Exercises XI

January 13th

E11.1 *Dirac invariants*

The covering map $h : SL(2,\mathbb{C}) \to \mathcal{L}_+^\uparrow$, $g \mapsto \rho(g) = \Lambda$ was defined in such a way that

$$g\sigma_\mu g^+ = \Lambda^\nu{}_\mu \sigma_\nu, \quad \mu = 0,1,2,3$$

holds. For $\widehat{\sigma}_0 = \sigma_0$, $\widehat{\sigma}_k = -\sigma_k$ the following relation is satisfied (cf. E10.2.3):

$$(g^+)^{-1}\widehat{\sigma}_\mu g^{-1} = \Lambda^\nu{}_\mu \widehat{\sigma}_\nu, \quad \mu = 0,1,2,3.$$

Thus the Poincaré group can be represented on 2–spinors ξ, η in two different ways:

$$\rho(g,a)\xi = g\xi(\Lambda^{-1}(x-a)), \quad \widehat{\rho}(g,a)\eta = (g^+)^{-1}\eta(\Lambda^{-1}(x-a)).$$

If $\psi = \binom{\xi}{\eta}$ solves the Dirac equation $\mathrm{i}\gamma^\mu \frac{\partial}{\partial x^\mu}\psi = \frac{mc}{\hbar}\psi$ also $D(g,a)\psi$ will do, where

$$D(g,a)\psi = \begin{pmatrix} \rho\,\xi \\ \widehat{\rho}\,\eta \end{pmatrix} = \begin{pmatrix} g & 0 \\ 0 & (g^+)^{-1} \end{pmatrix} \psi(\Lambda^{-1}(x-a)) = S(g)\psi(\Lambda^{-1}(x-a)).$$

In this Weyl representation the γ matrices have the special form

$$\gamma^0 = \mathrm{i}\begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}, \quad \gamma^k = -\mathrm{i}\begin{pmatrix} 0 & \sigma^k \\ \sigma^k & 0 \end{pmatrix}.$$

(1) Show $\gamma^5 := \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3 = \left(\begin{smallmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{smallmatrix}\right)$ in this representation. Deduce $[S(g), \gamma^5] = 0$.
(2) Deduce from the definition of the covering map that $S(g)^{-1}\gamma^\mu S(g) = \Lambda^\mu{}_\nu \gamma^\nu$.
(3) Show that $S(g)^+\gamma^0 S(g) = \gamma^0$.

We define the *adjoint spinor*

$$\overline{\psi}(x) = \psi^+(x)\gamma^0$$

and hence the following bilinear (why are they real?) expressions:

$$\begin{array}{rl} \text{scalar} & s = \overline{\psi}(x)\psi(x) \\ \text{pseudo scalar} & \widetilde{s} = \mathrm{i}\overline{\psi}(x)\gamma^5\psi(x) \\ \text{vector current} & j^\mu = \overline{\psi}(x)\gamma^\mu\psi(x) \\ \text{axial vector current} & \widetilde{j}^\mu = \overline{\psi}(x)\gamma^5\gamma^\mu\psi(x) \\ \text{antisymmetric tensor} & B^{\mu\nu} = \overline{\psi}(x)\gamma^\mu\gamma^\nu\psi(x) \quad (\mu > \nu) \end{array} \qquad (*)$$

(4) Check that the expressions above indeed transform as scalar, vector and tensor, respectively, under proper orthocrone Lorentz transformations.

We define the *parity* $\mathcal{P}$, the *time reversal* $\mathcal{T}$ and the *charge conjugation* $\mathcal{C}$ as

$$\begin{aligned}
\mathcal{P}\psi(x) &= \gamma^0\psi(Px) \\
\mathcal{T}\psi(x) &= \gamma^2\gamma^0\psi^*(Tx) = \begin{pmatrix} -\sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix}\psi^*(Tx) \\
\mathcal{C}\psi(x) &= \gamma^2\gamma^5\psi^*(x)
\end{aligned}$$

