Exercises II

October 28th

E2.1 *Wigner–Eckart Theorem*

If $2\ell+1$ operators $T_{\ell M}$, $M=-\ell,-\ell+1,\ldots,\ell$ satisfy the commutation relations

$$[J_\pm, T_{\ell M}] = \hbar\sqrt{\ell(\ell+1)-M(M\pm 1)}\,T_{\ell(M\pm 1)} \quad\text{and}\quad [J_3,T_{\ell M}] = \hbar M\,T_{\ell M},$$

then these are the components of an *irreducible tensor operator of* ℓ^{th} *order* T_ℓ. In addition, one can define *cartesian* components, e.g. for $\ell=1$:

$$T_1 = \frac{1}{\sqrt{2}}(T_{1,-1}-T_{1,1}),\quad T_2 = \frac{\mathrm{i}}{\sqrt{2}}(T_{1,1}+T_{1,-1}),\quad T_3 = T_{1,0}.$$

(1) Show for the cartesian components of such a *vector operator* (since $\ell=1$) that

$$[J_m,T_n] = \mathrm{i}\hbar\sum_{k=1}^{3}\varepsilon_{mnk}T_k.$$

The *Wigner–Eckart theorem* proved in the lecture states: matrix elements of $T_{\ell M}$ with eigenstates of angular momentum depend on the m quantum numbers only via Clebsch–Gordan coefficients, where the factor $\langle j||T_\ell||j'\rangle$ is called *reduced matrix element* and is independent of m quantum numbers:

$$\langle j'm'|T_{\ell M}|jm\rangle = \langle jm\ell M|j\ell j'm'\rangle\,\frac{1}{\sqrt{2j+1}}\,\langle j'||T_\ell||j\rangle.$$

(2) Without using the theorem, show that the following condition for a non-vanishing matrix element holds:

$$m+M=m'.$$

(3) For a *scalar operator* $S=T_{00}$ ($\ell=0$), show that

$$\langle j'm'|S|jm\rangle \sim \delta_{jj'}\delta_{mm'} \quad\text{and does not denpend on } m.$$

(4) For a vector operator, show that a non-vanishing matrix element requires

$$|\Delta j| = |j-j'| \le 1 \quad\text{und}\quad |\Delta m| = |m-m'| \le 1,$$

with the additional condition that $j=j'=0$ is not allowed.

(5) Show that angular momentum is a vector operator, with $J_{\pm 1} = \mp\frac{1}{\sqrt{2}}J_\pm$, $J_0=J_3$.

(6) For a components V_M of a vector operator, show the so-called *projection theorem*

$$\langle jm'|V_M|jm\rangle = \frac{1}{\hbar^2 j(j+1)}\langle j\widetilde{m}|\vec{J}\cdot\vec{V}|j\widetilde{m}\rangle\langle jm'|J_M|jm\rangle.$$

Here, $\widetilde{m}$ is arbitrary, i.e. $\vec{J}\cdot\vec{V}$ is proportional to $\mathbb{1}$ on the space with fixed j.

(7) In assignment A3 of the exam Theoretical Physics II we considered a hydrogen atom in an electric quadrupole field, giving rise to $H'=c\,r^2\,Y_{20}$ as perturbation of the Hamiltonian. Show that all matrix elements $\langle\psi|H'|\psi'\rangle$ between eigenstates ψ,ψ' in the first and second main shell vanish, except $\Delta E_m = \langle\psi_m|H'|\psi_m\rangle$, where for brevity the states with angular momentum quantum number $\ell=1$ and magnetic quantum number m in the second main shell will be denoted by ψ_m. In addition, show that $\Delta E_1 = \Delta E_{-1} = -\frac{1}{2}\Delta E_0$.

Homework II

Return: November 4th

H2.1 *Landé factor*

Consider an atom in a magnetic field $\vec{B} = B\vec{e}_z$ (constant in time, homogeneous in space). Let the magnetic field be so weak that the perturbation Hamiltonia is given by $H' = \omega(L_z + 2S_z)$, where $\omega = \frac{eB}{2mc}$ is called Larmor frequency. We already know the term involving orbit angular momentum from the linear Zeeman effect (Theoretical Physics II, H7.1), the g factor 2 of the spin coupling follows from its relativistic nature and can be deduced in the Dirac theory.

(1) Let the Hamiltonian H of the atom contain the $\vec{L}\cdot\vec{S}$ term, the spin–orbit coupling (Theoretical Physics II, H9.2). Why are then H, $\vec{L}^{\,2}$, $\vec{S}^{\,2}$, $\vec{J}^{\,2}$ and J_3 a complete set of commuting observables?

(2) Why does an energy level have the degeneracy $2j+1$ at least?

(3) With the projection theorem from E2.1, show that $H' = g_J\omega J_3$ holds on the subspace of states with fixed E, j, ℓ and s, where

$$g_J = \frac{3}{2} + \frac{s(s+1) - \ell(\ell+1)}{2j(j+1)}$$

is the so-called Landé factor. How do the energy eigenvalues look like?

(10 points)

H2.2 *Central force field*

A particle in a central force potential has bound states $|n\ell m\rangle$ with energy $E_{n\ell}$.

(1) Using *only* the Wigner–Eckart theorem, relate, as much as possible, the matrix elements $\langle n'\ell'm'| \mp \frac{x\pm iy}{\sqrt{2}}|n\ell m\rangle$ and $\langle n'\ell'm'|z|n\ell m\rangle$.

(2) State under which conditions the matrix elements are non-vanishing.

(3) Do the same problem using wave functions $\psi(\vec{r}) = R_{nl}(r)Y_{\ell m}(\vartheta,\varphi)$.

(4) Consider $M_{mn} = \langle n|\frac{1}{|\vec{r}-\vec{a}|}|m\rangle$ for constant $\vec{a}$, where $|m\rangle$ and $|n\rangle$ are Coulomb eigenfunctions. Which multipoles of $\frac{1}{|\vec{r}-\vec{a}|}$ can contribute to M_{mn}?

(10 points)

H2.3 *Quadrupole moment*

(1) Write xy, xz, $x^2 - y^2$ as components of a spherical tensor of rank 2.

(2) The expectation value $Q = e\langle jj|3z^2 - r^2|jj\rangle$ is called *quadrupole moment*. Evaluate $e\langle jm'|x^2 - y^2|jj\rangle$ in terms of Q and appropriate Clebsch–Gordan coefficients.

(6 points)

H2.4 *More operatoren in Schwinger's model of angular momentum*

In Schwinger's model (cf. the lecture) one considers two pairs of mutually commuting raising and lowering operators a, a^+ and b, b^+. Then the angular momentum algebra is satisfied by $J_+ = \hbar a^+ b$, $J_- = \hbar b^+ a$ and $J_3 = \hbar(a^+a - b^+b)$. Discuss the physical significance of $K_+ = a^+b^+$, $K_- = ab$. Give the non-vanishing matrix elements of $K_\pm$.

(4 points)