Exercises XIII

January 27th

E13.1 *The Lamb Shift*

If one solves the Dirac equation in the Coulomb potential (cf. H11.1), then the energy eigenvalues Energieeigenwerte in a main shell will still be degenerated (besides the m degeneracy because of rotational invariance) addionally: the states with the same j will share the same energy. However, the ℓ degeneracy of the nonrelativistic problem will be removed by effects like the spin-orbit coupling (fine structure). This would also cancel the j degenary, but in fact there are more effects like the dispersion term (cf. H10.2), such that in the full solution the degeneracy will be restored.

However, in experiments it was observed that the states $2S_{j=\frac{1}{2}}$ and $2P_{j=\frac{1}{2}}$ do not have the same energy. This experiments was first done by Lamb and Retherford in 1947. We would like to comprehed the explanation given by Bethe also in 1947.

In addition, the interaction of the electrons and the proton spin creates a hyperfine splitting, but this energy scale is suppressed by the gyromagnetic ratio of the proton, i.e. by its large mass.

The effects fine structure, Lamb shift and hyperfine structure are distinguished by a factor of 10 in energy splitting. The absolute value of the fine structure splitting is in the range $10^{-4}\ldots 10^{-5}$ of typical atomic energies.

(1) Like in E4.1, consider the emission of a photon with wave number vector $\vec{k}$ and polarisation λ by an atom, making a transition from the state $|n\rangle$ to $|n'\rangle$ übergeht. In first order time dependent perturbation theory we obtained the transition rate, the lifetime etc. Now we compute that in *stationary* perturbation theory. In first order there is no contribution, since the interaction operator

$$H' = -\frac{e}{mc}\vec{A}(\vec{r},t)\cdot\vec{p} + \frac{e^2}{2mc^2}|\vec{A}(\vec{r},t)|^2 + e\,\underbrace{\Phi(\vec{r},t)}_{=0}$$

contains only 1- and 2-photon exchange processes and thus has no diagonal elements. However, in second order there will be a shift of the energy levels, since the emitted photon can be absorbed again, quantitatively we have:

$$\Delta E_n = \sum_{n',\vec{k},\lambda} \frac{|\langle n',\vec{k},\lambda|H'|n\rangle|^2}{E_n - E_{n'} - \hbar c|\vec{k}|}.$$

(2) We have already computed the matrix element of the interaction operator H':

$$\langle n',\vec{k},\lambda|H'|n\rangle = -\frac{\sqrt{\hbar}}{2\pi}\frac{1}{\sqrt{\omega_{\vec{k}}}}\langle n'|\vec{\jmath}(\vec{k})\cdot\vec{e}\,(\vec{k},\lambda)^*|n\rangle$$

where $\omega_{\vec{k}} = c|\vec{k}|$, $\vec{\jmath}(\vec{k}) = \frac{e}{m}\vec{p}\,\mathrm{e}^{-\mathrm{i}\vec{k}\cdot\vec{r}}$ is the "Fourier transformed" current density and $\vec{e}\,(\vec{k},\lambda)$ are the polarisation vectors of the photons. Then we have:

$$\Delta E_n = \int \mathrm{d}k\, k\frac{\hbar}{4\pi^2 c}\sum_{n'}\frac{\int \mathrm{d}\Omega\sum_\lambda |\langle n'|\vec{\jmath}(\vec{k})\cdot\vec{e}\,(\vec{k},\lambda)^*|n\rangle|^2}{E_n - E_{n'} - \hbar ck}.$$

(3) In the long wave approximation we stated $\vec{\jmath}(\vec{k}) = \vec{\jmath}(\vec{r}) = \frac{e}{m}\vec{p}$. The angular integration and the sum over the two polarisations gave rise to a factor of $\frac{8\pi}{3}$, such that:

$$\Delta E_n = \frac{2e^2\hbar}{3\pi m^2 c^3}\int_0^\infty \mathrm{d}\omega\, \omega \sum_{n'} \frac{|\langle n'|\vec{p}|n\rangle|^2}{E_n - E_{n'} - \hbar\omega}.$$

(4) The integration over ω results in an infinite value. In order to understand that, we do the same calculation for a free electron with momentum $\vec{q}$:

$$\Delta E_{\vec{q}} = \frac{2e^2\hbar}{3\pi m^2 c^3}\int_0^\infty \mathrm{d}\omega\, \omega \int \frac{\mathrm{d}^3 q'}{(2\pi)^3} \frac{|\langle \vec{q}\,'|\vec{p}|\vec{q}\rangle|^2}{E_{\vec{q}} - E_{\vec{q}\,'} - \hbar\omega} = -\frac{2e^2}{3\pi m^2 c^3}|\vec{q}|^2 \int_0^\infty \mathrm{d}\omega.$$

This gives a contribution which diverges with the same order as (3), but we see that the energy shift is proportional to $\vec{p}^{\,2}$, hence we can interpret this is as a shift of the electron mass. If the electron without any electromagnetic field has the kinetic energy $\frac{\vec{p}^{\,2}}{2m_0}$, i.e. mass m_0, due to the interaction it now will have the *renormalised* mass

$$\frac{1}{m} = \frac{1}{m_0} - \frac{4e^2}{3\pi m^2 c^3}\int_0^\infty \mathrm{d}\omega,$$

i.e. the kinetic energy is now $\frac{\vec{p}^{\,2}}{2m}$. Since we cannot switch off the interaction with the electromagnetic field, we will not be able to observe the "bare" mass m_0, but rather m is the value one can measure. Thus, we have to start with $H = \frac{\vec{p}^{\,2}}{2m_0} - \frac{e^2}{r} + H'$, then we have:

$$H = \frac{\vec{p}^{\,2}}{2m} - \frac{e^2}{r} + \underbrace{H' + \frac{2e^2\vec{p}^{\,2}}{3\pi m^2 c^3}\int_0^\infty \mathrm{d}\omega}_{H'_{\text{ren.}}}.$$