(5) Show that $\mathcal{P}\psi$ will solve the Dirac equation if ψ is a solution.
(6) Compute the action of $\mathcal{P}$ on $(*)$. How do s and j distinguish from $\widetilde{s}$ and $\widetilde{j}$?
(7) Show that time reversed solutions of the Weyl equations (i.e. $W\eta = 0$, $\widehat{W}\xi = 0$ for $m = 0$) are again solutions (as we learned in E10.2.6, this is not true for $\mathcal{P}$). Deduce that $\mathcal{T}\psi$ will be a solution of the Dirac equation if ψ solves the equation.
(8) Show that the charge conjugated spinor $\mathcal{C}\psi$ will be a solution of the Dirac equation if ψ solves the equation. How will it look like if an electromagnetic field is present?
(9) Calculate $\mathcal{PTC}$ and discuss this transformation.
(10) Compute the action of $\mathcal{T}$ and $\mathcal{C}$ on $(*)$.

Homework XI

Return: January 20th

H11.1 *Dirac equation in a spherically symmetric electric field* (30 points)

This problem was solved independently by G. Darwin and W. Gordon in 1928. A particle with spin $\frac{1}{2}$, charge q and mass m moves in a spherically symmetric electric field ($qA_0 = V(r)$, $\vec{A} = 0$). In general, the Dirac equation reads

$$\mathrm{i}\gamma^\mu\left(\frac{\partial}{\partial x^\mu} + \mathrm{i}\frac{q}{\hbar c}A_\mu\right)\psi = \frac{mc}{\hbar}\psi$$

and can be converted to the form

$$\mathrm{i}\frac{\partial}{\partial x^0}\psi = \left(-\mathrm{i}\sum_{k=1}^{3}\alpha^k\left(\frac{\partial}{\partial x^k} + \mathrm{i}\frac{q}{\hbar c}A_k\right) + \beta\frac{mc}{\hbar} + \frac{1}{\hbar c}V(r)\right)\psi.$$

(1) Show that the matrices α^k and β can be expressed by the γ^μ: $\alpha^k = \gamma^0\gamma^k$, $\beta = \gamma^0$. Compute $\alpha^k\alpha^\ell + \alpha^\ell\alpha^k$, β^2 and $\beta\alpha^k + \alpha^k\beta$.
(2) Take the ansatz $\psi(x) = \psi_0(\vec{r})\,\mathrm{e}^{-\mathrm{i}\frac{Et}{\hbar}}$ and deduce an eigenvalue equation for $\psi_0(\vec{r})$:

$$H\psi = \left(-i\hbar c\sum_{k=1}^{3}\alpha^k\frac{\partial}{\partial x^k} + \beta mc^2 + V(r)\right)\psi_0(\vec{r}) = E\psi_0(\vec{r}).$$

(3) Show: if $\psi_0(\vec{r})$ is a solution of this equation then also $\mathcal{P}\psi_0(\vec{r}) = \beta P\psi_0(\vec{r}) = \beta\psi_0(-\vec{r})$ will be. What is the meaning of $\mathcal{P}$? Why can one assume $\mathcal{P}\psi_0 = \pm\psi_0$?
(4) Show that the total angular momentum operator $\vec{J}$ commutes with H and $\mathcal{P}$.

According to (3) and (4) the eigenfunctions of H can be chosen as eigenfunctions of $\vec{J}^2$, J_3 and $\mathcal{P}$. For the spinor spherical harmonics $\Phi_{\ell j m_j}$ one has (cf. TP II, H9.1)

$$\vec{J}^2\Phi_{\ell j m_j} = \hbar^2 j(j+1)\Phi_{\ell j m_j}, \quad J_3\Phi_{\ell j m_j} = \hbar m_j \Phi_{\ell j m_j}, \quad P\Phi_{\ell j m_j} = (-1)^\ell \Phi_{\ell j m_j}.$$

For one value of j there are two values of ℓ: $j \pm \frac{1}{2}$, i.e. two different parities in space.

