E11.1 \textit{Dirac invariants}

The covering map $h : SL(2, \mathbb{C}) \to L_+^\mathbb{C}$, $g \mapsto \rho(g) = \Lambda$ was defined in such a way that

\[ g\sigma_\mu g^+ = \Lambda^\nu_\mu \sigma_\nu, \quad \mu = 0, 1, 2, 3 \]

holds. For $\hat{\sigma}_0 = \sigma_0$, $\hat{\sigma}_k = -\sigma_k$ the following relation is satisfied (cf. E10.2.3):

\[ (g^+)^{-1} \hat{\sigma}_\mu g^{-1} = \Lambda^\nu_\mu \hat{\sigma}_\nu, \quad \mu = 0, 1, 2, 3. \]

Thus the Poincaré group can be represented on 2-spinors $\xi, \eta$ in two different ways:

\[ \rho(g, a)\xi = g\xi(\Lambda^{-1}(x - a)), \quad \hat{\rho}(g, a)\eta = (g^+)^{-1}\eta(\Lambda^{-1}(x - a)). \]

If $\psi = (\xi \eta)$ solves the Dirac equation $i\gamma^\mu \frac{\partial}{\partial x^\mu} \psi = \frac{mc}{\hbar} \psi$ also $D(g, a)\psi$ will do, where

\[ D(g, a)\psi = \begin{pmatrix} \rho \xi \\ \hat{\rho} \eta \end{pmatrix} = \begin{pmatrix} g & 0 \\ 0 & (g^+)^{-1} \end{pmatrix} \psi(\Lambda^{-1}(x - a)) = S(g)\psi(\Lambda^{-1}(x - a)). \]

In this Weyl representation the $\gamma$ matrices have the special form

\[ \gamma^0 = i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \gamma^k = -i \begin{pmatrix} 0 & \sigma^k \\ \sigma^k & 0 \end{pmatrix}. \]

(1) Show $\gamma^5 := i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ in this representation. Deduce $[S(g), \gamma^5] = 0$.

(2) Deduce from the definition of the covering map that $S(g)^{-1}\gamma^\mu S(g) = \Lambda^\mu_\nu \gamma^\nu$.

(3) Show that $S(g)^+\gamma^0 S(g) = \gamma^0$.

We define the \textit{adjoint spinor}

\[ \overline{\psi}(x) = \psi^+(x)\gamma^0 \]

and hence the following bilinear (why are they real?) expressions:

<table>
<thead>
<tr>
<th>Type</th>
<th>Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>scalar</td>
<td>$s = \overline{\psi}(x)\psi(x)$</td>
</tr>
<tr>
<td>pseudo scalar</td>
<td>$\overline{s} = i\overline{\psi}(x)\gamma^5\psi(x)$</td>
</tr>
<tr>
<td>vector current</td>
<td>$j^\mu = \overline{\psi}(x)\gamma^\mu \psi(x)$</td>
</tr>
<tr>
<td>axial vector current</td>
<td>$\overline{j}^\mu = \overline{\psi}(x)\gamma^5\gamma^\mu \psi(x)$</td>
</tr>
<tr>
<td>antisymmetric tensor</td>
<td>$B^{\mu\nu} = \overline{\psi}(x)\gamma^\mu \gamma^\nu \psi(x)$ ((\mu &gt; \nu))</td>
</tr>
</tbody>
</table>

(4) Check that the expressions above indeed transform as scalar, vector and tensor, respectively, under proper orthocron Lorentz transformations.
We define the \textit{parity} \( P \), the \textit{time reversal} \( T \) and the \textit{charge conjugation} \( C \) as

\[
\begin{align*}
P \psi(x) &= \gamma^0 \psi(Px) \\
T \psi(x) &= \gamma^2 \gamma^0 \psi^*(Tx) = \begin{pmatrix} -\sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix} \psi^*(Tx) \\
C \psi(x) &= \gamma^2 \gamma^5 \psi^*(x)
\end{align*}
\]

(5) Show that \( P \psi \) will solve the Dirac equation if \( \psi \) is a solution.

