the coupled system

$$\begin{pmatrix} 0 & W \\ \widehat{W} & 0 \end{pmatrix}\begin{pmatrix} \psi \\ \widehat{\psi} \end{pmatrix} + \frac{mc}{\hbar}\begin{pmatrix} \psi \\ \widehat{\psi} \end{pmatrix} = 0.$$

(9) State why one cannot write down two decoupled equations for two component spinors in the case of nonvanishing mass, but one has to consider a four dimensional spinor instead.

(10) Show: if ψ and $\widehat{\psi}$ solve the system above, then also $\rho(g,a)\psi$ and $\widehat{\rho}(g,a)\widehat{\psi}$ will do. Show that ψ and $\widehat{\psi}$ satisfy the Klein–Gordon equation with mass m.

Homework X

Return: January 13th

H10.1 *Pionic atom*

Consider the Klein–Gordon equation for a spinless particle with mass m and charge $(-e)$ in the Coulomb field of a point-like charge Ze. We are interested in the stationary states, thus we take the ansatz $\psi(\vec{r},t) = \psi(\vec{r})\,\mathrm{e}^{-\frac{iE}{\hbar}t}$.

(1) Show

$$\left(\left(E + \frac{Ze^2}{r}\right)^2 - m^2c^4 + \hbar^2c^2\Delta\right)\psi(\vec{r}) = 0.$$

(2) Using the separation $\psi(x) = R_\ell(r)Y_{\ell m}(\vartheta,\varphi)$, show that with the variable $\varrho = \beta r$

$$\left(\frac{\partial^2}{\partial\varrho^2} + \frac{2}{\varrho}\frac{\partial}{\partial\varrho} - \frac{s(s+1)}{\varrho^2} + \frac{\lambda}{\varrho} - \frac{1}{4}\right)R_\ell(r) = 0$$

holds, where $\beta^2 = \frac{4(m^2c^4-E^2)}{\hbar^2c^2}$, $\lambda = \frac{2Z\alpha E}{\hbar c\beta}$, $s(s+1) = \ell(\ell+1) - Z^2\alpha^2$ and $\alpha = \frac{e^2}{\hbar c}$.

(3) Motivate the ansatz $R_\ell = \varrho^s W(\varrho)\,\mathrm{e}^{-\varrho/2}$ und zeige

$$\varrho W''(\varrho) + (2s + 2 - \varrho)W'(\varrho) + (\lambda - s - 1)W(\varrho) = 0.$$

(4) Compare (3) with Kummer's equation for the nonrelavistic Coulomb problem (TP II, E7.1) and read off the energy eigenvalues. Why does $E > 0$ hold?

(5) Expand the energy eigenvalues in orders of α and compare the result with the nonrelativistic case. In particular, discuss the cancellation of the ℓ-degeneracy.

(6) Estimate in how far one can assume a point-like nucleus for a π^- meson.

(7) Discuss the solutions for $E < 0$. When is it possible that bound states exist?

Rem.: One can handle this problem in that way, since for massive particles the little group is $SO(3)$ and thus the solutions are labelled by angular momentum.

(15 points)

H10.2 *Nonrelativistic limit of the Dirac equation*

In this assignment we show how the spin–orbit coupling (TP II, H9.2) and the g factor 2 (H2.1) for spin $\frac{1}{2}$ follow from the Dirac equation in a natural way.

(1) In the nonrelativistic case mc^2 is the largest energy in the problem. This motivates the ansatz $\psi = \exp(-\mathrm{i}\frac{mc^2}{\hbar}t)\binom{\varphi}{\chi}$, where φ and χ only slightly depend on time. Dirac equation: Using this, deduce from the Schrödinger–like form of the

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\begin{pmatrix}\varphi\\ \chi\end{pmatrix} = c\begin{pmatrix}\vec{\sigma}(\vec{p}-\frac{q}{c}\vec{A})\chi\\ \vec{\sigma}(\vec{p}-\frac{q}{c}\vec{A})\varphi\end{pmatrix} + q\Phi\begin{pmatrix}\varphi\\ \chi\end{pmatrix} - 2mc^2\begin{pmatrix}0\\ \chi\end{pmatrix}. \qquad (*)$$

Show that in lowest order, i.e. $\frac{1}{c}$, one has: $\chi = \frac{1}{2mc}\vec{\sigma}(\vec{p}-\frac{q}{c}\vec{A})\varphi$.

(2) Deduce the *Pauli equation* (why is it exact in order $\frac{1}{c}$?) for φ from (1):

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\varphi = \left(\frac{1}{2m}(\vec{p}-\frac{q}{c}\vec{A})^2 - \frac{q\hbar}{2mc}\vec{\sigma}(\vec{B}) + q\Phi\right)\varphi.$$

Show by taking $\vec{A} = \frac{1}{2}\vec{B}\times\vec{r}$ that the g factor for the spin is $g = 2$.

(3) For $\vec{A} = 0$ we would like to compute the $\frac{1}{c^2}$ corrections. For that purpose, plug the expression for χ from (1) into $(*)$ for the expressions neglected there and resolve this in favour of χ. Insert the result into the first equation of $(*)$:

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\varphi = \left(\left(1 - \frac{\vec{p}^2}{4m^2c^2}\right)\left(\frac{\vec{p}^2}{2m} + q\Phi\right) + \frac{q}{4m^2c^2}\vec{\sigma}(\vec{p})\Phi\,\vec{\sigma}(\vec{p})\right)\varphi =: H'\varphi.$$

Why does one obtains all corrections of order $\frac{1}{c^2}$ in this way?

(4) Why is the probability density $\varrho = |\varphi|^2 + |\chi|^2$ in order $\frac{1}{c^2}$ now given by $\varrho = |\varphi|^2 + \frac{\hbar^2}{4m^2c^2}|\vec{\sigma}(\vec{\nabla})\varphi|^2$? In order to proceed in the Schrödinger picture, we have to introduce a new φ_S such that $\varrho = |\varphi_S|^2$. Show that $\varphi_S = (1 + \frac{\vec{p}^2}{8m^2c^2})\varphi$.

(5) Why does $\mathrm{i}\hbar\frac{\partial}{\partial t}\varphi_S = H\varphi_S$ hold with $H = (1 + \frac{\vec{p}^2}{8m^2c^2})H'(1 - \frac{\vec{p}^2}{8m^2c^2})$? Show that

$$H = \frac{\vec{p}^2}{2m} + q\Phi + \widetilde{H}, \quad \widetilde{H} = \underbrace{-\frac{|\vec{p}|^4}{8m^3c^2}}_{\text{dispersion term}}\ \underbrace{-\frac{q\hbar}{4m^2c^2}\,\vec{\sigma}(\vec{E}\times\vec{p})}_{\text{spin-orbit term}}\underbrace{-\frac{e\hbar^2}{8m^2c^2}\,\mathrm{div}\vec{E}}_{\text{G.Darwin's term}}.$$

Interpret the dispersion and the spin–orbit term (for spherically symmetric Φ).

(15 points)