(5) Show that the eigenfunctions of $\vec{J}^2$, J_3 and $\mathcal{P}$ must have the following form:

$$\psi_0(\vec{r}) = \begin{pmatrix} \Phi_{\ell=j\mp\frac{1}{2}\, jm_j} \times \text{radial function } F(r) \\ \Phi_{\ell'=j\pm\frac{1}{2}\, jm_j} \times \text{radial function } G(r) \end{pmatrix}.$$

Two more properties of $\Phi_{\ell j m_j}$ shown in TP II, H9.1: given a spinor spherical harmonic $\Phi_{\ell j m_j}$ for fixed j and m_j, then one will obtain the one with the opposite parity by

$$\Phi_{\ell'=j\pm\frac{1}{2}\, jm_j} = \vec{\sigma}\left(\frac{\vec{r}}{r}\right)\Phi_{\ell=j\mp\frac{1}{2}\, jm_j}.$$

Furthermore, the spinor spherical harmonics are eigenfunctions of $K = 1 + \frac{1}{\hbar}\vec{\sigma}(\vec{L})$:

$$K\Phi_{\ell j m_j} = \kappa\Phi_{\ell j m_j}, \quad \kappa = \begin{cases} \ell+1, & \text{if } \ell = j - \frac{1}{2} \\ -\ell, & \text{if } \ell = j + \frac{1}{2} \end{cases}$$

(6) Using these properties, show that by taking the ansatz from (5) the radial functions $F(r) = f(r)$ and $G(r) = -\mathrm{i}g(r)$ satisfy the following differential equations:

$$\hbar c\left(f'(r) + \frac{1-\kappa}{r}f(r)\right) - (E + mc^2 - V(r))g(r) = 0$$

$$\hbar c\left(g'(r) + \frac{1+\kappa}{r}g(r)\right) + (E - mc^2 - V(r))f(r) = 0.$$

(7) Now, let $V(r) = -\frac{Ze^2}{r} = -\hbar c\frac{Z\alpha}{r}$. Show that $f, g \sim \mathrm{e}^{-\lambda r}$ for $r \to \infty$ with $\lambda = \frac{1}{\hbar c}\sqrt{m^2c^4 - E^2}$ and that $f, g \sim r^{\gamma-1}$ for $r \to 0$ with $\gamma = \sqrt{\kappa^2 - (Z\alpha)^2}$.

(8) Using the variable $\varrho = \lambda r$, consider new radial functions $\varphi_\pm(\varrho)$, defined by $\binom{f(\varrho)}{g(\varrho)} = V_+\varphi_+(\varrho) + V_-\varphi_-(\varrho)$, $V_\pm = \binom{1}{\pm b^{-1}}$, $b = \frac{mc^2+E}{\sqrt{m^2c^4-E^2}} = \sqrt{\frac{mc^2+E}{mc^2-E}}$. Show

$$\left(\varrho\frac{\mathrm{d}}{\mathrm{d}\varrho} + 1 + \frac{Z\alpha}{2}(b - b^{-1}) - \varrho\right)\varphi_+ + \left(-\kappa + \frac{Z\alpha}{2}(b + b^{-1})\right)\varphi_- = 0$$

$$\left(\varrho\frac{\mathrm{d}}{\mathrm{d}\varrho} + 1 - \frac{Z\alpha}{2}(b - b^{-1}) + \varrho\right)\varphi_- - \left(-\kappa - \frac{Z\alpha}{2}(b + b^{-1})\right)\varphi_+ = 0.$$

(9) Eliminate φ_- in (8) and show for the ansatz $\varphi_+ = \varrho^{\gamma-1}\mathrm{e}^{-\varrho}\varphi(y)$ (motivated by the investigation of the asymptotics in (7)) with $y = 2\varrho$ that

$$\left(y\frac{\mathrm{d}^2}{\mathrm{d}y^2} + (2\gamma + 1 - y)\frac{\mathrm{d}}{\mathrm{d}y} - \left(1 + \gamma - \frac{Z\alpha}{2}(1 - b^{-1})\right)\right)\varphi(y) = 0.$$

(10) Read off the energy eigenvalues from this Kummer equation:

$$E = mc^2\left(1 + \left(\frac{Z\alpha}{n+\gamma}\right)^2\right)^{-\frac{1}{2}}, \quad n \in \mathbb{N}$$

and compare with the nonrelativistic result. Which degenerations do still occur?