(6) Compute the action of \( P \) on \( \psi \). How do \( s \) and \( j \) distinguish from \( \bar{s} \) and \( \bar{j} \)?

(7) Show that time reversed solutions of the Weyl equations (i.e. \( \tilde{\psi} \tilde{W} \eta = 0, \tilde{\psi} \tilde{W} \xi = 0 \) for \( m = 0 \)) are again solutions (as we learned in E10.2.6, this is not true for \( P \)). Deduce that \( T \psi \) will be a solution of the Dirac equation if \( \psi \) solves the equation.

(8) Show that the charge conjugated spinor \( C \psi \) will be a solution of the Dirac equation if \( \psi \) solves the equation. How will it look like if an electromagnetic field is present?

(9) Calculate \( PT \) and discuss this transformation.

(10) Compute the action of \( T \) and \( C \) on \( \psi \).

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**Homework XI**

Return: January 20th

**H11.1 Dirac equation in a spherically symmetric electric field** (30 points)

This problem was solved independently by G. Darwin and W. Gordon in 1928. A particle with spin \( \frac{1}{2} \), charge \( q \) and mass \( m \) moves in a spherically symmetric electric field \( (qA_0 = V(r), \tilde{A} = 0) \). In general, the Dirac equation reads

\[
i \gamma^\mu \left( \frac{\partial}{\partial x^\mu} + i \frac{q}{\hbar c} A_\mu \right) \psi = \frac{mc}{\hbar} \psi
\]

and can be converted to the form

\[
i \frac{\partial}{\partial x^\mu} \psi = -i \sum_{k=1}^3 \alpha^k \left( \frac{\partial}{\partial x^k} + i \frac{q}{\hbar c} A_k \right) + \beta \frac{mc}{\hbar} + \frac{1}{\hbar c} V(r) \psi.
\]

(1) Show that the matrices \( \alpha^k \) and \( \beta \) can be expressed by the \( \gamma^\mu \): \( \alpha^k = \gamma^0 \gamma^k, \beta = \gamma^0 \). Compute \( \alpha^k \alpha^k + \alpha^k \alpha^k, \beta^2 \) and \( \beta \alpha^k + \alpha^k \beta \).

(2) Take the ansatz \( \psi(x) = \psi_0(\vec{r}) e^{-i E t / \hbar} \) and deduce an eigenvalue equation for \( \psi_0(\vec{r}) \):

\[
H \psi = \left( -i \hbar c \sum_{k=1}^3 \alpha^k \frac{\partial}{\partial x^k} + \beta mc^2 + V(r) \right) \psi_0(\vec{r}) = E \psi_0(\vec{r}).
\]

(3) Show: if \( \psi_0(\vec{r}) \) is a solution of this equation then also \( P \psi_0(\vec{r}) = \beta P \psi_0(\vec{r}) = \beta \psi_0(-\vec{r}) \) will be. What is the meaning of \( P \)? Why can one assume \( P \psi_0 = \pm \psi_0 \)?

(4) Show that the total angular momentum operator \( \vec{J} \) commutes with \( H \) and \( P \).
According to (3) and (4) the eigenfunctions of $H$ can be chosen as eigenfunctions of $\hat{J}^2$, $J_3$ and $\mathcal{P}$. For the spinor spherical harmonics $\Phi_{\ell jm}$ one has (cf. TP II, H9.1)

$$\hat{J}^2 \Phi_{\ell jm} = \hbar^2 j(j+1) \Phi_{\ell jm}, \quad J_3 \Phi_{\ell jm} = \hbar m_j \Phi_{\ell jm}, \quad \mathcal{P} \Phi_{\ell jm} = (-1)^\ell \Phi_{\ell jm},$$

For one value of $j$ there are two values of $\ell$: $j \pm \frac{1}{2}$, i.e. two different parities in space.