(5) The new $H'_{\text{ren.}}$ yields a contribution in first order perturbation theory, having the same order of magnitude as the second order term of H'. Thus, for the energy corrections with an e^2 as prefactor of the ω integral we have:

$$\Delta E_n = \frac{2e^2\hbar}{3\pi m^2 c^3}\int_0^\infty \mathrm{d}\omega\, \omega \left(\sum_{n'} \frac{|\langle n'|\vec{p}|n\rangle|^2}{E_n - E_{n'} - \hbar\omega} + \frac{\langle n|\vec{p}^{\,2}|n\rangle}{\hbar\omega}\right).$$

With the completeness relation $\langle n|\vec{p}^{\,2}|n\rangle = \sum_{n'} |\langle n'|\vec{p}|n\rangle|^2$ we have:

$$\Delta E_n = \frac{2e^2}{3\pi m^2 c^3}\sum_{n'} |\langle n'|\vec{p}|n\rangle|^2 \int_0^\infty \mathrm{d}\omega\, \frac{E_n - E_{n'}}{E_n - E_{n'} - \hbar\omega}.$$

(6) This integral is still logarithmically divergent, we make it finite by *regularisation* with a *cutoff* Λ as upper border at $\hbar\omega = mc^2$. We could have done the same thing for the divergent integrals in (3) and (4), but the result would much more depend

on this choice due to the linear divergence. The fact that one can really make that choice (provided, the cutoff is large compared to all energy scales occuring in the problem, in addition we use that there are no charged particle lighter than the electron in nature), corresponds to a second renormalisation, the renormalisation of the electric charge, done multiplicatively as running coupling constant:

$$e^2(\Lambda^2) = \frac{e_0^2}{1 - 12\frac{e_0^2}{\pi^2 \hbar c} \ln \frac{m^2c^4}{\Lambda^2}},$$

where e_0 is the bare charge. Note that $e(\Lambda = mc^2) = e_0$. Now we can compute the integral, here we have to take care of the pole in the denominator, such that we only consider the principal value, the additional imaginary part can be interpreted as decay width and is not of interest to us:

$$\Delta E_n = \frac{2e^2}{3\pi\hbar m^2 c^3} \sum_{n'} |\langle n'|\vec{p}|n\rangle|^2 \left(E_{n'} - E_n\right) \ln \frac{mc^2}{|E_{n'} - E_n|},$$

where the energy differences $E_{n'} - E_n$ are neglected compared to mc^2.

(7) With the following definition we eliminate the slowly varying logarithm from the sum over the intermediate states $|n'\rangle$:

$$\ln(\delta E_n)_{\text{av}} = \frac{\sum_{n'} |\langle n'|\vec{p}|n\rangle|^2 (E_{n'} - E_n) \ln |E_{n'} - E_n|}{\sum_{n'} |\langle n'|\vec{p}|n\rangle|^2 (E_{n'} - E_n)}.$$

Then we have:

$$\Delta E_n = \frac{2e^2}{3\pi\hbar m^2 c^3} \ln \frac{mc^2}{(\delta E_n)_{\text{av}}} \sum_{n'} |\langle n'|\vec{p}|n\rangle|^2 \left(E_{n'} - E_n\right).$$

The sum can now be converted to (H_0 is the Hamiltonian of the atom):

$$\sum_{n'} |\langle n'|\vec{p}|n\rangle|^2 \left(E_{n'} - E_n\right) = -\frac{1}{2} \sum_{m=1}^{3} \langle n|[p_m, [p_m, H_0]]|n\rangle.$$

The double commutator gives rise to a delta distribution in our case:

$$\Delta E_n = \frac{4e^4\hbar}{3m^2c^3} |\psi_n(0)|^2 \ln \frac{mc^2}{(\delta E_n)_{\text{av}}}.$$

Thus there is a contribution only for S states ($|\psi_{nS}(0)|^2 = \frac{1}{\pi(nr_0)^3}$):

$$\Delta E_{nS} = \frac{1}{3\pi} \frac{e^2}{2r_0} \left(\frac{2\alpha}{n}\right)^3 \ln \frac{mc^2}{(\delta E_{nS})_{\text{av}}},$$

where $\alpha = \frac{e^2}{\hbar c} \approx \frac{1}{137}$ and $\frac{e^2}{2r_0} \approx 13.6$ eV, to estimate the order of magnitude.

(8) $(\delta E_{nS})_{\text{av}}$ can be calculated numerically only, for the $2S$ state one obtains the huge (compared to the binding energy) value $(\delta E_{2S})_{\text{av}} = 248$ eV, with it a value of $\ln \frac{mc^2}{(\delta E_{2S})_{\text{av}}} = 7.63$ and so for ΔE_{2S} an energy shift corresponding to a frequency of 1034 MHz. In experiment one measures 1057 MHz, being fully reproduced in a real quantum electrodynamically calculation.

second check list for the final exam

We would like to suggest you to recapitulate (in addition to the first check list):

- different representations of the γ matrices: E10.1
- Weyl equation (in particular spinors under Lorentz transformations): E10.2
- Pauli equation (in particular rewriting the Dirac eqaution in Schrödinger-like form, g factor of 2), corrections like the spin-orbit coupling: H10.2
- bilinear invariants: E11.1
- Dirac equation for the H atom (general procedure, energy eigenvalues): H11.1