(5) Show that the eigenfunctions of $\hat{J}^2$, $J_3$ and $\mathcal{P}$ must have the following form:

$$\psi_0(\vec{r}) = \left( \Phi_{\ell = j+\frac{1}{2}, jm} \times \text{radial function } F(r) \right),$$

Two more properties of $\Phi_{\ell jm}$ shown in TP II, H9.1: given a spinor spherical harmonic $\Phi_{\ell jm}$ for fixed $j$ and $m_j$, then one will obtain the one with the opposite parity by

$$\Phi_{\ell' = j \pm \frac{1}{2}, jm} = \sigma \left( \frac{\vec{r}}{r} \right) \Phi_{\ell = j \mp \frac{1}{2}, jm}.$$

Furthermore, the spinor spherical harmonics are eigenfunctions of $K = 1 + \frac{\hbar}{\mc} (\hat{L})$:

$$K \Phi_{\ell jm} = \kappa \Phi_{\ell jm}, \quad \kappa = \begin{cases} \ell + 1, & \text{if } \ell = j - \frac{1}{2} \\ -\ell, & \text{if } \ell = j + \frac{1}{2} \end{cases}$$

(6) Using these properties, show that by taking the ansatz from (5) the radial functions $F(r) = f(r)$ and $G(r) = -ig(r)$ satisfy the following differential equations:

$$\hbar c \left( f'(r) + \frac{1 - \kappa}{r} f(r) \right) - (E + mc^2 - V(r))g(r) = 0,$$

$$\hbar c \left( g'(r) + \frac{1 + \kappa}{r} g(r) \right) + (E - mc^2 - V(r))f(r) = 0.$$

(7) Now, let $V(r) = -\frac{Ze^2}{r} = -\hbar c \frac{Z\alpha}{r}$. Show that $f, g \sim e^{-\lambda r}$ for $r \to \infty$ with $\lambda = \frac{1}{\hbar c} \sqrt{m^2 c^4 - E^2}$ and that $f, g \sim r^{\gamma - 1}$ for $r \to 0$ with $\gamma = \sqrt{\kappa^2 - (Z\alpha)^2}$.

(8) Using the variable $q = \lambda r$, consider new radial functions $\varphi_{\pm q}(q)$, defined by

$$\begin{pmatrix} f(q) \\ g(q) \end{pmatrix} = V_+ \varphi_+(q) + V_- \varphi_-(q), \quad V_+ = (\pm b - 1), \quad b = \frac{mc^2 + E}{\sqrt{m^2 c^4 - E^2}} = \sqrt{\frac{mc^2 + E}{m^2 c^4 - E^2}}.$$

Show

$$\left( \frac{\partial}{\partial q} + 1 + \frac{Z\alpha}{2} (b - b^{-1}) - q \right) \varphi_+ + \left( -\kappa + \frac{Z\alpha}{2} (b + b^{-1}) \right) \varphi_- = 0,$$

$$\left( \frac{\partial}{\partial q} + 1 - \frac{Z\alpha}{2} (b - b^{-1}) + q \right) \varphi_- - \left( -\kappa - \frac{Z\alpha}{2} (b + b^{-1}) \right) \varphi_+ = 0.$$

(9) Eliminate $\varphi_-$ in (8) and show for the ansatz $\varphi_+ = q^{\gamma - 1} e^{-q} \varphi(y)$ (motivated by the investigation of the asymptotics in (7)) with $y = 2q$ that

$$\left( y \frac{d^2}{dy^2} + (2\gamma + 1 - y) \frac{d}{dy} - \left( 1 + \gamma - \frac{Z\alpha}{2} (1 - b^{-1}) \right) \right) \varphi(y) = 0.$$

(10) Read off the energy eigenvalues from this Kummer equation:

$$E = mc^2 \left( 1 + \left( \frac{Z\alpha}{n + \gamma} \right)^2 \right)^{-\frac{1}{2}}, \quad n \in \mathbb{N}$$

and compare with the nonrelativistic result. Which degenerations do still occur?